Florida

HONORS

Prentice Hall

Algebra 2

Randall I. Charles
Basia Hall
Dan Kennedy
Allan E. Bellman
Sadie Chavis Bragg
William G. Handlin
Stuart J. Murphy
Grant Wiggins

Massachusetts • Chandler, Arizona • Glenview, Illinois • Upper Saddle River, New Jersey

Acknowledgments appear on page 806, which constitutes an extension of this copyright page.

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Contents *in Brief*

Welcome to Pearson's *Prentice Hall Algebra 2* student book. Throughout this textbook, you will find content that has been developed to cover all of the Sunshine State Standards for your Honors Algebra 2 class. A special section at the end of each lesson and chapter offers ongoing practice for the standards.

Series Authors

Randall I. Charles, Ph.D., is Professor Emeritus in the Department of Mathematics and Computer Science at San Jose State University, San Jose, California. He began his career as a high school mathematics teacher, and he was a mathematics supervisor for five years. Dr. Charles has been a member of several NCTM committees and is the former Vice President of the National Council of Supervisors of Mathematics. Much of his writing and research has been in the area of problem solving. He has authored more than 75 mathematics textbooks for kindergarten through college.

Dan Kennedy, Ph.D., is a classroom teacher and the Lupton Distinguished Professor of Mathematics at the Baylor School in Chattanooga, Tennessee. A frequent speaker at professional meetings on the subject of mathematics education reform, Dr. Kennedy has conducted more than 50 workshops and institutes for high school teachers. He is coauthor of textbooks in calculus and precalculus, and from 1990 to 1994 he chaired the College Board's AP Calculus Development Committee. He is a 1992 Tandy Technology Scholar and a 1995 Presidential Award winner.

Basia Hall currently serves as Manager of Instructional Programs for the Houston Independent School District. With 33 years of teaching experience, Ms. Hall has served as a department chair, instructional supervisor, school improvement facilitator, and professional development trainer. She has developed curricula for Algebra 1, Geometry, and Algebra 2 and co-developed the Texas state mathematics standards. A 1992 Presidential Awardee, Ms. Hall is past president of the Texas Association of Supervisors of Mathematics and is a state representative for the National Council of Supervisors of Mathematics (NCSM).

Consulting Authors

Stuart J. Murphy is a visual learning author and consultant. He is a champion of developing visual learning skills and using related strategies to help children become more successful students. He is the author of MathStart, a series of children's books that presents mathematical concepts in the context of stories. A graduate of the Rhode Island School of Design, he has worked extensively in educational publishing and has been on the authorship teams of a number of elementary and high school mathematics programs. He is a frequent presenter at meetings of the National Council of Teachers of Mathematics, the International Reading Association, and other professional organizations.

Grant Wiggins, Ed.D., is the President of Authentic Education in Hopewell, New Jersey. He earned his Ed.D. from Harvard University and his B.A. from St. John's College in Annapolis. Dr. Wiggins consults with schools, districts, and state education departments on a variety of reform matters; organizes conferences and workshops; and develops print materials and Web resources on curricular change. He is perhaps best known for being the coauthor, with Jay McTighe, of *Understanding by Design* and *The Understanding by Design Handbook*, the award-winning and highly successful materials on curriculum published by ASCD. His work has been supported by the Pew Charitable Trusts, the Geraldine R. Dodge Foundation, and the National Science Foundation.

Program Authors

Algebra 1 and Algebra 2

Allan E. Bellman, Ph.D., is a Lecturer/Supervisor in the School of Education at the University of California, Davis. Before coming to Davis, he was a mathematics teacher for 31 years in Montgomery County, Maryland. He has been an instructor for both the Woodrow Wilson National Fellowship Foundation and the T^3 program. He has been involved in the development of many products from Texas Instruments. Dr. Bellman has a particular expertise in the use of technology in education and speaks frequently on this topic. He was a 1992 Tandy Technology Scholar and has twice been listed in Who's Who Among America's Teachers.

Sadie Chavis Bragg, Ed.D., is Senior Vice President of Academic Affairs at the Borough of Manhattan Community College of the City University of New York. A former professor of mathematics, she is a past president of the American Mathematical Association of Two-Year Colleges (AMATYC), co-director of the AMATYC project to revise the standards for introductory college mathematics before calculus, and an active member of the Benjamin Banneker Association. Dr. Bragg has coauthored more than 50 mathematics textbooks for kindergarten through college.

William G. Handlin, Sr., is a classroom teacher and Department Chairman of Technology Applications at Spring Woods High School in Houston, Texas. Awarded Life Membership in the Texas Congress of Parents and Teachers for his contributions to the well-being of children, Mr. Handlin is also a frequent workshop and seminar leader in professional meetings throughout the world.

Geometry

Laurie E. Bass is a classroom teacher at the 9–12 division of the Ethical Culture Fieldston School in Riverdale, New York. A classroom teacher for more than 30 years, Ms. Bass has a wide base of teaching experience, ranging from Grade 6 through Advanced Placement Calculus. She was the recipient of a 2000 Honorable Mention for the Radio Shack National Teacher Awards. She has been a contributing writer for a number of publications, including software-based activities for the Algebra 1 classroom. Among her areas of special interest are cooperative learning for high school students and geometry exploration on the computer. Ms. Bass is a frequent presenter at local, regional, and national conferences.

Art Johnson, Ed.D., is a professor of mathematics education at Boston University. He is a mathematics educator with 32 years of public school teaching experience, a frequent speaker and workshop leader, and the recipient of a number of awards: the Tandy Prize for Teaching Excellence, the Presidential Award for Excellence in Mathematics Teaching, and New Hampshire Teacher of the Year. He was also profiled by the Disney Corporation in the American Teacher of the Year Program. Dr. Johnson has contributed 18 articles to NCTM journals and has authored over 50 books on various aspects of mathematics.

Reviewers *Florida*

Elizabeth Bosse
Math Department Co-Chair
New Smyrna Beach
High School
New Smyrna Beach, Florida

Michael Caballero
IB Math Teacher
Hillsborough High School
Tampa, Florida

Denise Chenowith
Math Teacher
Spoto High School
Riverview, Florida

Renee Gardner
Math Curriculum Support
Specialist
Miami-Dade Public Schools
Miami, Florida

Sheila Kotter
Math Teacher
River Ridge High School
New Port Richey, Florida

Suzie Davis Quido
Math Supervisor, retired
Pinellas County, Florida

Heliodoro Ramos
Math Teacher
Crestwood Middle School
Royal Palm Beach, Florida

Ellen Reilly
Math Teacher
Spanish River High School
Boca Raton, Florida

Erica Reynolds
Math Teacher
Estero High School
Estero, Florida

Sandra Urbano
Math Department Chair
Plantation High School
Plantation, Florida

Tennille Wallace
Math Supervisor
Flagler County
Palm Coast, Florida

From the **Authors**

Welcome

Math is a powerful tool with far-reaching applications throughout your life. We have designed a unique and engaging program that will enable you to tap into the power of mathematics and mathematical reasoning.

Developing mathematical skills and problem-solving strategies is an ongoing process—a journey both inside and outside the classroom. This course is designed to help make sense of the mathematics you encounter in and out of class each day.

You will learn important mathematical principles. You will also learn how the principles are connected to one another and to what you already know. You will learn to solve problems and learn the reasoning that lies behind your solutions.

Each chapter begins with the "big ideas" of the chapter and some essential questions that you will learn to answer. Through this question-and-answer process you will develop your ability to analyze problems independently and solve them in different applications.

Your skills and confidence will increase through practice and review. Work the examples so you understand the concepts and methods presented and the thinking behind them. Then do your homework. Ask yourself how new concepts relate to old ones. Make the connections!

Everyone needs help sometimes. You will find that this program has built-in opportunities, both in this text and online, to get help whenever you need it.

This course will also help you succeed on the tests you take in class and on other tests like the SAT, ACT, and state exams. The practice problems in each lesson will prepare you for the format and content of such tests. No surprises!

The reasoning habits and problem-solving skills you develop in this program will serve you in all your studies and in your daily life. They will prepare you for future success not only as a student, but also as a member of a changing technological society.

Best wishes,

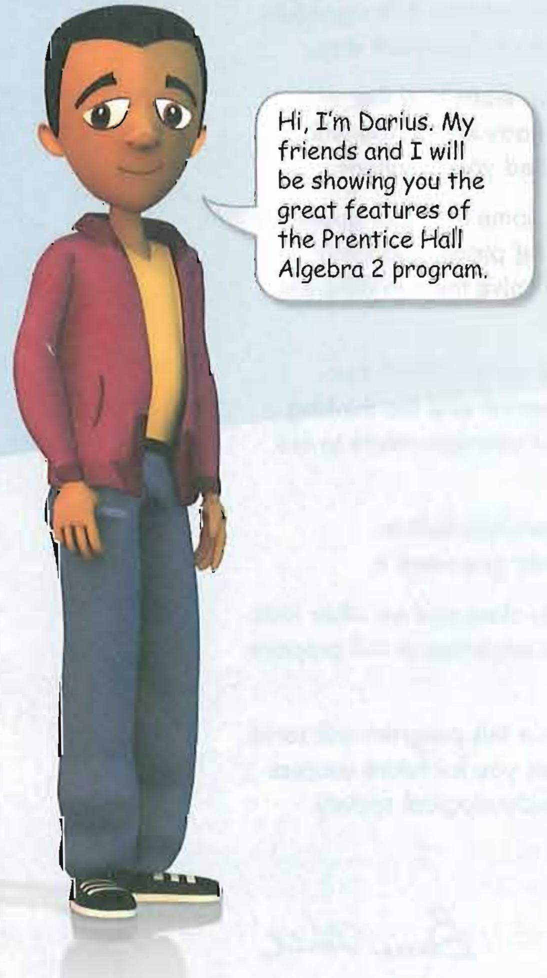
Dan Kennedy Randy Charles Basia Hall

Allan E. Bellman Sadie C. Bragg William G. Handlin

Laurie E. Bass Stuart J. Murphy

Power*Algebra.com*

Welcome to Algebra 2. *Prentice Hall Algebra 2* is part of an integrated digital and print environment for the study of high school mathematics. Take some time to look through the features of our mathematics program, starting with **PowerAlgebra.com,** the site of the digital features of the program.




PowerAlgebra.com

Your place to get all things digital

Download videos connecting math to your world.

Math definitions in English and Spanish

The online Solve It will get you in gear for each lesson.

Interactive! Vary numbers, graphs, and figures to explore math concepts.

Download Step-by-Step Problems with Instant Replay.

Get and view your assignments online.

Extra practice and review online

In each chapter opener, you will be invited to visit the **PowerAlgebra.com** site to access these online features. Look for these buttons throughout the lessons.

The Big Ideas are similar to the standards in each Body of Knowledge in the **Sunshine State Standards.** Both help you focus on key math concepts.

Sunshine State Standards

Big *Ideas*

We start with **Big Ideas.** Each chapter is organized around Big Ideas that convey the key mathematics concepts you will be studying in the program. Take a look at the Big Ideas on pages xx and xxi.

BIG ideas

1 **Modeling**

Essential Question How do you model a quantity that changes regularly over time by the same percentage?

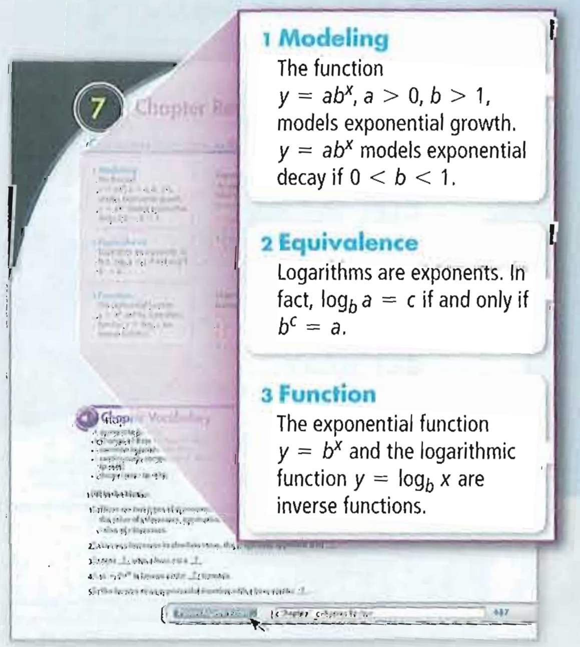
2 **Equivalence**

Essential Question How are exponents and logarithms related?

3 **Function**

Essential Question How are exponential functions and logarithmic functions related?

The **Big Ideas** are organizing ideas for all of the lessons in the program. At the beginning of each chapter, we'll tell you which Big Ideas you'll be studying. We'll also present an **Essential Question** for each Big Idea.



1 **Modeling**

The function $y = ab^x, a > 0, b > 1,$ models exponential growth. $y = ab^x$ models exponential decay if $0 < b < 1$.

2 **Equivalence**

Logarithms are exponents. In fact, $\log_b a = c$ if and only if $b^c = a$.

3 **Function**

The exponential function $y = b^x$ and the logarithmic function $y = \log_b x$ are inverse functions.

In the **Chapter Review** at the end of the chapter, you'll find the answers to the Essential Question for each Big Idea. We'll also remind you of the lesson(s) where you studied the concepts that support the Big Ideas.

Exploring *Concepts*

The lessons offer many opportunities to explore concepts in different contexts and through different media.

BIG ideas

Essential Question How do you model a quantity that changes regularly over time by the same percentage?

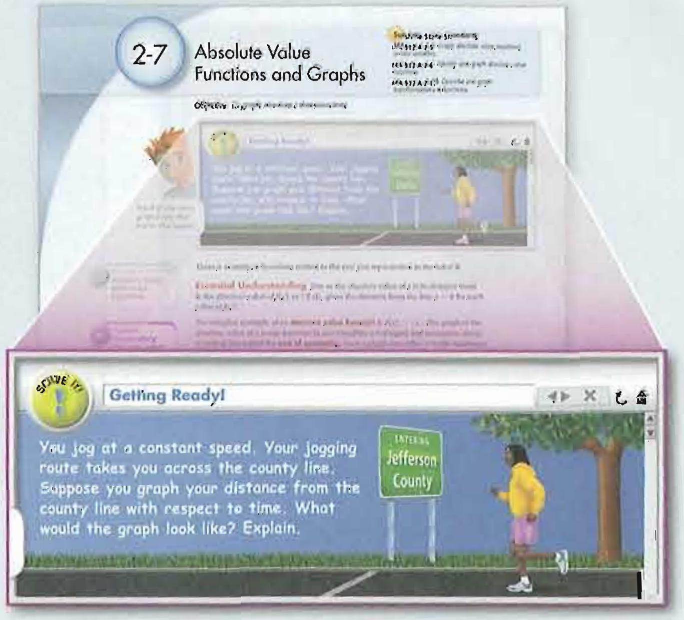
Essential Question How are exponents and logarithms related?

Essential Question How are exponential functions and logarithmic functions related?

Chapter Preview

7-1 Exploring Exponential Models
7-2 Properties of Exponential Functions
7-3 Logarithmic Functions as Inverses
7-4 Properties of Logarithms
7-5 Exponential and Logarithmic Equations
7-6 Natural Logarithms

For each chapter, there is a video that you can access at **PowerAlgebra.com.** The video presents concepts in a real-life context. And you can contribute your own math video.



2-7 Absolute Value Functions and Graphs

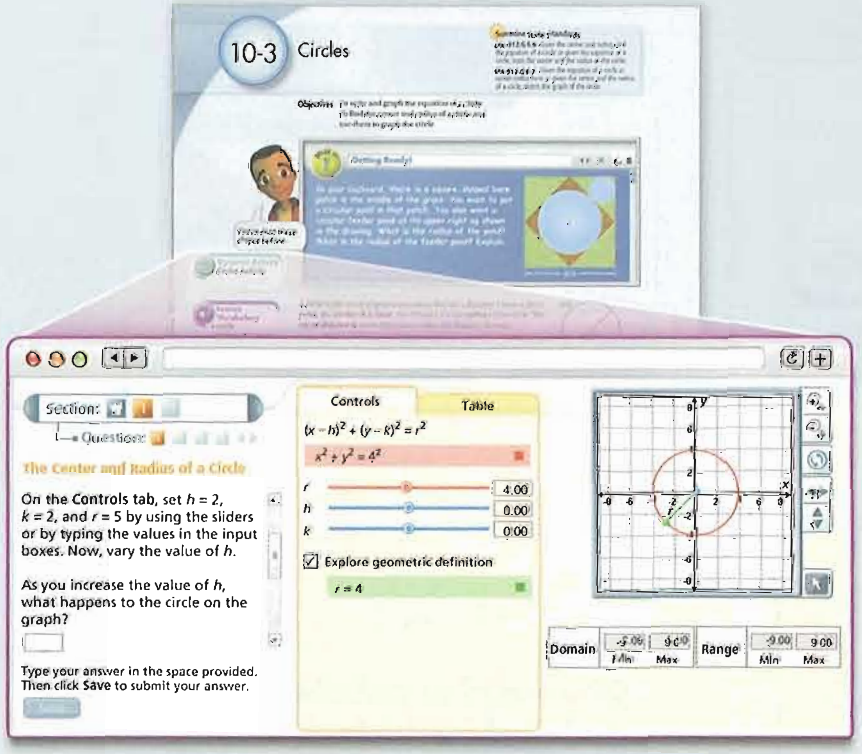
Getting Ready!

You jog at a constant speed. Your jogging route takes you across the county line. Suppose you graph your distance from the county line with respect to time. What would the graph look like? Explain.

Here's another cool feature. Each lesson opens with a **Solve It,** a problem that helps you connect what you know to an important concept in the lesson. Do you notice how the Solve It frame looks like it comes from a computer? That's because all of the Solve Its can be found at **PowerAlgebra.com.**

Exploring concepts in print and digitally helps you develop important **21st Century Skills,** such as technological literacy.

21st Century Skills



10-3 Circles

The Center and Radius of a Circle

On the Controls tab, set $h = 2$, $k = 2$, and $r = 5$ by using the sliders or by typing the values in the input boxes. Now, vary the value of h.

As you increase the value of h, what happens to the circle on the graph?

Type your answer in the space provided. Then click **Save** to submit your answer.

$(x - h)^2 + (y - k)^2 = r^2$

$x^2 + y^2 = 4^2$

r 4.00

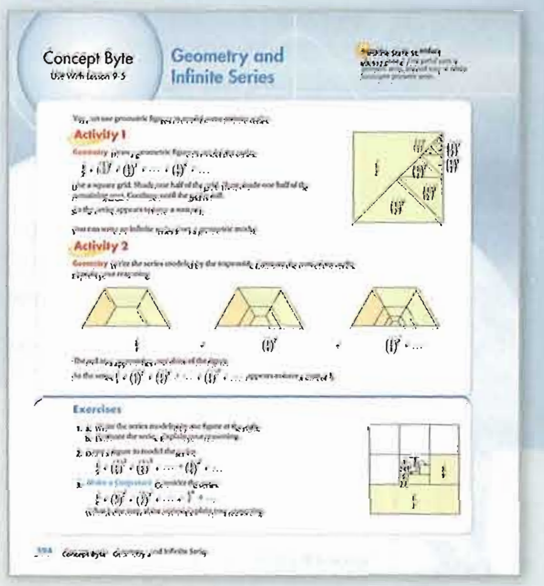
h 0.00

k 0.00

☑ Explore geometric definition

$r = 4$

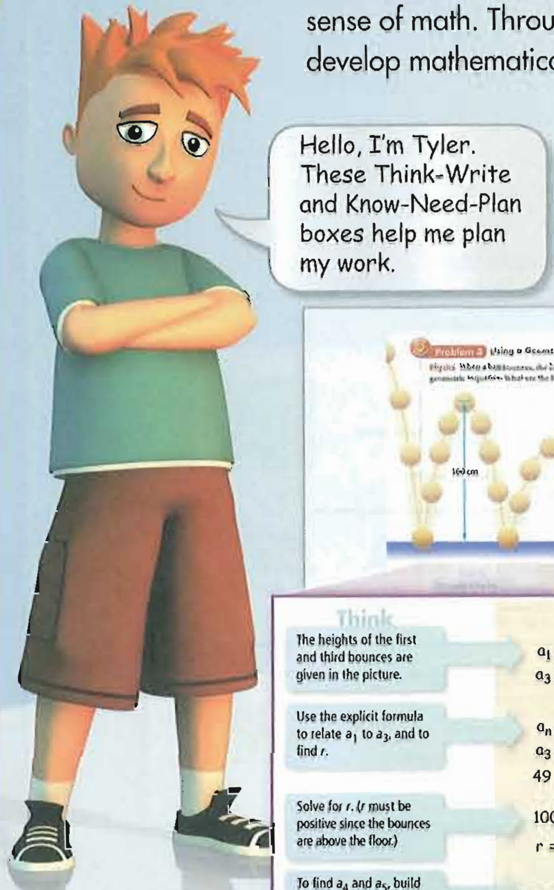
Want to do some more exploring? Look for this icon in your book. It lets you know that there is a **Dynamic Activity** at **PowerAlgebra.com.** With the Dynamic Activity, you can continue to explore the concept that is presented in the lesson.

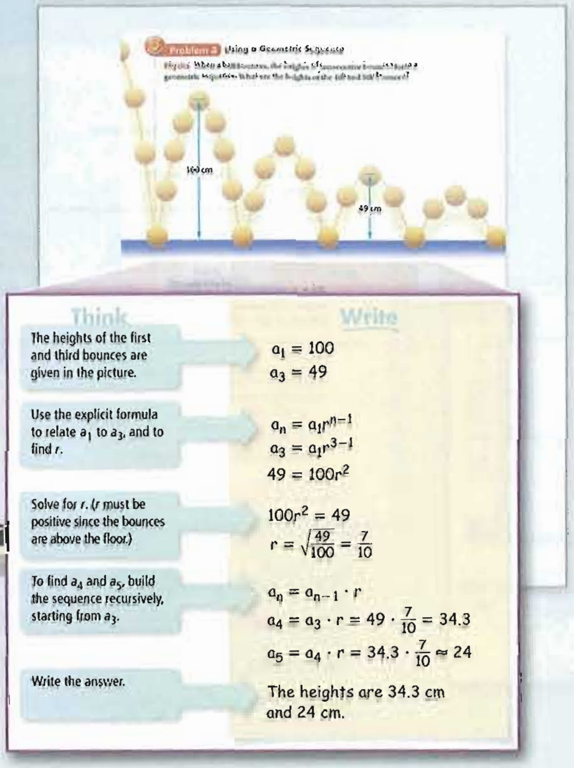


Try a **Concept Byte!** In a Concept Byte, you might explore technology, do a hands-on activity, or try a challenging extension.

Thinking *Mathematically*

Mathematical reasoning is the key to solving problems and making sense of math. Throughout the program you'll learn strategies to develop mathematical reasoning habits.

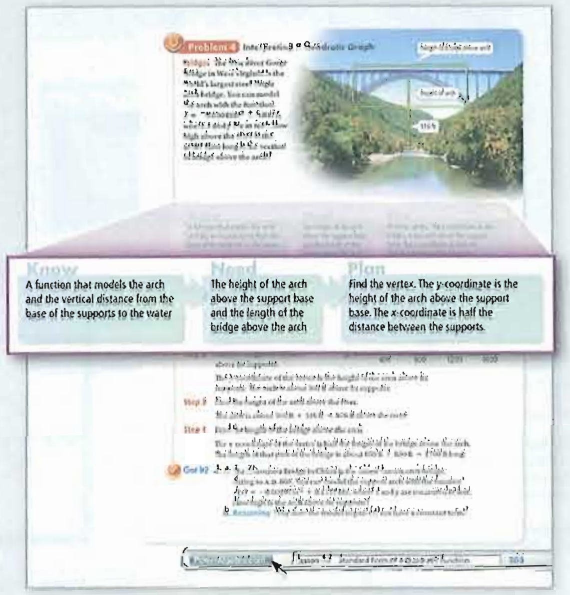




Think	Write
The heights of the first and third bounces are given in the picture.	$a_1 = 100$ $a_3 = 49$
Use the explicit formula to relate a_1 to a_3, and to find *r*.	$a_n = a_1r^{n-1}$ $a_3 = a_1r^{3-1}$ $49 = 100r^2$
Solve for *r*. (*r* must be positive since the bounces are above the floor.)	$100r^2 = 49$ $r = \sqrt{\frac{49}{100}} = \frac{7}{10}$
To find a_4 and a_5, build the sequence recursively, starting from a_3.	$a_n = a_{n-1} \cdot r$ $a_4 = a_3 \cdot r = 49 \cdot \frac{7}{10} = 34.3$ $a_5 = a_4 \cdot r = 34.3 \cdot \frac{7}{10} \approx 24$
Write the answer.	The heights are 34.3 cm and 24 cm.

The worked-out problems include call-outs that reveal the strategies and reasoning behind the solution. The **Think-Write** problems model the thinking behind each step of a solution.

Also, look for the boxes labeled **Plan** and **Think.**

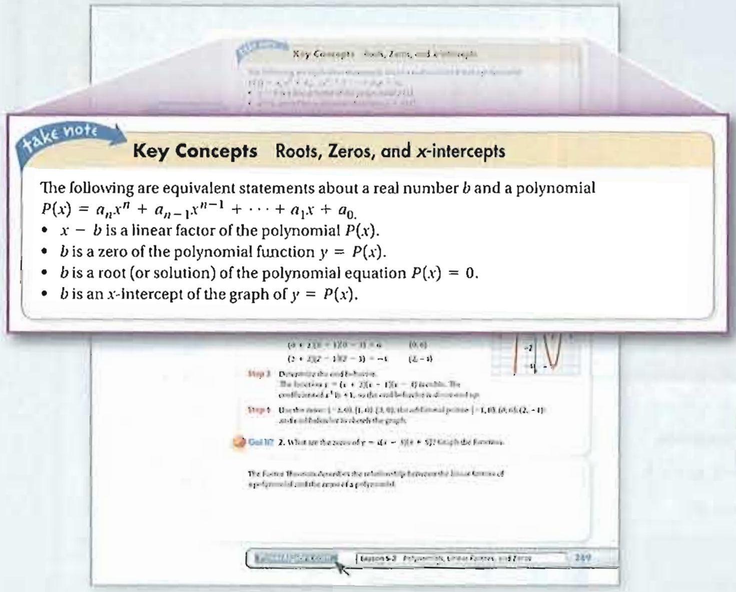


Know	Need	Plan
A function that models the arch and the vertical distance from the base of the supports to the water	The height of the arch above the support base and the length of the bridge above the arch	Find the vertex. The *y*-coordinate is the height of the arch above the support base. The *x*-coordinate is half the distance between the supports.

Other worked-out problems model a problem-solving plan that includes the steps of stating what you **Know,** identifying what you **Need,** and developing a **Plan.**

Good problem-solving strategies will help you score well on any state or national assessment such as **ADP Algebra 2 Exam.**

American Diploma Project Assessment Consortium

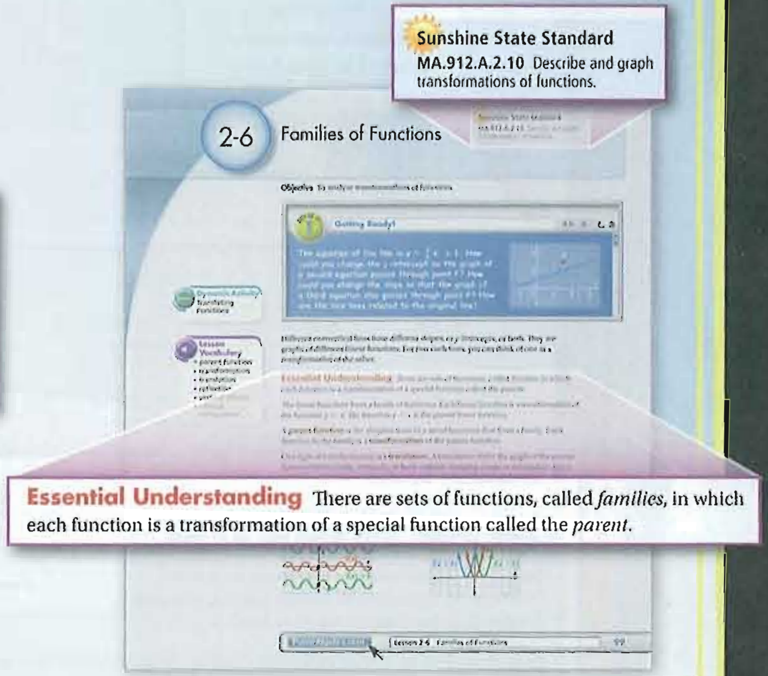


Key Concepts Roots, Zeros, and *x*-intercepts

The following are equivalent statements about a real number b and a polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.

- $x - b$ is a linear factor of the polynomial $P(x)$.
- b is a zero of the polynomial function $y = P(x)$.
- b is a root (or solution) of the polynomial equation $P(x) = 0$.
- b is an x-intercept of the graph of $y = P(x)$.

A **Take Note** box highlights key concepts in a lesson. You can use these boxes to review concepts throughout the year.



Sunshine State Standard
MA.912.A.2.10 Describe and graph transformations of functions.

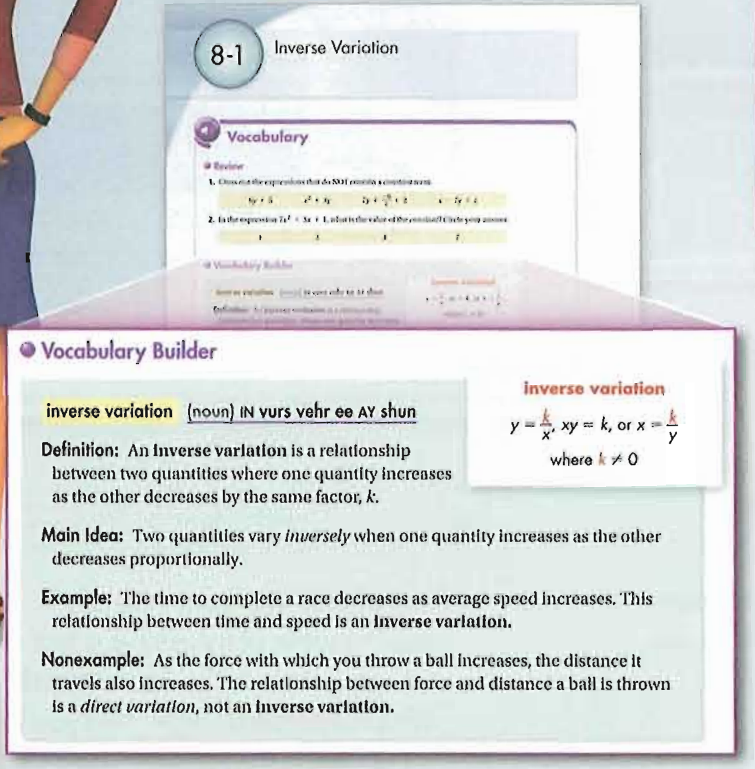
2-6 Families of Functions

Essential Understanding There are sets of functions, called *families*, in which each function is a transformation of a special function called the *parent*.

Part of thinking mathematically is making sense of the concepts that are being presented. The **Essential Understandings** help you build a framework for the Big Ideas. The small box at the top corner of the first page of a lesson tells you the benchmarks from the **Sunshine State Standards** you will be studying.

Active *Learning*

Through active learning, you become a successful, independent problem solver. The **Student Companion** has graphic organizers and other tools to help you master skills and problem solving.



8-1 Inverse Variation

Vocabulary Builder

inverse variation (noun) IN vurs vehr ee AY shun

Definition: An **inverse variation** is a relationship between two quantities where one quantity increases as the other decreases by the same factor, k.

Main Idea: Two quantities vary *inversely* when one quantity increases as the other decreases proportionally.

Example: The time to complete a race decreases as average speed increases. This relationship between time and speed is an **inverse variation.**

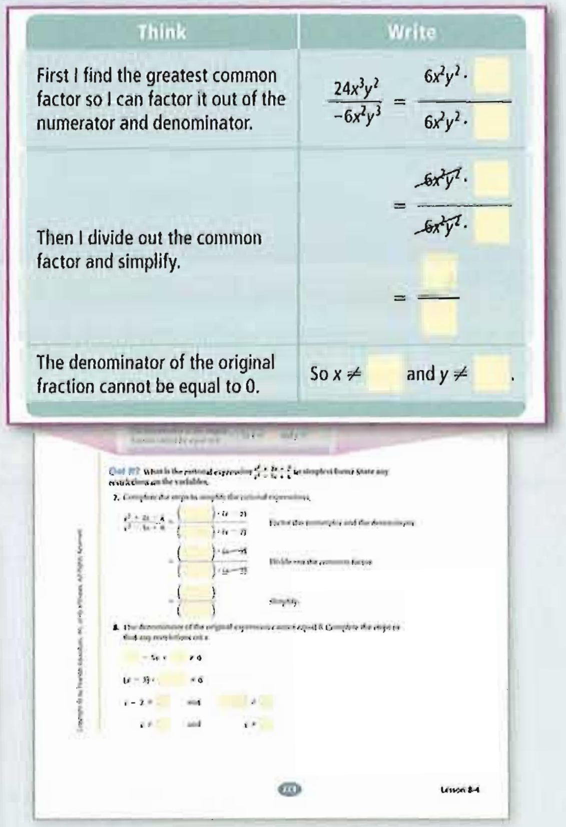
Nonexample: As the force with which you throw a ball increases, the distance it travels also increases. The relationship between force and distance a ball is thrown is a *direct variation*, not an **inverse variation.**

inverse variation

$y = \frac{k}{x}$, $xy = k$, or $x = \frac{k}{y}$

where $k \neq 0$

The Companion has a **Vocabulary Builder** for each lesson. After reading the definitions, examples and nonexamples in the Vocabulary Builder, you use the vocabulary in realistic contexts.

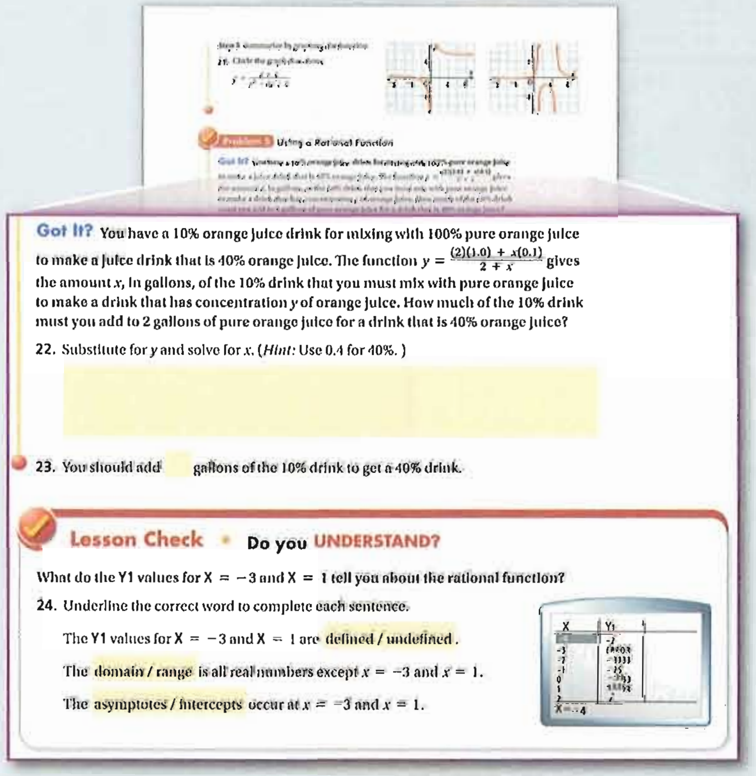


Think	Write
First I find the greatest common factor so I can factor it out of the numerator and denominator.	$\frac{24x^3y^2}{-6x^2y^3} = \frac{6x^2y^2 \cdot \square}{6x^2y^2 \cdot \square}$
Then I divide out the common factor and simplify.	$= \frac{\cancel{6x^2y^2} \cdot \square}{\cancel{6x^2y^2} \cdot \square}$ $= \frac{\square}{\square}$
The denominator of the original fraction cannot be equal to 0.	So $x \neq \square$ and $y \neq \square$.

The **Think-Write** format allows you to organize your thinking in order to solve a problem.

Efficient and effective problem solvers
are likely to score well on state
and national assessments and
be better prepared for
college studies.

SAT® and ACT®



Got It? You have a 10% orange juice drink for mixing with 100% pure orange juice to make a juice drink that is 40% orange juice. The function $y = \frac{(2)(1.0) + x(0.1)}{2 + x}$ gives the amount x, in gallons, of the 10% drink that you must mix with pure orange juice to make a drink that has concentration y of orange juice. How much of the 10% drink must you add to 2 gallons of pure orange juice for a drink that is 40% orange juice?

22. Substitute for y and solve for x. (*Hint:* Use 0.4 for 40%.)

23. You should add ____ gallons of the 10% drink to get a 40% drink.

Lesson Check • **Do you UNDERSTAND?**

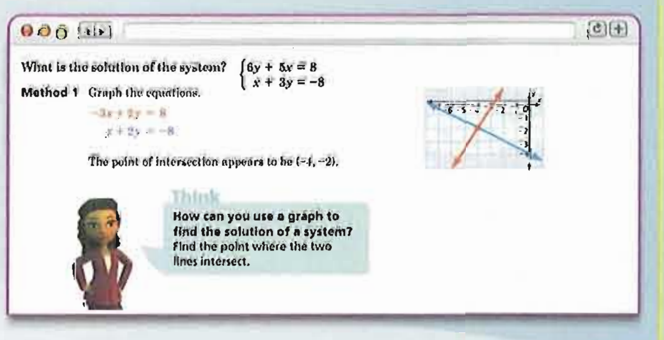
What do the Y1 values for X = −3 and X = 1 tell you about the rational function?

24. Underline the correct word to complete each sentence.

The Y1 values for X = −3 and X = 1 are defined / undefined .

The domain / range is all real numbers except $x = -3$ and $x = 1$.

The asymptotes / intercepts occur at $x = -3$ and $x = 1$.



What is the solution of the system? $\begin{cases} 6y + 5x = 8 \\ x + 3y = -8 \end{cases}$

Method 1 Graph the equations.

The point of intersection appears to be (−4, −2).

Think

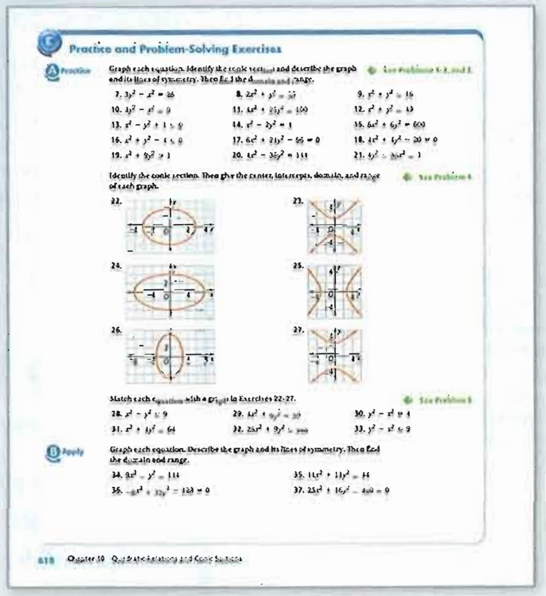
How can you use a graph to find the solution of a system?
Find the point where the two lines intersect.

Not sure you "got it" yet? Try out the **Online Problems** at **PowerAlgebra.com.** You will find some problems with stepped-out solutions as well as some helpful math tools, such as the graphing utility.

Use the **Got Its** and **Lesson Checks** to actively participate in the presentation of a lesson. These will help you make sure you understand a lesson before you do your homework.

Practice *Makes Perfect*

Ask any professional and you'll be told that the one requirement for becoming an expert is practice, practice, practice.

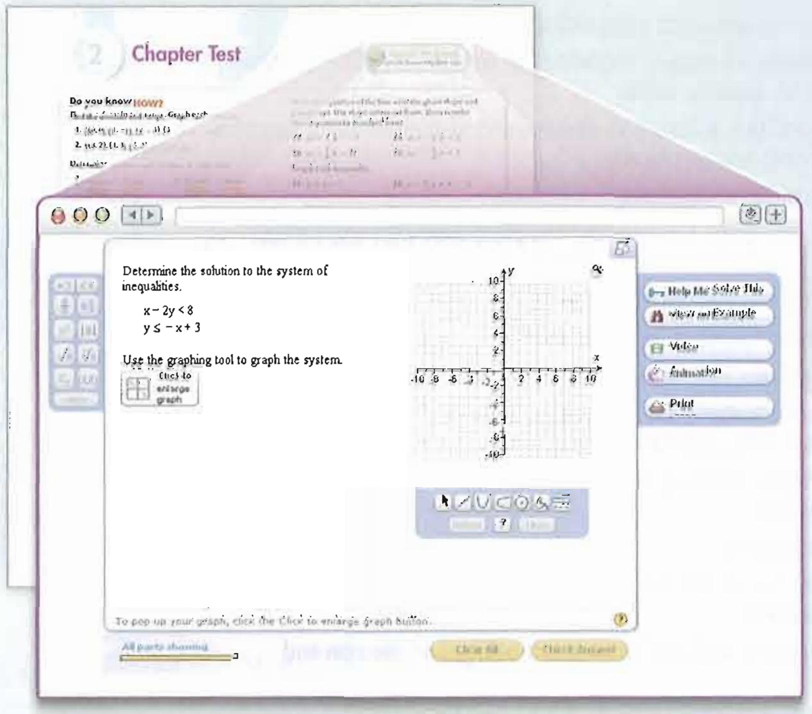

Practice and Problem-Solving Exercises

We give you lots of practice!
There are **Practice** exercises for each concept or skill. Having difficulty with any of them? The green arrow tells you what problem with a worked-out solution to revisit in the lesson.
In the **Apply** section, you apply the concepts or skills to different situations or contexts.

You can test your knowledge using the **Self-Quiz** for each lesson at **PowerAlgebra.com**

For every lesson, you can take a quiz online. This will help you be ready to take some tests and state or national assessments online.

Sunshine State Standards Practice



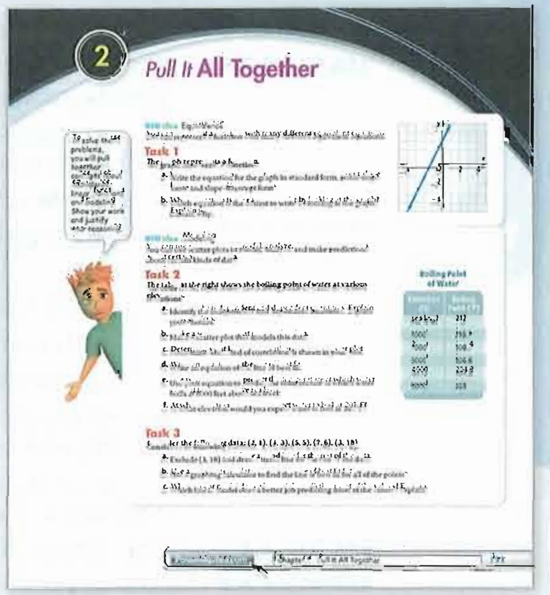
Determine the solution to the system of inequalities.

$x - 2y < 8$
$y \leq -x + 3$

Use the graphing tool to graph the system.

To pop up your graph, click the Click to enlarge graph button.

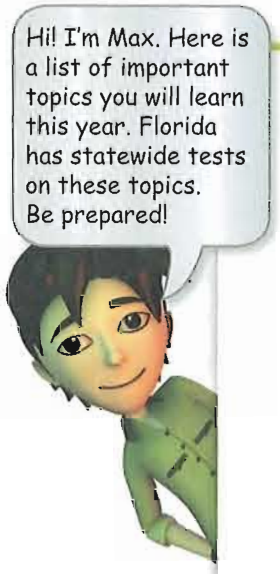
Want more practice? Look for this icon in your book. Check out all of the opportunities in **MathXL® for School.** Your teacher can assign you some practice exercises or you can choose some on your own. And you'll know right away if you got the right answer!



But the best practice occurs when you **Pull It All Together** — understanding of concepts, mathematical thinking, and problem solving — to solve interesting problems. And look: there are those Big Ideas again.

Sunshine State *Standards*

Honors Algebra 2



LA.910.1.6.1	The student will use new vocabulary that is introduced and taught directly.
LA.910.4.2.1	The student will write in a variety of informational/expository forms, including a variety of technical documents (e.g., how-to manuals, procedures, assembly directions).
MA.912.A.1.6	Identify the real and imaginary parts of complex numbers and perform basic operations.
MA.912.A.2.5	Graph absolute value equations and inequalities in two variables.
MA.912.A.2.6	Identify and graph common functions (including but not limited to linear, rational, quadratic, cubic, radical, absolute value).
MA.912.A.2.7	Perform operations (addition, subtraction, division, and multiplication) of functions algebraically, numerically, and graphically.
MA.912.A.2.8	Determine the composition of functions.
MA.912.A.2.9	Recognize, interpret, and graph functions defined piece-wise with and without technology.
MA.912.A.2.10	Describe and graph transformations of functions.
MA.912.A.2.11	Solve problems involving functions and their inverses.
MA.912.A.2.12	Solve problems using direct, inverse, and joint variations.
MA.912.A.3.14	Solve systems of linear equations and inequalities in two and three variables using graphical, substitution, and elimination methods.
MA.912.A.3.15	Solve real-world problems involving systems of linear equations and inequalities in two and three variables.
MA.912.A.4.3	Factor polynomial expressions.
MA.912.A.4.4	Divide polynomials by monomials and polynomials with various techniques, including synthetic division.
MA.912.A.4.5	Graph polynomial functions with and without technology and describe end behavior.
MA.912.A.4.6	Use theorems of polynomial behavior (including but not limited to the Fundamental Theorem of Algebra, Remainder Theorem, the Rational Root Theorem, Descartes' Rule of Signs, and the Conjugate Root Theorem) to find the zeros of a polynomial function.
MA.912.A.4.7	Write a polynomial equation for a given set of real and/or complex roots.
MA.912.A.4.8	Describe the relationships among the solutions of an equation, the zeros of a function, the *x*-intercepts of a graph, and the factors of a polynomial expression with and without technology.
MA.912.A.4.9	Use graphing technology to find approximate solutions for polynomial equations.
MA.912.A.4.10	Use polynomial equations to solve real-world problems.
MA.912.A.4.11	Solve a polynomial inequality by examining the graph with and without the use of technology.
MA.912.A.4.12	Apply the Binomial Theorem.

MA.912.A.5.6	Identify removable and non-removable discontinuities, and vertical, horizontal, and oblique asymptotes of a graph of a rational function, find the zeros, and graph the function.
MA.912.A.6.2	Add, subtract, multiply, and divide radical expressions (square roots and higher).
MA.912.A.6.3	Simplify expressions using properties of rational exponents.
MA.912.A.6.4	Convert between rational exponent and radical forms of expressions.
MA.912.A.6.5	Solve equations that contain radical expressions.
MA.912.A.7.3	Solve quadratic equations over the real numbers by completing the square.
MA.912.A.7.4	Use the discriminant to determine the nature of the roots of a quadratic equation.
MA.912.A.7.5	Solve quadratic equations over the complex number system.
MA.912.A.7.6	Identify the axis of symmetry, vertex, domain, range and intercept(s) for a given parabola.
MA.912.A.7.7	Solve non-linear systems of equations with and without using technology.
MA.912.A.7.10	Use graphing technology to find approximate solutions of quadratic equations.
MA.912.A.8.1	Define exponential and logarithmic functions and determine their relationship.
MA.912.A.8.2	Define and use the properties of logarithms to simplify logarithmic expressions and to find their approximate values.
MA.912.A.8.3	Graph exponential and logarithmic functions.
MA.912.A.8.5	Solve logarithmic and exponential equations.
MA.912.A.8.6	Use the change of base formula.
MA.912.A.8.7	Solve applications of exponential growth and decay.
MA.912.A.9.1	Write the equations of conic sections in standard form and general form, in order to identify the conic section and to find its geometric properties (foci, asymptotes, eccentricity, etc.).
MA.912.A.9.2	Graph conic sections with and without using graphing technology.
MA.912.A.10.3	Decide whether a given statement is always, sometimes, or never true (statements involving linear or quadratic expressions, equations, or inequalities, rational or radical expressions, or logarithmic or exponential functions).
MA.912.D.11.1	Define arithmetic and geometric sequences and series.
MA.912.D.11.2	Use sigma notation to describe series.
MA.912.D.11.3	Find specified terms of arithmetic and geometric sequences.
MA.912.D.11.4	Find partial sums of arithmetic and geometric series, and find sums of infinite convergent geometric series. Use sigma notation where applicable.
MA.912.G.6.6	Given the center and the radius, find the equation of a circle in the coordinate plane or given the equation of a circle in center-radius form, state the center and the radius of the circle.
MA.912.G.6.7	Given the equation of a circle in center-radius form or given the center and the radius of a circle, sketch the graph of the circle.

BIGideas

These Big Ideas are the organizing ideas for the study of important areas of mathematics: algebra, geometry, and statistics.

Algebra

Properties

- In the transition from arithmetic to algebra, attention shifts from arithmetic operations (addition, subtraction, multiplication, and division) to use of the *properties* of these operations.
- All of the facts of arithmetic and algebra follow from certain properties.

Variable

- Quantities are used to form expressions, equations, and inequalities.
- An expression refers to a quantity but does not make a statement about it. An equation (or an inequality) is a statement about the quantities it mentions.
- Using variables in place of numbers in equations (or inequalities) allows the statement of relationships among numbers that are unknown or unspecified.

Equivalence

- A single quantity may be represented by many different expressions.
- The facts about a quantity may be expressed by many different equations (or inequalities).

Solving Equations & Inequalities

- Solving an equation is the process of rewriting the equation to make what it says about its variable(s) as simple as possible.
- Properties of numbers and equality can be used to transform an equation (or inequality) into equivalent, simpler equations (or inequalities) in order to find solutions.
- Useful information about equations and inequalities (including solutions) can be found by analyzing graphs or tables.
- The numbers and types of solutions vary predictably, based on the type of equation.

Proportionality

- Two quantities are *proportional* if they have the same ratio in each instance where they are measured together.
- Two quantities are *inversely proportional* if they have the same product in each instance where they are measured together.

Function

- A function is a relationship between variables in which each value of the input variable is associated with a unique value of the output variable.
- Functions can be represented in a variety of ways, such as graphs, tables, equations, or words. Each representation is particularly useful in certain situations.
- Some important families of functions are developed through transformations of the simplest form of the function.
- New functions can be made from other functions by applying arithmetic operations or by applying one function to the output of another.

Modeling

- Many real-world mathematical problems can be represented algebraically. These representations can lead to algebraic solutions.
- A function that models a real-world situation can be used to make estimates or predictions about future occurrences.

Statistics and Probability

Data Collection and Analysis

- Sampling techniques are used to gather data from real-world situations. If the data are representative of the larger population, inferences can be made about that population.
- Biased sampling techniques yield data unlikely to be representative of the larger population.
- Sets of numerical data are described using measures of central tendency and dispersion.

Data Representation

- The most appropriate data representations depend on the type of data—quantitative or qualitative, and univariate or bivariate.
- Line plots, box plots, and histograms are different ways to show distribution of data over a possible range of values.

Probability

- Probability expresses the likelihood that a particular event will occur.
- Data can be used to calculate an experimental probability, and mathematical properties can be used to determine a theoretical probability.
- Either experimental or theoretical probability can be used to make predictions or decisions about future events.
- Various counting methods can be used to develop theoretical probabilities.

Geometry

Visualization

- Visualization can help you see the relationships between two figures and connect properties of real objects with two-dimensional drawings of these objects.

Transformations

- Transformations are mathematical functions that model relationships with figures.
- Transformations may be described geometrically or by coordinates.
- Symmetries of figures may be defined and classified by transformations.

Measurement

- Some attributes of geometric figures, such as length, area, volume, and angle measure, are measurable. Units are used to describe these attributes.

Reasoning & Proof

- Definitions establish meanings and remove possible misunderstanding.
- Other truths are more complex and difficult to see. It is often possible to verify complex truths by reasoning from simpler ones using deductive reasoning.

Similarity

- Two geometric figures are similar when corresponding lengths are proportional and corresponding angles are congruent.
- Areas of similar figures are proportional to the squares of their corresponding lengths.
- Volumes of similar figures are proportional to the cubes of their corresponding lengths.

Coordinate Geometry

- A coordinate system on a line is a number line on which points are labeled, corresponding to the real numbers.
- A coordinate system in a plane is formed by two perpendicular number lines, called the x- and y-axes, and the quadrants they form. The coordinate plane can be used to graph many functions.
- It is possible to verify some complex truths using deductive reasoning in combination with the distance, midpoint, and slope formulas.

1

Expressions, Equations, and Inequalities

FL Standards

Chapter 1

MA.912.A.3.3 Solve literal equations for a specified variable.

MA.912.A.3.6 Solve and graph the solutions of absolute value equations and inequalities with one variable.

MA.912.A.10.3 Decide whether a linear equation or inequality is always, sometimes, or never true.

Chapter 2

MA.912.A.2.6 Identify and graph common functions (including linear and absolute value).

MA.912.A.2.9 Recognize, interpret, and graph functions defined piece-wise, with and without technology.

MA.912.A.2.10 Describe and graph transformations of functions.

MA.912.A.2.12 Solve problems using direct variation.

2

Functions, Equations, and Graphs

Visual See It!

Reasoning Try It!

Practice Do It!

3

Linear Systems

FL Standards

Chapter 3

MA.912.A.3.14 Solve systems of linear equations and inequalities in two and three variables using graphical, substitution, and elimination methods.

MA.912.A.3.15 Solve real-world problems involving systems of linear equations and inequalities in two and three variables.

Chapter 4

MA.912.A.1.6 Identify the real and imaginary parts of complex numbers and perform basic operations.

MA.912.A.4.3 Factor polynomial expressions.

MA.912.A.7.3 Solve quadratic equations over the real numbers by completing the square.

MA.912.A.7.6 Identify the axis of symmetry, vertex, domain, range, and intercept(s) for a given parabola.

4

Quadratic Functions and Equations

CHAPTER 1

Expressions, Equations, and Inequalities

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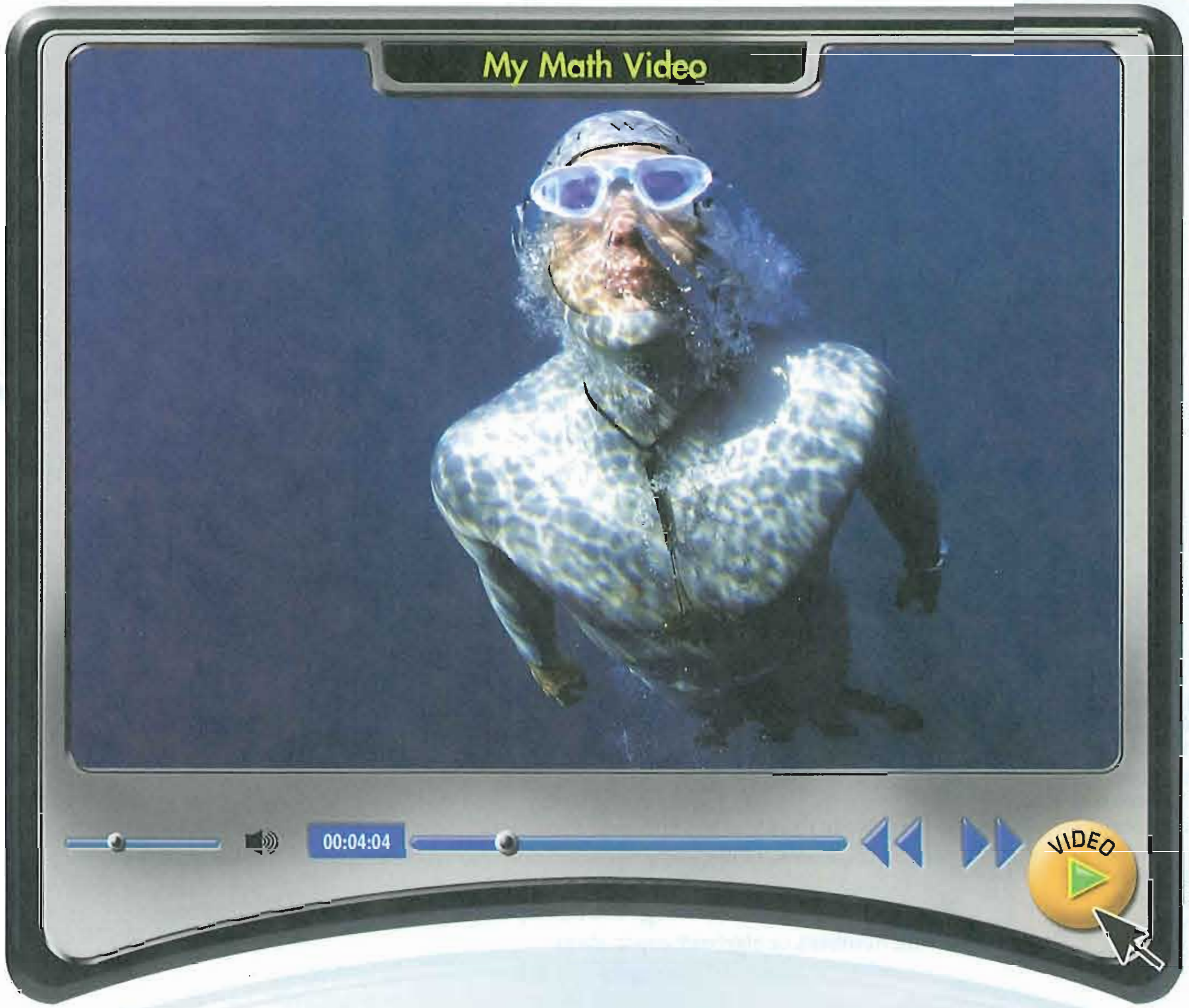
You can use variables to represent the distance or time of the swimmer in the video.

How can you use algebraic expressions to represent patterns? How can you solve equations and inequalities? How can you solve absolute value equations? You will learn how in this chapter.

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
absolute value, *p. 41*	valor absoluto
algebraic expression, *p. 5*	expresión algebraica
compound inequality, *p. 36*	desigualdad compuesta
like terms, *p. 21*	términos semejantes
literal equation, *p. 29*	ecuación literal
term, *p. 20*	término
variable, *p. 5*	variable



BIG ideas

1 **Variable**

Essential Question How do variables help you model real-world situations?

2 **Properties**

Essential Question How can you use the properties of real numbers to simplify algebraic expressions?

3 **Solving Equations and Inequalities**

Essential Question How do you solve an equation or inequality?

Chapter Preview

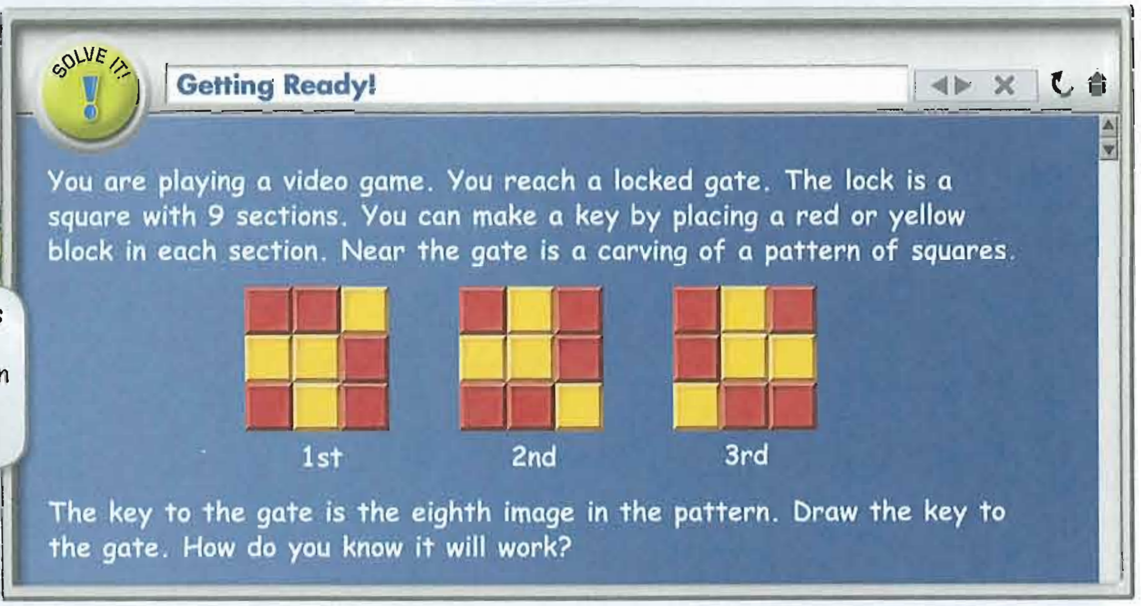
1-1 Patterns and Expressions

Sunshine State Standards

Prepares for MA.912.D.11.1 Define arithmetic and geometric sequences and series.

Prepares for MA.912.D.11.3 Find specified terms of arithmetic and geometric sequences.

Objective To identify and describe patterns



You are playing a video game. You reach a locked gate. The lock is a square with 9 sections. You can make a key by placing a red or yellow block in each section. Near the gate is a carving of a pattern of squares.

The key to the gate is the eighth image in the pattern. Draw the key to the gate. How do you know it will work?

Lesson Vocabulary
- constant
- variable quantity
- variable
- numerical expression
- algebraic expression

In the Solve It, you identified and used a geometric pattern. In this lesson, you will identify patterns in pictures, tables, and graphs and describe them using numbers and variables.

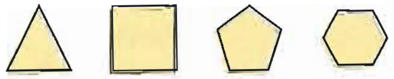
Essential Understanding You can represent some patterns using diagrams, words, numbers, or algebraic expressions.

Problem 1 Identifying a Pattern

Think

How can you identify a pattern?
Look for the same type of change between consecutive figures.

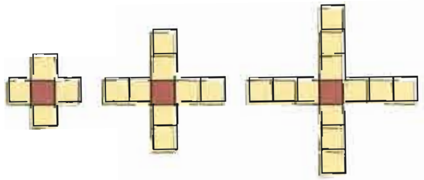
Look at the figures from left to right. What is the pattern? What would the next figure in the pattern look like?



The pattern shows regular polygons with the number of sides increasing by one.

The last figure shown above has six sides, so the next figure would have seven sides. This is a heptagon: .

Got It? **1.** Look at the figures from left to right. What is the pattern? Draw the next figure in the pattern.



A mathematical *quantity* is anything that can be measured or counted. The *value* of the quantity is its measure or the number of items that are counted. Quantities whose values do not change are called **constants**. In other situations, the value of a quantity can change. Quantities whose values change or vary are called **variable quantities**.

take note

Key Concepts Variables and Expressions

Definition	Examples	
A **variable** is a symbol, usually a letter, that represents one or more numbers.	n	x
A **numerical expression** is a mathematical phrase that contains numbers and operation symbols.	$3 + 5$	$(8 - 2) + 5$
An **algebraic expression** is a mathematical phrase that contains one or more variables.	$3n + 5$	$(8x - 2) + 5n$

Tables are a convenient way to organize data and discover patterns. They work much like an "input/output" machine: a machine that takes one value as an input, processes it, and gives a value as an output. A process column in the table provides a way to understand what happens to the input values.

Problem 2 Expressing a Pattern With Algebra

Use a pattern to answer each question.

A How many toothpicks are in the 20th figure?

Use a table. Look for a pattern that relates the figure number to the number of toothpicks.

Figure Number (Input)	Process Column	Number of Toothpicks (Output)
1	1(4)	4
2	2(4)	8
3	3(4)	12
⋮	⋮	⋮
n	■	■

To get the output, multiply the input by 4.

Pattern: Multiply the figure number by 4 to get the number of toothpicks. So, there are $20(4) = 80$ toothpicks in the 20th figure.

B What is an expression that describes the number of toothpicks in the *n*th figure?

Use the pattern from part (A). There are $4n$ toothpicks in the nth figure.

Think

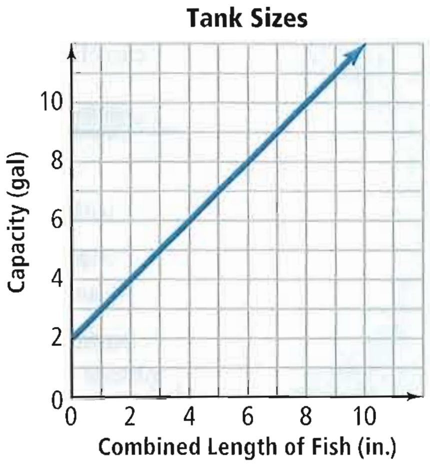
What would the process look like for the *n*th row?
Multiply the figure number, n, by 4.

Got It? 2. How many tiles are in the 25th figure in this pattern? Show a table of values with a process column.

Problem 3 Using a Graph

Aquarium You want to set up an aquarium and need to determine what size tank to buy. The graph shows tank sizes using a rule that relates the capacity of the tank to the combined lengths of the fish it can hold.

If you want five 2-in. platys, four 1-in. guppies, and a 3-in. loach, which is the smallest capacity tank you can buy: 15-gallon, 20-gallon, or 25-gallon? Use a table to find the answer.



Plan

How can you use the given graph?
You can use the graph to make a table and find a pattern relating capacity and combined fish length.

Think

Choose some points on the graph.

Write

(0, 2), (5, 7), (10, 12)

Think

Make a table using the input and output values shown in the ordered pairs.

Find a pattern in the process column. Each output is 2 more than the corresponding input.

Write

Input	Process Column	Output
0	0 + 2	2
5	5 + 2	7
10	10 + 2	12

Think

You want 5 platys, 4 guppies, and 1 loach. So, you will have a total of 17 in. of fish. Find the output when the input is 17.

Write

$$\begin{aligned} \text{output} &= \text{input} + 2 \\ &= 17 + 2 \\ &= 19 \end{aligned}$$

Think

Write the answer in words.

Write

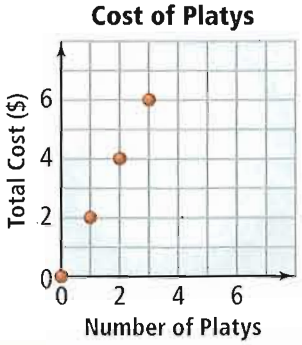
You need to buy the 20-gal tank.

Got It? 3. The graph at the right shows the total cost of platys at the aquarium shop. Use a table to answer the questions.

a. How much do six platys cost?

b. How much do ten platys cost?

c. **Reasoning** Why is the graph in Problem 3 a line while the graph at the right is a set of points?



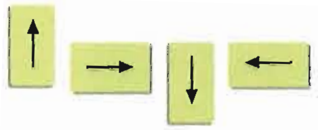
Lesson Check

Do you know HOW?

Describe a rule for each pattern.

1. 35, 70, 105, 140, . . .

2.



Make a table to represent each pattern. Use a process column.

3. 2, 4, 6, 8, . . .

4.

Do you UNDERSTAND?

5. Explain the strategy you use to identify a pattern.

6. **Compare and Contrast** How are tables of values like pictorial representations? How are they different?

7. **Error Analysis** Your friend looks for a pattern in the table below and claims that the output equals the input divided by 2. Is your friend correct? Explain.

Input	3	6.8	8	10	25
Output	2	3.4	4	5	12.5

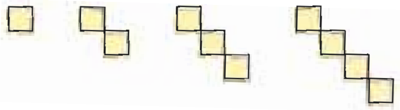
Practice and Problem-Solving Exercises

A Practice

Describe each pattern using words. Draw the next likely figure in each pattern.

See Problem 1.

8.



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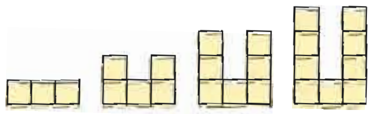

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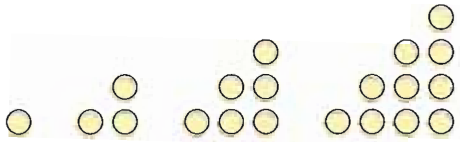
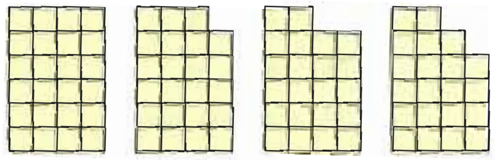


11.

Make a table with a process column to represent each pattern. Write an expression for the number of tiles or circles in the *n*th figure.

See Problem 2.

12.



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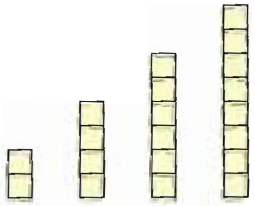

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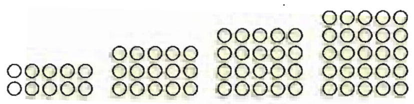


15.

Copy and complete each table. Include a process column.

16.

Input	Output
1	0
2	1
3	2
4	3
5	■
6	■
⋮	⋮
n	■

17.

Input	Output
1	3
2	4
3	5
4	6
5	■
6	■
⋮	⋮
n	■

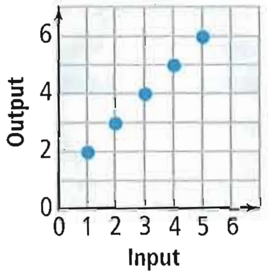
18.

Input	Output
1	−3
2	−6
3	−9
4	−12
5	■
6	■
⋮	⋮
n	■

Identify a pattern by making a table. Include a process column.

See Problem 3.

19.



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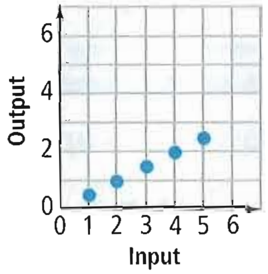
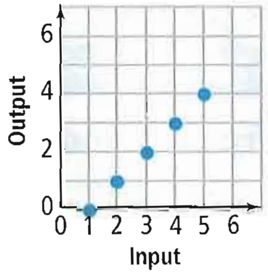
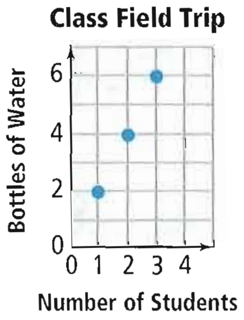


21.



The graph shows the number of bottles of water needed for students going on a field trip.



22. How many bottles of water are needed if 5 students attend?

23. How many bottles of water are needed if 20 students attend?

24. How many bottles of water are needed if n students attend?

B Apply

Identify a pattern and find the next three numbers in the pattern.

25. 6, 12, 18, 24, . . .

26. 1, 4, 3, 6, 5, . . .

27. 3, 6, 10, 15, . . .

28. 2, 6, 10, 14, . . .

29. 1, 3, 9, 27, 81, . . .

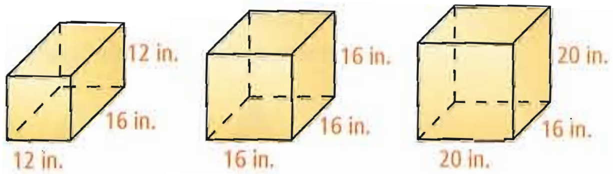
30. 4, 20, 100, 500, . . .

31. Identify a pattern and draw the next three figures in the pattern.

32. **Open-Ended** Write a rule so that for every input, the output is an even number.

33. **Think About a Plan** A moving company sells different sizes of boxes as shown. The extra-large box is one size larger than the third box shown. What is its volume?

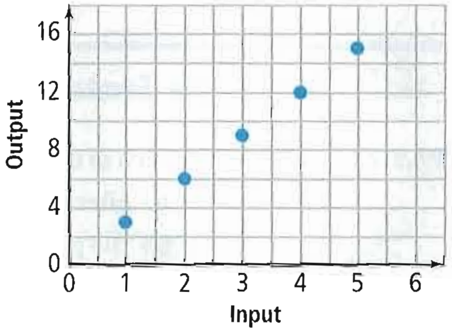
- Identify a pattern of the dimension changes.
- What is the formula for the volume of a rectangular prism?



34. Use the graph shown.

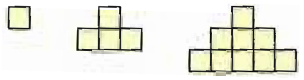
a. Identify a pattern of the graph by making a table of the inputs and outputs.

b. What are the outputs for inputs 6, 7, and 8?



35. **Collecting** Jay has a rare baseball card collection. He currently owns 10 baseball cards. Each month, he purchases a new card for his collection. Write a model to represent the number of cards in Jay's collection after n months.

36. Use the figures below.



a. Draw the next two figures.

b. Copy and complete the table to find the number of squares in each figure.

c. What is the number of squares in the nth figure? Explain your reasoning.

Figure (Input)	Process Column	Number of Squares (Output)
1	■	■
2	■	■
3	■	■
4	■	■
5	■	■

Copy and complete each table.

37.

Input	Output
1	5
2	9
3	13
4	17
5	■
⋮	⋮
n	■

38.

Input	Output
1	2
2	−3
3	−8
4	−13
5	■
⋮	⋮
n	■

39.

Input	Output
1	3
2	−1
3	−5
4	−9
5	■
⋮	⋮
n	■

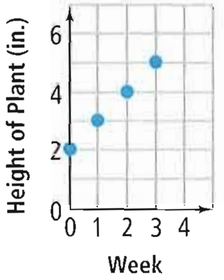
Challenge

40. **Open-Ended** Write the first five numbers of two different patterns in which 12 is the third number.

41. **Reasoning** For the past 4 years, Jesse has grown 3 inches each year. He is now 15 years old and is 5 feet 6 inches tall. He predicts that when he is 20 years old, he will be 6 feet 9 inches tall. What would you tell Jesse about his prediction?

42. This pattern shows the first five steps in constructing the Sierpinski Triangle. Use a pattern to describe the figures.

Sunshine State Standards Practice


Plant Growth

MA.912.A.2.2 **43.** Which of the following is the best statement about the graph?

(A) After 4 weeks, the plant will be 7 inches tall.
(B) The plant was 1 inch tall at the beginning of the experiment.
(C) After 2 weeks, the plant was 3 inches tall.
(D) After 6 weeks, the plant will be 8 inches tall.

MA.912.D.11.3 **44.** Which is the 7th number in this pattern?

8, 13, 18, 23, . . .

(F) 28
(G) 33
(H) 38
(I) 43

MA.912.D.11.3 **45. Short Response** Look at the pattern shown.

144, 72, 36, . . .

a. What is a rule for the pattern?
b. What is the first non-integer number in this pattern?

Mixed Review

Simplify each expression. See p. 677.

46. $3.6 + (-1.7)$
47. $1.2 - 5$
48. $(-3)(-9)$
49. $0(-8)$
50. $-2.8 \div 7$
51. $-35 \div (-5)$

Get Ready! **To prepare for Lesson 1-2, do Exercises 52–57.**

Write each number as a percent. See p. 674.

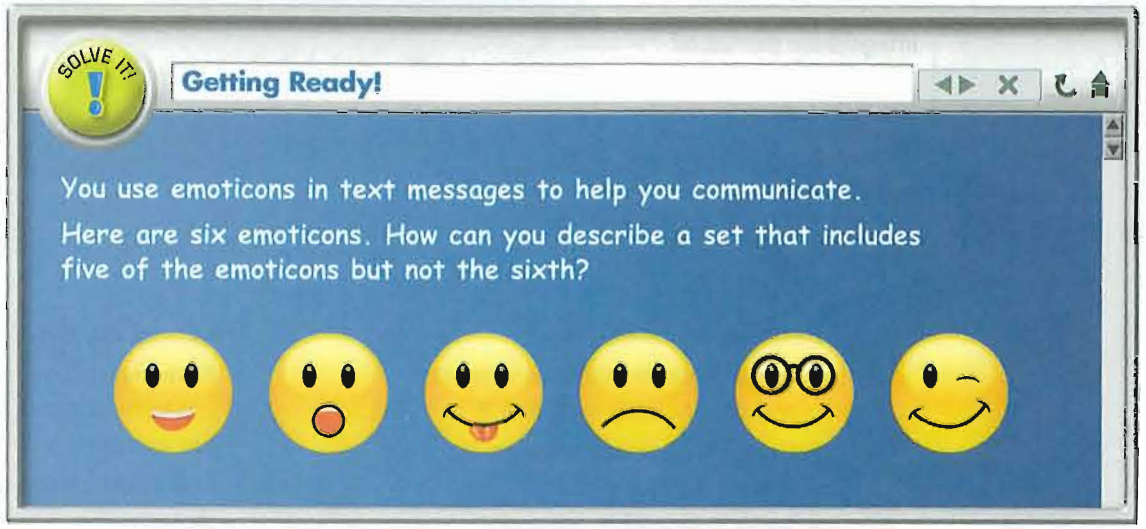
52. 0.5
53. 0.25
54. $\frac{1}{3}$
55. $1\frac{2}{5}$
56. 1.72
57. 1.23

1-2 Properties of Real Numbers

Sunshine State Standard

Prepares for MA.912.A.1.6 Identify the real and imaginary parts of complex numbers and perform basic operations.

Objectives To graph and order real numbers
To identify properties of real numbers



You use emoticons in text messages to help you communicate.

Here are six emoticons. How can you describe a set that includes five of the emoticons but not the sixth?

Dynamic Activity
Real Number Line

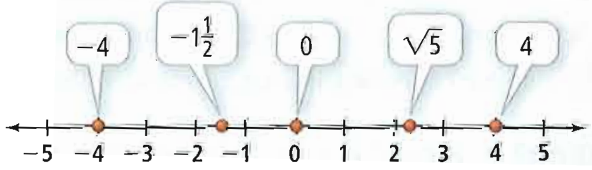
Lesson Vocabulary
- opposite
- additive inverse
- reciprocal
- multiplicative inverse

In the Solve It, you classified sets of emoticons. In this lesson, you will classify real numbers into special subsets.

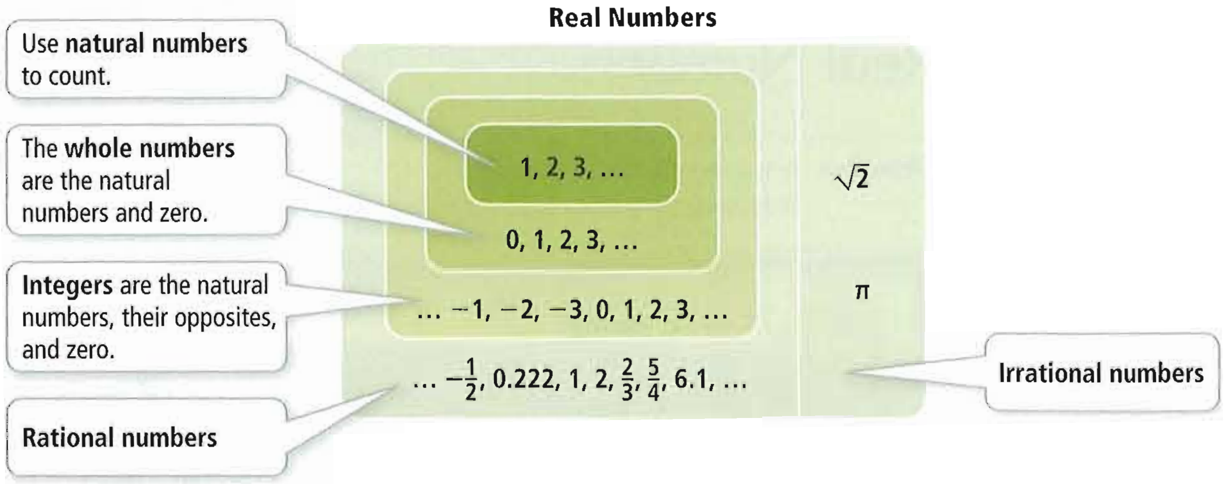
Essential Understanding The set of real numbers has several special subsets related in particular ways.

Algebra involves operations on and relations among numbers, including real numbers and imaginary numbers. (You will learn about imaginary numbers in Chapter 4.) Rational numbers and irrational numbers form the set of real numbers.

You can graph every real number as a point on the number line.



The diagram shows how subsets of the real numbers are related.



Rational numbers

- are all numbers you can write as a quotient of integers $\frac{a}{b}$, $b \neq 0$.
- include terminating decimals. For example, $\frac{1}{8} = 0.125$.
- include repeating decimals. For example, $\frac{1}{3} = 0.\overline{3}$.

Irrational numbers

- have decimal representations that neither terminate nor repeat. For example, $\sqrt{2} = 1.414213\ldots$.
- cannot be written as quotients of integers.

You classify a variable by naming the subset that gives you the most information about the numbers the variable represents.

Problem 1 Classifying a Variable

Multiple Choice Your school is sponsoring a charity race. Which set of numbers best describes the number of people *p* who participate in the race?

Ⓐ natural numbers
Ⓑ integers
Ⓒ rational numbers
Ⓓ irrational numbers

The number of people *p* is a natural number. The correct answer is A.

Think

Can you eliminate any answer choices? You can't have part of a person, so you can eliminate the rational and irrational numbers. The number of people can't be negative, so you can eliminate the integers.

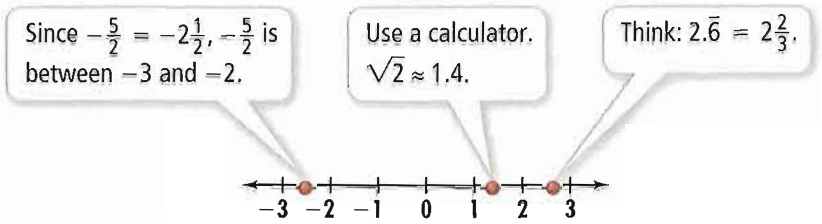
Got It? 1. In Problem 1, if each participant made a donation *d* of \$15.50 to a local charity, which subset of real numbers best describes the amount of money raised?

Problem 2 Graphing Numbers on the Number Line

Plan

How do you graph a number on the number line?
If the number is an integer, determine whether it is positive or negative. If it's not an integer, determine which integer it's closest to.

What is the graph of the numbers $-\frac{5}{2}$, $\sqrt{2}$, and $2.\overline{6}$?



Since $-\frac{5}{2} = -2\frac{1}{2}$, $-\frac{5}{2}$ is between -3 and -2.

Use a calculator. $\sqrt{2} \approx 1.4$.

Think: $2.\overline{6} = 2\frac{2}{3}$.

Got It? **2.** What is the graph of the numbers $\sqrt{3}$, $-1.\overline{4}$, and $\frac{1}{3}$?

The number line is helpful for ordering several real numbers. For two numbers, however, it is easier to show order, or compare, using one of the inequality symbols $>$ or $<$.

Problem 3 Ordering Real Numbers

Think

Why compare $\sqrt{17}$ to the square root of a perfect square?
It makes it easier to determine which two integers $\sqrt{17}$ is between.

How do $\sqrt{17}$ and 3.8 compare? Use $>$ or $<$.

Compare both numbers to $\sqrt{16}$.

$\sqrt{16} < \sqrt{17}$ 16 is less than 17.

$3.8 < \sqrt{16}$ $\sqrt{16} = 4$ and $3.8 < 4$.

Therefore, $3.8 < \sqrt{17}$, or $\sqrt{17} > 3.8$.

Check Use a calculator.

$\sqrt{17} \approx 4.123$

$3.8 < 4.123$ ✔

Got It? **3. a.** How do $\sqrt{26}$ and 6.25 compare? Use $>$ or $<$.

b. Reasoning Let a, b, and c be real numbers such that $a < b$ and $b < c$. How do a and c compare? Explain.

Essential Understanding The properties of real numbers are relationships that are true for all real numbers (except, in one case, zero).

One property of real numbers excludes a single number, zero. Zero is the *additive identity* for the real numbers, and zero is the one real number that has no *multiplicative inverse.*

The **opposite** or **additive inverse** of any number a is $-a$.
The sum of a number and its opposite is 0, the additive identity.

Examples $12 + (-12) = 0$ $-7 + 7 = 0$

The **reciprocal** or **multiplicative inverse** of any nonzero number a is $\frac{1}{a}$.
The product of a number and its reciprocal is 1, the multiplicative identity.

Examples $8\left(\frac{1}{8}\right) = 1$ $-5\left(-\frac{1}{5}\right) = 1$

take note **Properties** Properties of Real Numbers

Let a, b, and c represent real numbers.

Property	**Addition**	**Multiplication**
Closure	$a + b$ is a real number.	ab is a real number.
Commutative	$a + b = b + a$	$ab = ba$
Associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Identity	$a + 0 = a, 0 + a = a$ 0 is the additive identity.	$a \cdot 1 = a, 1 \cdot a = a$ 1 is the multiplicative identity.
Inverse	$a + (-a) = 0$	$a \cdot \frac{1}{a} = 1, a \neq 0$
Distributive	$a(b + c) = ab + ac$	

Problem 4 Identifying Properties of Real Numbers

Plan

How can you analyze an equation?
Determine whether it
- uses addition or multiplication
- reorders or regroups the numbers
- uses an identity

Which property does the equation illustrate?

A $\left(-\frac{2}{3}\right)\left(-\frac{3}{2}\right) = 1$

The product of the numbers is 1.

Inverse Property of Multiplication

B $(3 \cdot 4) \cdot 5 = (4 \cdot 3) \cdot 5$

The equation reorders 3 and 4.

Commutative Property of Multiplication

Got It? **4. a.** Which property does the equation $3(g + h) + 2g = (3g + 3h) + 2g$ illustrate?

b. **Reasoning** Use properties of real numbers to show that $a + [3 + (-a)] = 3$. Justify each step of your solution.

Lesson Check

Do you know HOW?

Write an example from daily life that uses each type of real number.

1. whole numbers

2. integers

3. rational numbers

Identify the property illustrated by the equation.

4. $5 + (-5) = 0$

5. $2 \cdot (4 \cdot 5) = (2 \cdot 4) \cdot 5$

Do you UNDERSTAND?

6. **Vocabulary** Identify another name for a reciprocal.

7. **Compare and Contrast** How is the Additive Identity Property similar to the Multiplicative Identity Property? How is it different?

8. **Reasoning** There are grouping symbols in the equation $(5 + w) + 8 = (w + 5) + 8$, but it does not illustrate the Associative Property of Addition. Explain.

9. Give an example of a number that is not a rational number. Explain why it is not rational.

Practice and Problem-Solving Exercises

A Practice

Classify each variable according to the set of numbers that best describes its values. **See Problem 1.**

10. the number of times n a ball bounces; the height h from which the ball is dropped

11. the year y; the median selling price p for a house that year

12. the circumference C of a circle found by using the formula $C = 2\pi r$

Graph each number on a number line. **See Problem 2.**

13. 0 **14.** $-\sqrt{24}$ **15.** -2 **16.** $2\frac{1}{2}$ **17.** $-4\frac{2}{3}$

18. 3.5 **19.** -1.4 **20.** $\sqrt{10}$ **21.** $-2\frac{1}{5}$ **22.** 4.8

Compare the two numbers. Use > or <. **See Problem 3.**

23. $16, \sqrt{16}$ **24.** $-4, -\sqrt{4}$ **25.** $\sqrt{5}, \sqrt{7}$

26. $-\sqrt{3}, -\sqrt{5}$ **27.** $5, \sqrt{22}$ **28.** $-\sqrt{38}, 6$

29. $4, \sqrt{12}$ **30.** $-8, \sqrt{70}$ **31.** $\sqrt{63}, 7.5$

32. $4.7, \sqrt{26}$ **33.** $\sqrt{75}, 9$ **34.** $12, -\sqrt{150}$

Name the property of real numbers illustrated by each equation. **See Problem 4.**

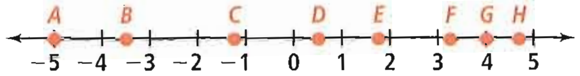
35. $\pi(a + b) = \pi a + \pi b$ **36.** $-10 + 4 = 4 + (-10)$

37. $(2\sqrt{7}) \cdot \sqrt{3} = 2(\sqrt{7} \cdot \sqrt{3})$ **38.** $29 \cdot \pi = \pi \cdot 29$

39. $-\sqrt{5} + 0 = -\sqrt{5}$ **40.** $\frac{4}{7} \cdot \frac{7}{4} = 1$

B Apply

Estimate the numbers graphed at the labeled points.



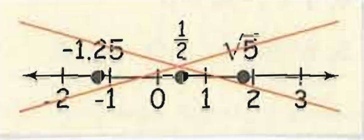
41. point A **42.** point B **43.** point C **44.** point D

45. point E **46.** point F **47.** point G **48.** point H

49. Think About a Plan A cube-shaped jewelry box has a surface area of 300 square inches. What are the dimensions of the jewelry box?

- Write an algebraic expression to find the total surface area of a cube. What is the surface area of one side of a cube?
- How is the side length of a square related to its area?

50. Error Analysis A student labeled the points on the number line as shown. Explain the student's error.



Science **The formula $I = \sqrt{\frac{W}{R}}$ gives the electric current I in amperes that flows through an appliance, where W is the power in watts and R is the resistance in ohms. Which set of numbers best describes the value of I for the given values of W and R?**

51. $W = 100, R = 25$ **52.** $W = 100, R = 5$ **53.** $W = 500, R = 100$

54. $W = 50, R = 200$ **55.** $W = 250, R = 100$ **56.** $W = 240, R = 100$

Write the numbers in decreasing order.

57. $1, -3, -\sqrt{2}, 8, \frac{1}{3}$ **58.** $\sqrt{14}, \frac{5}{2}, -\frac{9}{16}, 1, 11$ **59.** $-17, -0.06, -3\sqrt{3}, 5.73, \frac{1}{4}$

Reasoning **An example is a *counterexample* to a general statement if it makes the statement false. Show that each of the following statements is false by finding a counterexample.**

60. The reciprocal of each natural number is a natural number.

61. The opposite of each whole number is a whole number.

62. There is no integer that has a reciprocal that is an integer.

63. The product of two irrational numbers is an irrational number.

64. All square roots are irrational numbers.

65. Writing Write an example of each of the 11 properties of real numbers shown on page 14.

66. Restaurant Five friends each ordered a sandwich and a drink at a restaurant. Each sandwich costs the same amount and each drink costs the same amount. What are two ways to compute the bill? What property of real numbers is illustrated by the two methods?

67. Open-Ended Write an algebraic problem that requires the use of the real-number properties to solve. Then solve the problem.

Challenge

68. **Writing** Are there two integers with a product of -12 and a sum of -3? Explain.

69. Your friend used the Distributive Property and got the expression $5x + 10y - 35$. What algebraic expression could your friend have started with?

70. **Geometry** π is an irrational number you can use to calculate the circumference or area of a circle.
  - a. Find the value of π on your calculator. Can you obtain an exact representation? Explain.
  - b. The value of π is often represented as $\frac{22}{7}$. How does this representation compare to the decimal representation your calculator gives using the π key?

71. Does zero have a multiplicative inverse? Explain.

Sunshine State Standards Practice

MA.912.A.1.2

72. Which of the following shows the numbers π, $\sqrt{8}$, and 3.5 in the correct order from greatest to least?

Ⓐ $\pi, \sqrt{8}, 3.5$ Ⓑ $3.5, \pi, \sqrt{8}$ Ⓒ $\sqrt{8}, \pi, 3.5$ Ⓓ $\sqrt{8}, 3.5, \pi$

MA.912.A.2.2

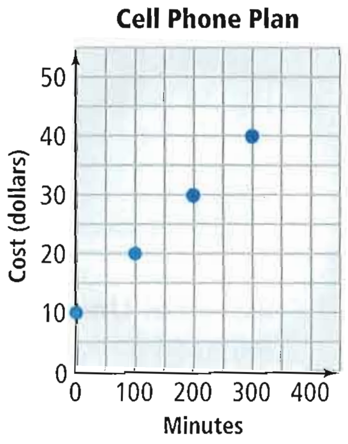
73. Which of the following is the best statement about the graph?

Ⓕ A 400-minute plan costs \$40.

Ⓖ A 100-minute plan costs \$10.

Ⓗ A 1000-minute plan costs \$110.

Ⓘ A 200-minute plan costs \$35.



MA.912.A.1.2

74. **Short Response** Why is the opposite of the reciprocal of 5 the same as the reciprocal of the opposite of 5?

Mixed Review

Identify a pattern and find the next three terms in the pattern. See Lesson 1-1.

75. 4, 8, 12, 16, . . . 76. 8, 9, 10, 11, . . . 77. $-4, -3, -2, -1, \ldots$

Get Ready! **To prepare for Lesson 1-3, do Exercises 78–83.**

Use the order of operations to simplify each expression. See p. 677.

78. $3 \div 4 + 6 \div 4$

79. $5[(2 + 5) \div 3]$

80. $\frac{8 + 5 \times 2}{12}$

81. $(40 + 24) \div 8 - (2^2 - 1)$

82. $40 + 24 \div 8 - 2^2 - 1$

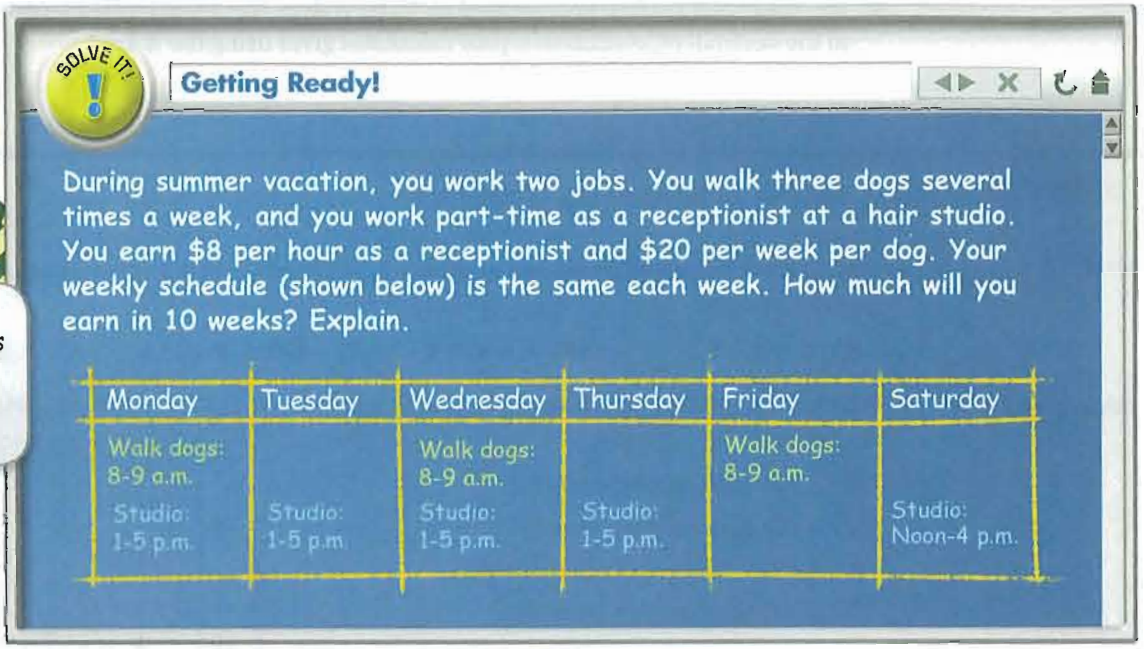
83. $(40 + 24) \div (8 - 2^2) - 1$

1-3 Algebraic Expressions

Sunshine State Standard
Prepares for MA.912.A.2.7 Perform operations (addition, subtraction, division and multiplication) of functions algebraically and numerically.

Objectives To evaluate algebraic expressions
To simplify algebraic expressions

Ten weeks is a long time! Perhaps you can solve a simpler problem first.



SOLVE IT! **Getting Ready!**

During summer vacation, you work two jobs. You walk three dogs several times a week, and you work part-time as a receptionist at a hair studio. You earn $8 per hour as a receptionist and $20 per week per dog. Your weekly schedule (shown below) is the same each week. How much will you earn in 10 weeks? Explain.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Walk dogs: 8-9 a.m.		Walk dogs: 8-9 a.m.		Walk dogs: 8-9 a.m.	
Studio: 1-5 p.m.	Studio: 1-5 p.m.	Studio: 1-5 p.m.	Studio: 1-5 p.m.		Studio: Noon-4 p.m.

Lesson Vocabulary
- evaluate
- term
- coefficient
- constant term
- like terms

Essential Understanding You can represent some mathematical phrases and real-world quantities using algebraic expressions.

Problem 1 Modeling Words With an Algebraic Expression

Think
What does "seven fewer than t" mean?
"Seven fewer than t" means your answer will be less than t.

Multiple Choice Which algebraic expression models the word phrase *seven fewer than a number t*?

Ⓐ $t + 7$ Ⓑ $-7t$ Ⓒ $t - 7$ Ⓓ $7 - t$

"Seven fewer than" suggests subtraction. Begin with the number t and subtract 7. This can be represented by the expression $t - 7$. The correct answer is C.

Got It? **1.** Which algebraic expression models the word phrase *two times the sum of a and b*?

Ⓕ $a + b$ Ⓗ $2(a + b)$
Ⓖ $2a + b$ Ⓘ $a + 2b$

To model a situation with an algebraic expression, do the following:

- Identify the actions that suggest operations.
- Define one or more variables to represent the unknown(s).
- Represent the actions using the variables and the operations.

Problem 2 Modeling a Situation

Plan

How can you identify the variable? Determine which quantity is unknown.

Savings **You start with \$20 and save \$6 each week. What algebraic expression models the total amount you save?**

Relate	starting amount	plus	amount saved	times	number of weeks
Define	Let w = the number of weeks.				
Write	20	$+$	6	$\cdot$	w

The expression $20 + 6w$ models the situation.

Got It? **2.** You had \$150, but you are spending \$2 each day. What algebraic expression models this situation?

To **evaluate** an algebraic expression, substitute a number for each variable in the expression. Then simplify using the order of operations.

Problem 3 Evaluating Algebraic Expressions

Plan

What operations should you start with? Do operations that occur in grouping symbols first. Parentheses are grouping symbols.

What is the value of the expression for the given values of the variables?

A $7(a + 4) + 3b - 8$ **for** $a = -4$ **and** $b = 5$

$7(-4 + 4) + 3(5) - 8$	Substitute the value for each variable.
$= 7(0) + 3(5) - 8$	Perform operations within grouping symbols.
$= 0 + 15 - 8$	Multiply.
$= 15 - 8$	Add and subtract from left to right.
$= 7$	

B $\frac{x}{2} + y^2$ **for** $x = 1$ **and** $y = \frac{1}{2}$

$\frac{1}{2} + \left(\frac{1}{2}\right)^2$	Substitute the value for each variable.
$= \frac{1}{2} + \frac{1}{4}$	Simplify the power.
$= \frac{3}{4}$	Add.

Got It? **3. a.** What is the value of the expression $\frac{2(x^2 - y^2)}{3}$ for $x = 6$ and $y = -3$?

b. **Reasoning** Will the value of the expression change if the parentheses are removed? Explain.

Problem 4 Writing and Evaluating an Expression

Sports In football, a touchdown (TD) is worth six points, an extra-point kick (EPK) one point, and a field goal (FG) three points. What algebraic expression models the total number of points that a football team scores in a game, assuming each scoring play is one of the three given types? Suppose a football team scores 3 touchdowns, 2 extra-point kicks, and 4 field goals. How many points did the team score?

Know	Need	Plan
• Number of points each scoring play is worth • Number of each type of score	• Algebraic expression to model points scored • Total number of points scored	• Determine the variables. • Write an expression. • Evaluate the expression.

Think

How many points come from touchdowns?
The number of points from touchdowns is six times the number of touchdowns.

Relate points per TD · number of TDs + points per EPK · number of EPKs + points per FG · number of FGs

Define Let t = the number of touchdowns.
Let k = the number of extra-point kicks.
Let f = the number of field goals.

Write $6 \cdot t + 1 \cdot k + 3 \cdot f$

The expression $6t + 1k + 3f$ models the team's total score.

The football team scores 3 touchdowns, 2 extra-point kicks, and 4 field goals, so $t = 3$, $k = 2$, and $f = 4$.

$6(3) + 1(2) + 3(4)$ Substitute the value for each variable.
$= 18 + 2 + 12$ Multiply.
$= 32$ Add.

The team scored 32 points.

Got It? 4. In basketball, teams can score by making two-point shots, three-point shots, and one-point free throws. What algebraic expression models the total number of points that a basketball team scores in a game? If a team makes 10 two-point shots, 5 three-point shots, and 7 free throws, how many points does it score in all?

An expression that is a number, a variable, or the product of a number and one or more variables is a **term**. A **coefficient** is the numerical factor of a term. A **constant term** is a term with no variables. You can add terms to form longer expressions. The expression below has three terms.

$-4ax + 7w - 6$

coefficients

constant term
Think of $7w - 6$ as $7w + (-6)$.
The constant term is -6.

Like terms have the same variables raised to the same powers.

like terms   like terms

$3x^2 + 5x^2 + 9y^3z + 2yz - 4y^3z$

You can simplify an algebraic expression that has like terms. You combine like terms using the properties of real numbers (Lesson 1-2). An expression and its simplified form are equivalent. Their values are equal for all values of their variables.

take note

Concept Summary Properties for Simplifying Algebraic Expressions

Let a, b, and c represent real numbers.

Definition of Subtraction	$a - b = a + (-b)$
Definition of Division	$a \div b = \frac{a}{b} = a \cdot \frac{1}{b}, b \neq 0$
Distributive Property for Subtraction	$a(b - c) = ab - ac$
Multiplication by 0	$0 \cdot a = 0$
Multiplication by −1	$-1 \cdot a = -a$
Opposite of a Sum	$-(a + b) = -a + (-b) = -a - b$
Opposite of a Difference	$-(a - b) = -a + b = b - a$
Opposite of a Product	$-(ab) = -a \cdot b = a \cdot (-b)$
Opposite of an Opposite	$-(-a) = a$

ONLINE PROBLEMS

Problem 5 Simplifying Algebraic Expressions

Combine like terms. What is a simpler form of each expression?

Think

Are $7x^2$ and $3y^2$ like terms?

No; they have different variables.

A $7x^2 + 3y^2 + 2y^2 - 4x^2$

$7x^2 + 3y^2 + 2y^2 - 4x^2$	Identify like terms.
$= 7x^2 - 4x^2 + 3y^2 + 2y^2$	Commutative Property of Addition
$= (7 - 4)x^2 + (3 + 2)y^2$	Distributive Property
$= 3x^2 + 5y^2$	Combine like terms.

B $-(3k + m) + 2(k - 4m)$

$-3k - m + 2k - 8m$	Opposite of a Sum and Distributive Property
$= -k - 9m$	Combine like terms.

Got It? **5.** Combine like terms. What is a simpler form of each expression?

a. $-4j^2 - 7k + 5j + j^2$

b. $-(8a + 3b) + 10(2a - 5b)$

Lesson Check

Do you know HOW?

Write an algebraic expression that models each word phrase.

1. the quotient of the sum of 2 and a number b, and 3

2. the sum of the product of a number k and 4, and a number m

Evaluate each algebraic expression for $x = 3$ and $y = -2$.

3. $2x - 3y$

4. $5x + y$

5. $y - x$

6. $x + 4y$

Do you UNDERSTAND?

7. Error Analysis A student simplified the expression as shown.

~~$3p^2q + 2p - (5q + p - 2p^2q) = q^2p + 3p - 5q$~~

Identify the errors and correct them.

8. Vocabulary Explain the difference between a constant and a coefficient.

9. Compare and Contrast How are algebraic expressions and numerical expressions alike? How are they different? Include examples to justify your reasoning.

Practice and Problem-Solving Exercises

A Practice

Write an algebraic expression that models each word phrase. **See Problem 1.**

10. four more than a number b

11. the product of 8 and the sum of a number x and 3

12. the quotient of the difference between 5 and a number n, and 2

Write an algebraic expression that models each situation. **See Problem 2.**

13. Jenny had \$130, but she is spending \$10 per week.

14. The piggy bank contained \$25, and \$1.50 is added each day.

15. You had 250 minutes left on your cell phone, and you talk an hour a week.

Evaluate each expression for the given values of the variables. **See Problem 3.**

16. $4a + 7b + 3a - 2b + 2a$; $a = -5$ and $b = 3$

17. $-k^2 - (3k - 5n) + 4n$; $k = -1$ and $n = -2$

18. $-5(x + 2y) + 15(x + 2y)$; $x = 7$ and $y = -7$

19. $4(2m - n) - 3(2m - n)$; $m = -15$ and $n = -18$

Physics **The expression $16t^2$ models the distance in feet that an object falls during the first t seconds after being dropped. What is the distance the object falls during each time?**

20. 0.25 second

21. 0.5 second

22. 2 seconds

23. 10 seconds

Investing The expression $1000(1.1)^t$ represents the value of a \$1000 investment that earns 10% interest per year, compounded annually for t years. What is the value of a \$1000 investment at the end of each period?

24. 2 years
25. 3 years
26. 4 years
27. 5 years

Write an algebraic expression to model the total score in each situation. Then evaluate the expression to find the total score.

See Problem 4.

28. In the first set, the volleyball team made only 8 shots worth one point each.

29. In the last baseball game, there were two 3-run home runs and 4 hits that each scored 2 runs.

Simplify by combining like terms.

See Problem 5.

30. $5a - a$
31. $5 + 10s - 8s$
32. $-5a - 4a + b$
33. $2a + 3b + 4a$
34. $6r + 3s + 2s + 4r$
35. $0.5x - x$
36. $7b - (3a - 8b)$
37. $5 + (4g - 7)$
38. $-(3x - 4y) + x$

B Apply

Evaluate each expression for the given value of the variable.

39. $x + 2x - x - 1; x = 2$
40. $2z + 3 + 5 - 3z; z = -3$
41. $3(2a + 5) + 2(3 - a); a = 4$
42. $\frac{5(2k - 3) - 3(k + 4)}{3k + 2}; k = -2$
43. $y^2 + 3; y = \sqrt{7}$
44. $5c^3 - 6c^2 - 2c; c = -5$

45. Think About a Plan Tran's truck gets very poor gas mileage. If Tran pays \$84 to fill his truck with gas and is able to drive m miles on a full tank, what expression shows his gas cost per mile?

- What operation does "per" indicate?
- Check your expression by substituting 200 miles for m. Does your answer make sense?

Simplify by combining like terms.

46. $-a^2 + 2b^2 + \frac{1}{4}a^2$
47. $x + \frac{x^2}{2} + 2x^2 - x$
48. $\frac{y^2}{4} + \frac{y}{3} + \frac{y^2}{3} - \frac{y}{5}$
49. $-(2x + y) - 2(-x - y)$
50. $x(3 - y) + y(x + 6)$
51. $\frac{1}{2}(x^2 - y^2) - \frac{5}{2}(x^2 - y^2)$

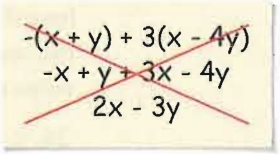
Write an algebraic expression to model each situation.

52. Class Project The freshman class will be selling carnations as a class project. What is the class's income after it pays the florist a flat fee of \$200 and sells x carnations for \$2 each?

53. Jobs You have a summer job at a car wash. You earn \$8.50 per hour and are expected to pay a one-time fee of \$15 for the uniform. If you work x hours per week, how much will you make during the first week?

54. Reasoning Suppose you need to subtract a from b but mistakenly subtract b from a instead. How is the answer you get related to the correct answer? Explain.

55. **Error Analysis** John simplified the expression as shown. Describe and correct his error.



~~-(x + y) + 3(x - 4y)~~
~~-x + y + 3x - 4y~~
~~2x - 3y~~

56. **Open-Ended** Write an example of an algebraic expression that has a nonnegative value regardless of the value of the variable.

Name the property of real numbers illustrated by the equation.

57. $2(s - t) = 2s - 2t$

58. $-[-(x - 10)] = x - 10$

59. $-(2t - 11) = 11 - 2t$

60. $-(a - b) = (-1)(a - b)$

Challenge

61. Simplify $2(b - a) + 5(b - a)$ and justify each step in your simplification.

62. **a.** Evaluate the expression $2(2x^2 - x) - 3(x^2 - x) + x^2 - x$ for $x = 3$. Do *not* simplify the expression before evaluating it.

b. Simplify the expression in part (a) and then evaluate your answer for $x = 3$.

c. **Writing** Explain why the values in parts (a) and (b) should be the same.

Sunshine State Standards Practice

MA.912.A.1.3

63. Which expression best represents a simpler form of $4m + 3(m + n)$?

Ⓐ $7m + 3n$ Ⓑ $4m + 3mn$ Ⓒ $3m + 4n$ Ⓓ $7m^2 + 3n$

MA.912.A.10.1

64. A driver drove 12 miles and made a pit stop. After that, the driver continued driving at a constant speed of 65 miles per hour for t hours. Which of the following represents the total distance driven?

Ⓕ $12 + 65t$ Ⓖ $65t$ Ⓗ $12t + 65$ Ⓘ $12(t + 65)$

MA.912.A.1.4

65. What is the value of r when $s = -1$, $t = 4$, and $u = \frac{1}{5}$?

$$r = 3s^2 + 5(t - 2u)$$

Ⓐ 7 Ⓑ 15 Ⓒ 21 Ⓓ $22\frac{1}{5}$

MA.912.A.1.2

66. **Short Response** Compare $\sqrt{26}$ and 4.9. Explain your answer.

Mixed Review

Order the numbers from least to greatest.

See Lesson 1-2.

67. $-1.5, -0.5, -\sqrt{2}, -1.4$

68. $-\frac{3}{8}, \frac{1}{2}, -\frac{3}{4}, -\frac{5}{6}$

69. $\sqrt{2}, -20, 0.2, \frac{1}{2}$

70. $\frac{3}{4}, -3, -0.5, -\frac{1}{4}$

Get Ready! **To prepare for Lesson 1-4, do Exercises 71–74.**

Simplify each expression.

See Lesson 1-3.

71. $4x + 3x - 4$

72. $-\frac{p}{3} + \frac{q}{3} - \frac{2p}{3} - q$

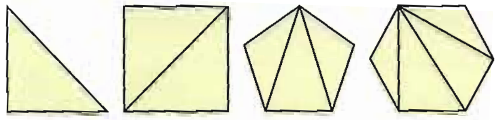
73. $-2(4 + b) + 4(b - 5)$

74. $(k - m) - (m - k)$

1 Mid-Chapter Quiz

Do you know HOW?

1. Draw the next figure in the pattern.



Identify a pattern and find the next number in the pattern.

2. $-405, -135, -45, -15, \ldots$

3. $\frac{2x}{3}, \frac{x}{3}, \frac{x}{6}, \frac{x}{12}, \ldots$

4. 101, 92, 83, 74, . . .

5. 0.4, 1.2, 3.6, 10.8, . . .

Name the property of real numbers illustrated by the equation.

6. $7(x - y) = 7x - 7y$

7. $\sqrt{7} \cdot 1 = \sqrt{7}$

8. $-2 \cdot \left(-\frac{1}{2}\right) = 1$

9. $2.3(3.4 \cdot 12.9) = (2.3 \cdot 3.4)(12.9)$

Write an algebraic expression to model each word phrase.

10. eight times the sum of a and b

11. four more than the product of x and y

12. six less than the quotient of d and g

13. ten less than twice the product of s and t

Simplify each expression.

14. $-x^2 + 2y - 3x^2 + 10$

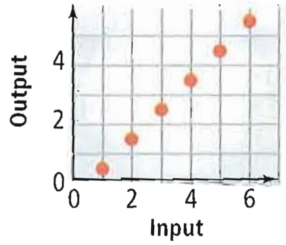
15. $-2(d + 2e) + 5(3d - 8e)$

16. $-(a + 2b) + 4(a + 2b) - 2(a + 2b)$

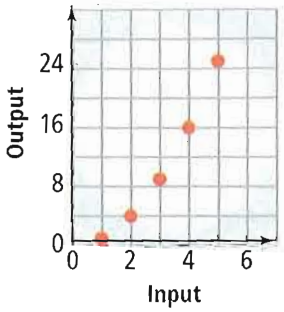
17. $-3x + 14x + 7x^2 - 3x + 4x(x + 1)$

Identify a pattern by making a table of the inputs and outputs. Include a process column.

18.



19.



Evaluate each expression for $a = 4$, $b = -3$, and $c = 10$.

20. $7a - 5b$

21. $4a + b - |2c|$

22. $|a - b - c^2|$

Write an algebraic expression to model each situation.

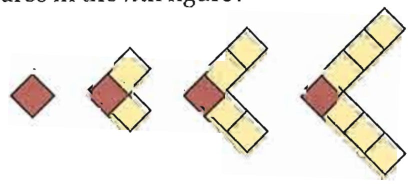
23. You have 16 tomatoes, and your tomato plants produce 5 tomatoes each day.

24. Your car's gas tank holds 25 gallons, and you use 1.5 gallons of gas each day.

Do you UNDERSTAND?

25. **Writing** Explain why every integer is also a rational number.

26. **Reasoning** What expression describes the number of squares in the nth figure?



27. **Reasoning** Is there a Closure Property of Subtraction that applies to whole numbers? Explain.

1-4 Solving Equations

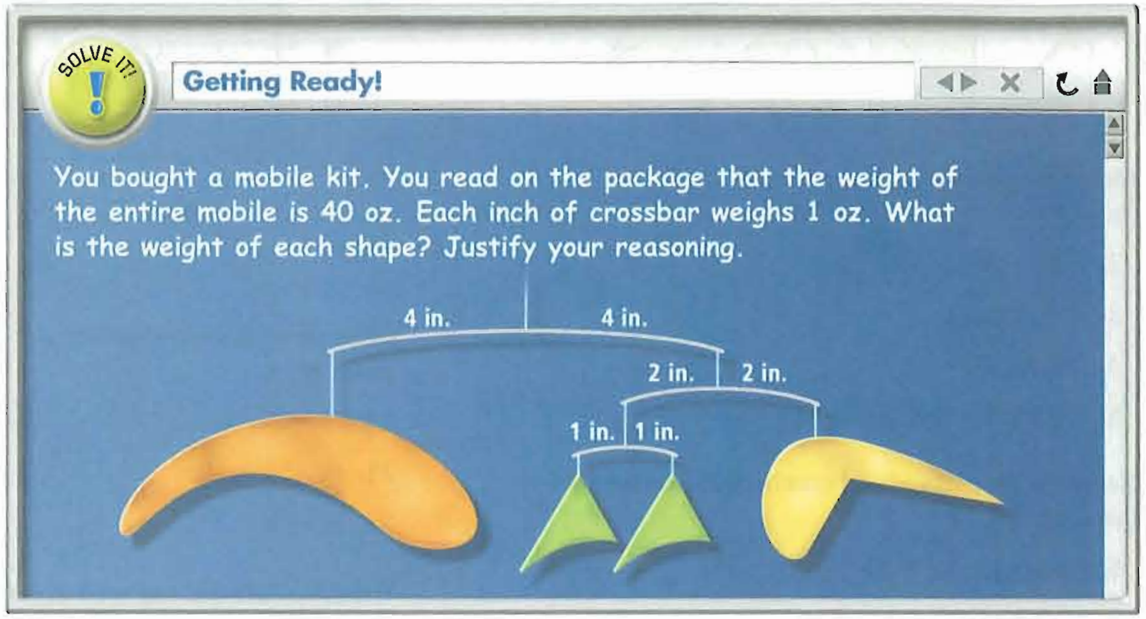
Sunshine State Standards

MA.912.A.3.3 Solve literal equations for a specified variable.

MA.912.A.10.3 Decide whether a linear equation is always, sometimes, or never true.

Objectives To solve equations

To solve problems by writing equations



Getting Ready!

You bought a mobile kit. You read on the package that the weight of the entire mobile is 40 oz. Each inch of crossbar weighs 1 oz. What is the weight of each shape? Justify your reasoning.

Lesson Vocabulary
- equation
- solution of an equation
- inverse operations
- identity
- literal equation

An **equation** is a statement that two expressions are equal. In this lesson you will use equations to model and solve problems.

Essential Understanding You can use the properties of equality and inverse operations to solve equations.

take note

Properties **Properties of Equality**

Assume a, b, and c represent real numbers.

Property	Definition	Example
Reflexive	$a = a$	$5 = 5$
Symmetric	If $a = b$, then $b = a$.	If $\frac{1}{2} = 0.5$, then $0.5 = \frac{1}{2}$.
Transitive	If $a = b$ and $b = c$, then $a = c$.	If $2.5 = 2\frac{1}{2}$ and $2\frac{1}{2} = \frac{5}{2}$, then $2.5 = \frac{5}{2}$.
Substitution	If $a = b$, then you can replace a with b and vice versa.	If $a = b$ and $9 + a = 15$, then $9 + b = 15$.

take note

Properties Properties of Equality, Continued

Assume a, b, and c represent real numbers.

Property	Definition	Example
Addition	If $a = b$, then $a + c = b + c$.	If $x = 12$, then $x + 3 = 12 + 3$.
Subtraction	If $a = b$, then $a - c = b - c$.	If $x = 12$, then $x - 3 = 12 - 3$.
Multiplication	If $a = b$, then $a \cdot c = b \cdot c$.	If $x = 12$, then $x \cdot 3 = 12 \cdot 3$.
Division	If $a = b$, then $a \div c = b \div c$ (with $c \neq 0$).	If $x = 12$, then $x \div 3 = 12 \div 3$.

Solving an equation that contains a variable means finding all values of the variable that make the equation true. Such a value is a **solution of the equation**. To find a solution, isolate the variable on one side of the equation using *inverse operations*.

Inverse operations are operations that "undo" each other. Addition and subtraction have this inverse relationship, as do multiplication and division.

Plan

How can you isolate the variable?
To isolate the variable, you have to remove the +4 from the left side of the equation.

Problem 1 Solving a One-Step Equation

What is the solution of $x + 4 = -12$?

Subtraction is the inverse operation of addition, so subtract 4 from each side.

$x + 4 = -12$

$x + 4 - 4 = -12 - 4$ Subtraction Property of Equality

$x = -16$ Simplify.

Check $-16 + 4 \overset{?}{=} -12$

$-12 = -12$ ✔

Got It? **1.** What is the solution of $12b = 18$?

Plan

How do you solve an equation with the variable on both sides?
Choose a side for the variable and remove it from the other side.

Problem 2 Solving a Multi-Step Equation

GRIDDED RESPONSE

What is the solution of $-27 + 6y = 3(y - 3)$?

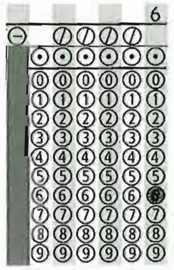
$-27 + 6y = 3(y - 3)$

$-27 + 6y = 3y - 9$ Distributive Property

$6y = 3y + 18$ Add 27 to each side.

$3y = 18$ Subtract $3y$ from each side.

$y = 6$ Divide each side by 3.



Got It? **2.** What is the solution of $3(2x - 1) - 2(3x + 4) = 11x$?

Problem 3 Using an Equation to Solve a Problem

Flowers "Flower carpets" incorporate hundreds of thousands of brightly-colored flowers as well as grass, tree bark, and sometimes fountains to form intricate designs and motifs. The flower carpet shown here, from Grand Place in Brussels, Belgium, has a perimeter of 200 meters. What are the dimensions of the flower carpet?

Plan

How can you relate the dimensions to the perimeter?
Use the formula for the perimeter of a rectangle.

Relate 2 · width plus 2 · length equals perimeter

Define Let x = the width.

Then $3x$ = the length.

Write $2 \cdot x + 2 \cdot 3x = 200$

$2x + 2 \cdot 3x = 200$

$2x + 6x = 200$ Multiply.

$8x = 200$ Combine like terms.

$\frac{8x}{8} = \frac{200}{8}$ Divide each side by 8.

$x = 25$ Simplify.

Find the length: $3x = 3 \cdot 25 = 75$.

The width is 25 meters. The length is 75 meters.

Got It? **3.** Suppose the flower carpet from Problem 3 had a perimeter of 320 meters. What would the dimensions of the flower carpet be?

An equation does not always have one solution. An equation has no solution if no value of the variable makes the equation true. An equation that is true for every value of the variable is an **identity**.

Essential Understanding Sometimes, no value of the variable makes an equation true. For identities, all values of the variable make the equation true.

Problem 4 Equations With No Solution and Identities

Think

What does it mean for an equation to be sometimes true?
An equation is sometimes true if it is true for some, but not all, values of the variable.

Is the equation *always, sometimes,* or *never* true?

A $11 + 3x - 7 = 6x + 5 - 3x$

$$4 + 3x = 3x + 5$$

$$4 = 5$$

Never true!

The last equation is not true, so no value of x makes the first two equations true. The original equation has no solution. It is never true.

B $6x + 5 - 2x = 4 + 4x + 1$

$$4x + 5 = 4x + 5$$

$$4x = 4x$$

$$0 = 0 \checkmark$$

Always true!

The last equation is true, so any value of x makes the first three equations true. The original equation is always true. It is an identity.

Got It? **4.** Is the equation *always, sometimes,* or *never* true?

a. $7x + 6 - 4x = 12 + 3x - 8$ **b.** $2x + 3(x - 4) = 2(2x - 6) + x$

A **literal equation** is an equation that uses at least two different letters as variables. You can solve a literal equation for any one of its variables by using the properties of equality. You solve for a variable "in terms of" the other variables.

Problem 5 Solving a Literal Equation

Plan

How do you solve a literal equation for one of its variables?
Use inverse operations to isolate the indicated variable.

The equation $C = \frac{5}{9}(F - 32)$ relates temperatures in degrees Fahrenheit F and degrees Celsius C. What is F in terms of C?

$$C = \frac{5}{9}(F - 32)$$

$$\frac{9}{5}C = F - 32 \quad \text{Multiply each side by } \tfrac{9}{5}.$$

$$\frac{9}{5}C + 32 = F \quad \text{Add 32 to each side.}$$

$$F = \frac{9}{5}C + 32 \quad \text{Symmetric Property}$$

Got It? **5. a.** The equation $K = C + 273$ relates temperatures kelvins K and degrees Celsius C. What is C in terms of K?

b. **Reasoning** Is the equation relating temperatures in kelvins and degrees Celsius *always, sometimes,* or *never* true? Explain your answer.

Lesson Check

Do you know HOW?

Solve each equation.

1. $w - 15 = 8.2$

2. $\frac{x}{3} = -30$

3. $2y - 1 = y + 11$

Solve each equation for k.

4. $r - 2k = 15$

5. $6k - 2z = 12$

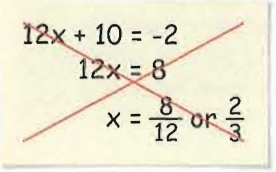
6. $4k + h = -2k - 14$

Do you UNDERSTAND?

7. Vocabulary Explain what it means to find a solution of an equation.

8. Reasoning Suppose you solve an equation and find that your school needs 4.3 buses for a class trip. Explain how to interpret this solution.

9. Error Analysis Find the error(s) in the steps shown.



$12x + 10 = -2$

$12x = 8$

$x = \frac{8}{12}$ or $\frac{2}{3}$

Practice and Problem-Solving Exercises

A Practice

Solve each equation.

See Problem 1.

10. $h - 12 = 6$

11. $-\frac{x}{3} = 27$

12. $4t = 48$

13. $22 + r = 36$

Solve each equation. Check your answer.

See Problem 2.

14. $7w + 2 = 3w + 94$

15. $15 - g = 23 - 2g$

16. $43 - 3d = d + 9$

17. $5y + 1.8 = 4y - 3.2$

18. $6a - 5 = 4a + 2$

19. $7y + 4 = 3 - 2y$

20. $5c - 9 = 8 - 2c$

21. $4y - 8 - 2y + 5 = 0$

22. $6(n - 4) = 3n$

23. $2 - 3(x + 4) = 8$

24. $5(2 - g) = 0$

25. $2(x + 4) = 8$

Write an equation to solve each problem.

See Problem 3.

26. Bus Travel Two buses leave Houston at the same time and travel in opposite directions. One bus averages 55 mi/h and the other bus averages 45 mi/h. When will they be 400 mi apart?

27. Aviation Two planes left an airport at noon. One flew east and the other flew west at twice the speed. After 3 hours the planes were 2700 mi apart. How fast was each plane flying?

28. Geometry The length of a rectangle is 3 cm greater than its width. The perimeter is 24 cm. What are the dimensions of the rectangle?

Determine whether the equation is *always*, *sometimes*, or *never* true.

See Problem 4.

29. $5x + 3 - 2x = 7x + 3$

30. $2(5x + 4) = 10x + 6$

31. $\frac{2}{3}x + 4 = 2x$

32. $6x - 12 + 2x = 3 + 8x - 15$

Solve each formula for the indicated variable.

See Problem 5.

33. $A = \frac{1}{2}bh$, for h

34. $s = \frac{1}{2}gt^2$, for g

35. $V = lwh$, for w

36. $I = prt$, for r

Solve each equation for x.

37. $ax + bx = c$

38. $\frac{x}{a} - 5 = b$

39. $\frac{x - 2}{2} = m + n$

40. $\frac{2}{5}(x + 1) = g$

B Apply

Solve each equation.

41. $0.2(x + 3) - 4(2x - 3) = 0.9$

42. $12 - 3(2w + 1) = 7w - 3(7 + w)$

43. $(m - 2) - 5 = 8 - 2(m - 4)$

44. $7(a + 1) - 3a = 5 + 4(2a - 1)$

45. **Think About a Plan** The measures of an angle and its complement differ by 22°. What are the measures of the angles?

- What is true about the sum of the measures of an angle and its complement?
- When modeling the problem with an equation, how can you algebraically represent that the two angle measures differ by 22°?

Solve each formula for the indicated variable.

46. $R(r_1 + r_2) = r_1r_2$, for R

47. $A = \frac{1}{2}h(b_1 + b_2)$, for b_2

48. $S = 2\pi r^2 + 2\pi rh$, for h

49. $h = vt - 5t^2$, for v

50. $v = s^2 + \frac{1}{2}sh$, for h

51. $R(r_1 + r_2) = r_1r_2$, for r_2

52. **Writing** Suppose you write and solve an equation to determine the amount of money m you have in your bank account after several weeks. You find that $m = -36$. What does this solution mean?

53. **Geometry** The measure of the supplement of an angle is 20° more than three times the measure of the original angle. Find the measures of the angles.

54. Find 4 consecutive odd integers with a sum of 184.

Solve each equation for x.

55. $c(x + 2) - 5 = b(x - 3)$

56. $a(3tx - 2b) = c(dx - 2)$

57. $b(5px - 3c) = a(qx - 4)$

58. $\frac{a}{b}(2x - 12) = \frac{c}{d}$

59. $\frac{3ax}{5} - 4c = \frac{ax}{5}$

60. $\frac{a - c}{x - a} = m$

Write an equation to solve each problem.

61. **Swimming** A city park is opening a new swimming pool. You can pay a daily entrance fee of \$3 or purchase a membership for the 12-week summer season for \$82 and pay only \$1 per day to swim. How many days would you have to swim to make the membership worthwhile?

62. **Rocket** The first stage of a rocket burns 28 s longer than the second stage. If the total burning time for both stages is 152 s, how long does each stage burn?

63. **Error Analysis** Your friend says that the equations shown are two ways to write the same formula. Is your friend correct? Explain your answer.

$$\xcancel{s = \frac{n}{n+1} \qquad \frac{s}{s-1} = n}$$

Challenge

64. Assume that a, b, and c are integers and $a \neq 0$.

a. Proof Prove that the solution of the linear equation $ax - b = c$ must be a rational number.

b. Writing Describe the values of a, b, and c for which the solutions of $ax^2 + b = c$ are rational.

65. A tortoise crawling at a rate of 0.1 mi/h passes a resting hare. The hare wants to rest another 30 min before chasing the tortoise at a rate of 5 mi/h. How many feet must the hare run to catch the tortoise?

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.3.5 **66.** A mural made of triangular tiles has an area of 1750 square centimeters. Each triangle has a height that is 3 centimeters longer than its base of 2 centimeters. How many triangular tiles are there?

MA.912.D.11.3 **67.** The table shows the population of bacteria in a petri dish at various times. If the pattern continues, what will the bacteria population be at 6:00 P.M.?

Bacteria Population				
Time	8:00 A.M.	10:00 A.M.	12:00 P.M.	2:00 P.M.
Population	100	200	400	800

MA.912.A.3.1 **68.** To the nearest tenth, what is the value of t in the following equation?

$$4(t - 30) = 5 - 3t$$

MA.912.A.3.5 **69.** A 10-foot-tall basketball hoop is 4 feet shorter than twice the height of a flag pole. What is the height in feet of the flag pole?

Mixed Review

Evaluate each expression for $x = -4$ and $y = 3$. See Lesson 1-3.

70. $x - 2y + 3$ **71.** $x + x \div y$ **72.** $3x - 4y - x$ **73.** $x + 2y \div x$

Write an algebraic expression for each phrase. See Lesson 1-3.

74. 5 more than a number x

75. the product of 16 and a number x

76. 3 times the difference of 12 and a number x

Get Ready! **To prepare for Lesson 1-5, do Exercises 77–79.**

State whether each inequality is *true* or *false*. See Lesson 1-2.

77. $5 < 12$ **78.** $5 < -12$ **79.** $5 \geq 5$

1-5 Solving Inequalities

Sunshine State Standards

Prepares for MA.912.A.3.6 Solve and graph the solutions of absolute value inequalities with one variable.

MA.912.A.10.3 Decide whether a linear inequality is always, sometimes, or never true.

Objectives To solve and graph inequalities
To write and solve compound inequalities

You want to download some new songs on your MP3 player. Each song will use about 4.3 MB of space. The amount of storage space on your MP3 player is shown at the right. At most how many songs can you download? Explain. (Hint: 1 GB = 1000 MB)

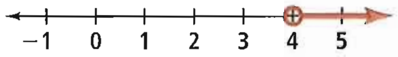
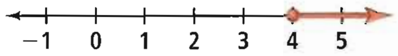
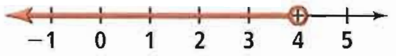
Dynamic Activity
Solving Inequalities
Compound Inequalities

Lesson Vocabulary
- compound inequality

Words like "at most" and "at least" suggest a relationship in which two quantities may not be equal. You can represent such a relationship with a mathematical inequality.

Essential Understanding Just as you use properties of equality to solve equations, you can use properties of inequality to solve inequalities.

take note Key Concept Writing and Graphing Inequalities

Inequality	Word Sentence	Graph
$x > 4$	*x* is greater than 4.	
$x \geq 4$	*x* is greater than or equal to 4.	
$x < 4$	*x* is less than 4.	
$x \leq 4$	*x* is less than or equal to 4.	

In the graphs above, the point at 4 is a boundary point because it separates the graph of the inequality from the rest of the number line. An open dot at 4 means that 4 *is not* a solution. A closed dot at 4 means that 4 *is* a solution.

Problem 1 Writing an Inequality From a Sentence

What inequality represents the sentence, "5 fewer than a number is at least 12"?

5 fewer than a number is at least 12.

"Fewer" indicates subtraction.

"At least" indicates greater than or equal to.

$x - 5 \geq 12$

Plan

How can you translate a sentence into an inequality? Look for key words, such as "at least" or "greater than."

Got It? **1.** What inequality represents the sentence, "The quotient of a number and 3 is no more than 15"?

The solutions of an inequality are the numbers that make it true. The properties you use for solving inequalities are similar to the properties you use for solving equations. However, when you multiply or divide each side of an inequality by a negative number, you must reverse the inequality symbol.

take note

Properties Properties of Inequalities

Let a, b, c, and d represent real numbers.

Property	Definition	Example
Transitive	If $a > b$ and $b > c$, then $a > c$.	$5 > 3$ and $3 > 1$, so $5 > 1$
Addition	If $a > b$, then $a + c > b + c$.	$4 > 2$, so $4 + 1 > 2 + 1$
Subtraction	If $a > b$, then $a - c > b - c$.	$7 > 4$, so $7 - 3 > 4 - 3$
Multiplication	If $a > b$ and $c > 0$, then $ac > bc$.	$6 > 5$ and $3 > 0$, so $6(3) > 5(3)$
	If $a > b$ and $c < 0$, then $ac < bc$.	$3 > 2$ and $-4 < 0$, so $3(-4) < 2(-4)$
Division	If $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$.	$9 > 3$ and $3 > 0$, so $\frac{9}{3} > \frac{3}{3}$
	If $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$.	$12 > 6$ and $-6 < 0$, so $\frac{12}{-6} < \frac{6}{-6}$

Here's Why It Works The steps below show that if $a > b$, then $-a < -b$. Therefore, you need to reverse the inequality symbol when multiplying each side of the inequality $a > b$ by -1.

$a > b$

$a - b > 0$ Subtract b from each side.

$-b - (-a) > 0$ $a - b = -b + a = -b - (-a)$.

$-b > -a$ Add $-a$ to each side.

$-a < -b$ Rewrite the inequality with $-a$ on the left side.

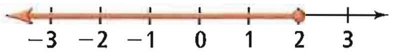
Problem 2 Solving and Graphing an Inequality

Plan

How is solving an inequality like solving an equation? You isolate the variable by doing the same things to each side of the inequality.

What is the solution of $-3(2x - 5) + 1 \geq 4$? Graph the solution.

$-3(2x - 5) + 1 \geq 4$	
$-6x + 15 + 1 \geq 4$	Distributive Property
$-6x + 16 \geq 4$	Simplify.
$-6x \geq -12$	Subtraction Property of Inequality
$x \leq 2$	Divide each side by -6. Reverse the inequality symbol.



Got It? **2.** What is the solution of $-2(x + 9) + 5 \geq 3$? Graph the solution.

Problem 3 Using an Inequality

Plan

How will an inequality help answer this question? The cost of the first plan must be less than the cost of the second plan, so an inequality can be used to determine the number of movies.

Movie Rentals **A movie rental company offers two subscription plans. You can pay \$36 a month and rent as many movies as desired, or you can pay \$15 a month and \$1.50 to rent each movie. How many movies must you rent in a month for the first plan to cost less than the second plan?**

Think	Write
Assign a variable.	Let n = the number of movie rentals in one month.
Write an expression for the cost of each plan for a month.	first plan: 36 second plan: 15 + 1.5n
The first plan must cost less than the second plan.	$36 < 15 + 1.5n$
Solve.	$21 < 1.5n$ $14 < n$
Write the answer in words.	You must rent more than 14 movies in a month for the first plan to cost less .

Got It? **3.** A digital music service offers two subscription plans. The first has a \$9 membership fee and charges \$1 per download. The second has a \$25 membership fee and charges \$.50 per download. How many songs must you download for the second plan to cost less than the first plan?

Problem 4 No Solution or All Real Numbers as Solutions

Think

How did you determine that an *equation* has no solution?
If you solve an equation and obtain a false statement, then the equation has no solution.

Is the inequality *always, sometimes,* or *never* true?

A $-2(3x + 1) > -6x + 7$

$-6x - 2 > -6x + 7$	Distributive Property
$-2 > 7$	Add $6x$ to each side.

The last inequality $-2 > 7$ is false, so $-2(3x + 1) > -6x + 7$ is always false. It has no solution, so it is never true.

B $5(2x - 3) - 7x \leq 3x + 8$

$10x - 15 - 7x \leq 3x + 8$	Distributive Property
$3x - 15 \leq 3x + 8$	Combine like terms.
$-15 \leq 8$	Subtract $3x$ from each side.

The inequality $-15 \leq 8$ is true, so $5(2x - 3) - 7x \leq 3x + 8$ is always true. All real numbers are solutions.

Got It? **4.** Is $4(2x - 3) < 8(x + 1)$ *always, sometimes,* or *never* true?

You can join two inequalities with the word *and* or the word *or* to form a **compound inequality**. To solve a compound inequality containing *and*, find all values of the variable that make both inequalities true.

Problem 5 Solving an *And* Inequality

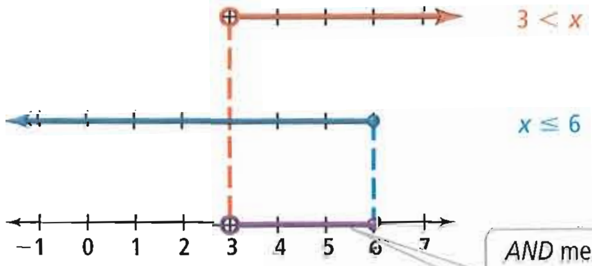
Think

How do you graph a compound inequality with *and*?
Find the intersection of the solutions of the two inequalities.

What is the solution of $7 < 2x + 1$ and $3x \leq 18$? Graph the solution.

$7 < 2x + 1$ and $3x \leq 18$

$3 < x$ and $x \leq 6$ Solve each inequality.



AND means that a solution makes *BOTH* inequalities true.

Got It? **5. a.** What is the solution of $5 \leq 3x - 1$ and $2x < 12$? Graph the solution.

b. Reasoning Is the compound inequality in Problem 5 *always, sometimes,* or *never* true? Explain your reasoning.

You can collapse a compound *and* inequality, like $5 < x + 1$ and $x + 1 < 13$, into a simpler form, $5 < x + 1 < 13$. You read $5 < x + 1 < 13$ as "$x + 1$ is greater than 5 and less than 13."

To solve a compound inequality containing *or*, find all values of the variable that make at least one of the inequalities true.

Problem 6 Solving an *Or* Inequality

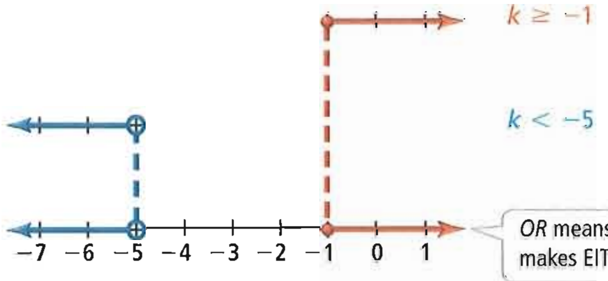
Think

How does the solution to an *or* inequality differ from the solution to an *and* inequality?
The solution to an *or* inequality includes all of the solutions of each inequality, not just the solutions of both inequalities.

What is the solution of $7 + k \geq 6$ or $8 + k < 3$? Graph the solution.

$7 + k \geq 6$ or $8 + k < 3$

$k \geq -1$ or $k < -5$ Subtraction Property of Inequality



OR means that a solution makes EITHER inequality true.

Got It? **6.** What is the solution of each compound inequality? Graph the solution.

a. $7w + 3 > 11$ or $4w - 1 < -13$

b. $16 < 5x + 1$ or $3x + 9 < 6$

Lesson Check

Do you know HOW?

Write an inequality that represents each sentence.

1. Rachel's hair is at least as long as Julia's.

2. The wind speeds of tropical storms are at least 40 mi/h, but less than 74 mi/h.

Solve each inequality. Graph the solution.

3. $-4(3x + 2) \geq 16$

4. $3 < 5x - 2 < 7$

5. $7x - 3 > 18$ or $3x - 2 \leq -2$

Do you UNDERSTAND?

6. Reasoning Make up an example to help explain why you must reverse the inequality symbol when you multiply or divide by a negative number.

7. Compare and Contrast Describe how the properties of inequality are similar to the properties of equality and how they differ.

8. Write an inequality for which the solution is the set of all real numbers.

9. Error Analysis Your classmate says that you cannot write a compound inequality that has no solution. Do you agree? If so, explain why. If not, give a counterexample.

Practice and Problem-Solving Exercises

A Practice

Write the inequality that represents the sentence. **See Problem 1.**

10. The sum of a number and 5 is less than -7.

11. The product of a number and 8 is at least 25.

12. Six less than a number is greater than 54.

13. The quotient of a number and 12 is no more than 6.

Solve each inequality. Graph the solution. **See Problem 2.**

14. $-12 \geq 24x$

15. $-7k < 63$

16. $8a - 15 > 73$

17. $57 - 4t \geq 13$

18. $-18 - 5y \geq 52$

19. $14 - 4y \geq 38$

20. $4(x + 3) \leq 44$

21. $2(m - 3) + 7 < 21$

22. $4(n - 2) - 6 > 18$

23. $-2(w + 4) + 9 < -11$

Solve each problem by writing an inequality. **See Problem 3.**

24. The length of a picture frame is 3 in. greater than the width. The perimeter is less than 52 in. Describe the dimensions of the frame.

25. The lengths of the sides of a triangle are in the ratio 5 : 6 : 7. Describe the length of the longest side if the perimeter is less than 54 cm.

26. Find the lesser of two consecutive integers with a sum greater than 16.

27. The cost of a field trip is \$220 plus \$7 per student. If the school can spend at most \$500, how many students can go on the field trip?

Is the inequality *always, sometimes,* or *never* true? **See Problem 4.**

28. $9(x + 2) > 9(x - 3)$

29. $6x - 13 < 6(x - 2)$

30. $-6(2x - 10) + 12x \leq 180$

31. $-7(3x - 7) + 21x \geq 50$

32. $3 + 5x < 5(x + 1)$

33. $2(x + 6) < 30$

34. $4x - 8 > 1 + 4(x + 3)$

35. $9x + 2(2 + x) < 5 + 9x$

Solve each compound inequality. Graph the solution. **See Problems 5 and 6.**

36. $2x > -10$ and $9x < 18$

37. $3x \geq -12$ and $8x \leq 16$

38. $6x \geq -24$ and $9x < 54$

39. $7x > -35$ and $5x \leq 30$

40. $4x < 16$ or $12x > 144$

41. $3x \geq 3$ or $9x < 54$

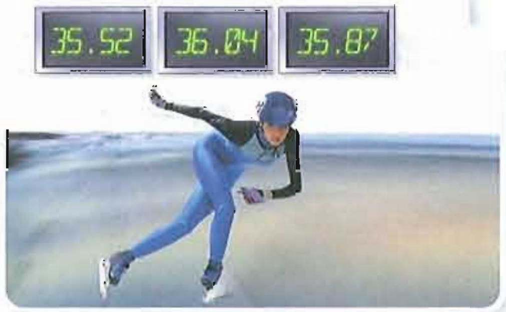
42. $8x > -32$ or $-6x \geq 48$

43. $9x \leq -27$ or $4x \geq 36$

B Apply

44. Think About a Plan The diagram shows the scores in seconds of a skater's first three trials in a speed-skating event. What is the maximum time she can score on her last trial so that her average time on all four trials is under 36 seconds?

- What do you need to find an average?
- What inequality can you use to model the situation?



Solve each inequality. Graph the solution.

45. $2 - 3z \geq 7(8 - 2z) + 12$

46. $6(x - 2.5) \geq 8 - 6(3.5 + x)$

47. $\frac{2}{3}(x - 12) \leq x + 8$

48. $\frac{3}{5}(x - 12) > x - 24$

49. $3[4x - (2x - 7)] < 2(3x - 5)$

50. $6[5y - (3y - 1)] \geq 4(3y - 7)$

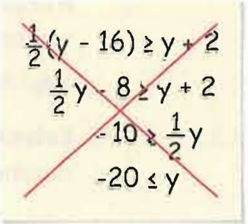
51. Grades Your math test scores are 68, 78, 90, and 91. What is the lowest score you can earn on the next test and still achieve an average of at least 85?

52. Chemistry The pH level of a popular shampoo is between 6.0 and 6.5 inclusive. What compound inequality shows the pH levels of this shampoo? Graph the solution.

53. Geometry The sum of the lengths of any two sides of a triangle is greater than the length of the third side. In $\triangle ABC$, $BC = 4$ and $AC = 8 - AB$. What can you conclude about AB?

54. Writing Write a word problem that can be solved using $25 + 0.5x \leq 60$.

55. Error Analysis A classmate solved the inequality $\frac{1}{2}(y - 16) \geq y + 2$ as shown. Prove that his answer is incorrect by checking a number that is less than -20. (Select a number that makes the computation easy.) What was his error?



$\frac{1}{2}(y - 16) \geq y + 2$

$\frac{1}{2}y - 8 \geq y + 2$

$-10 \geq \frac{1}{2}y$

$-20 \leq y$

56. Construction A contractor estimated that her expenses for a construction project would be between \$700,000 and \$750,000. She has already spent \$496,000. How much more can she spend and remain within her estimate?

Justifying Steps Justify each step by identifying the property used.

57. $3x \leq 4(x - 1) - 8$

$3x \leq 4x - 4 - 8$

$3x \leq 4x - 12$

$-x \leq -12$

$x \geq 12$

58. $\frac{1}{2}(y + 3) > \frac{1}{3}(4 - y)$

$3(y + 3) > 2(4 - y)$

$3y + 9 > 8 - 2y$

$5y + 9 > 8$

$5y > -1$

$y > -0.2$

Solve each compound inequality. Graph the solution.

59. $-6 < 2x - 4 < 12$

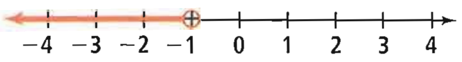
60. $4x \leq 12$ and $-7x \leq 21$

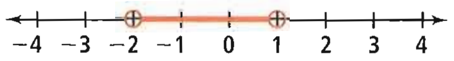
61. $15x > 30$ or $18x < -36$

Challenge

Open-Ended **Write an inequality with a solution that matches the graph. At least two steps should be needed to solve your inequality.**

62.


63.


64.


65.


66. **Reasoning** Consider the compound inequality $x < 8$ and $x > a$.

- **a.** Are there any values of a such that all real numbers are solutions of the compound inequality? If so, what are they?
- **b.** Are there any values of a such that no real numbers are solutions of the compound inequality? If so, what are they?
- **c.** Repeat parts (a) and (b) for the compound inequality $x < 8$ or $x > a$.

Sunshine State Standards Practice

MA.912.A.3.4

67. What is the solution of $1 < 2x + 3 < 9$?

Ⓐ $-1 > x < 2$ Ⓑ $2 < x < 3$ Ⓒ $-1 < x < 2$ Ⓓ $-1 < x < 3$

MA.912.A.3.3

68. Which expression best represents the value of x in $y = mx + b$?

Ⓕ $\frac{b - y}{m}$ Ⓖ $\frac{y + b}{m}$ Ⓗ $m(y - b)$ Ⓘ $\frac{y - b}{m}$

MA.912.A.3.5

69. The hourly rate of a waiter is $4 plus tips. On a particular day, the waiter worked 8 hours and received more than $150 in pay. Which could be the amount of tips the waiter received?

Ⓐ $18.75 Ⓑ $32 Ⓒ $118 Ⓓ $120.75

MA.912.A.3.2

70. **Extended Response** Solve $3(x - 2) + 8 = 12$. Identify each property of real numbers or equality you use.

Mixed Review

Simplify each expression. See Lesson 1-3.

71. $(2a - 4) + (5a + 9)$

72. $3(x + 3y) - 5(x - y)$

73. $\frac{1}{3}(b + 12) - \frac{1}{4}(b + 12)$

74. $0.4(k - 0.1) + 0.5(3.3 - k)$

Get Ready! **To prepare for Lesson 1-6, do Exercises 75–78.**

Solve each equation. Check your answers. See Lesson 1-4.

75. $7x - 6(11 - 2x) = 10$

76. $10x - 7 = 2(13 + 5x)$

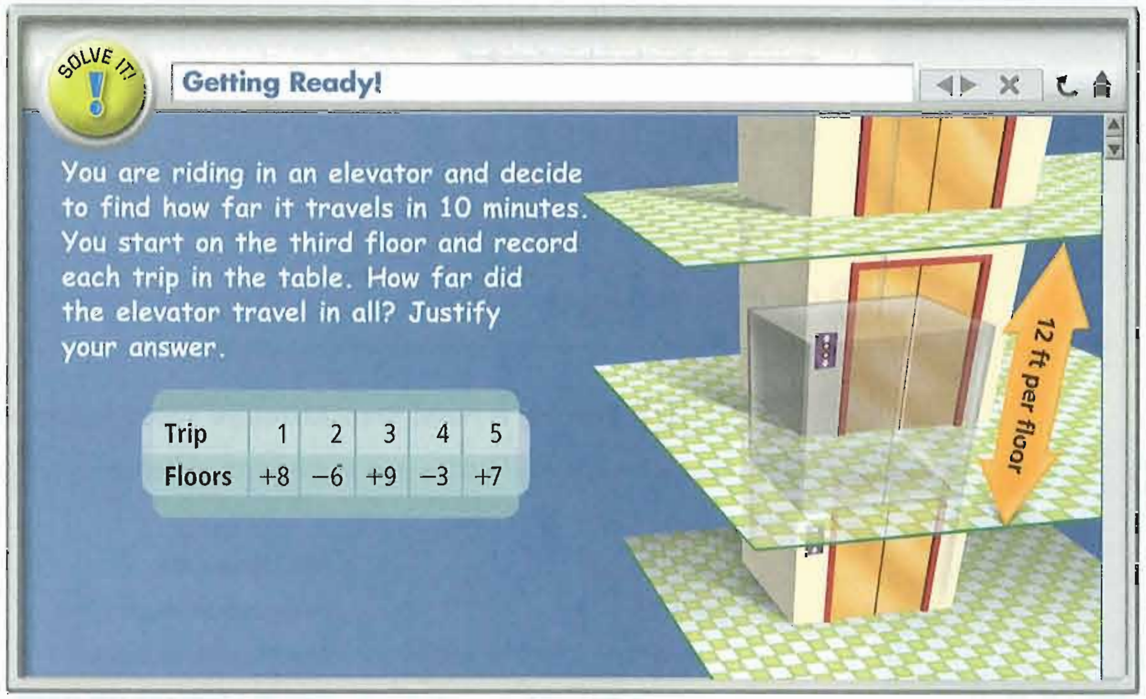
77. $4y - \frac{1}{10} = 3y + \frac{4}{5}$

78. $0.4x + 1.18 = -3.1(2 - 0.01x)$

1-6 Absolute Value Equations and Inequalities

Sunshine State Standard
MA.912.A.3.6 Solve and graph the solutions of absolute value equations and inequalities with one variable.

Objective To write and solve equations and inequalities involving absolute value



Getting Ready!

You are riding in an elevator and decide to find how far it travels in 10 minutes. You start on the third floor and record each trip in the table. How far did the elevator travel in all? Justify your answer.

Trip	1	2	3	4	5
Floors	+8	−6	+9	−3	+7

In the Solve It, signed numbers represent distance and direction. Sometimes, only the size of a number (its *absolute value*), not the direction, is important.

Lesson Vocabulary
- absolute value
- extraneous solution

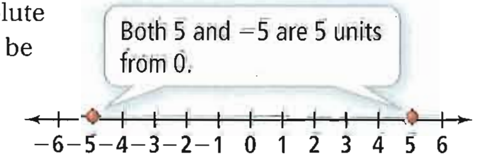
Essential Understanding An absolute value quantity is nonnegative. Since opposites have the same absolute value, an absolute value equation can have two solutions.

take note

Key Concept Absolute Value

Definition	Numbers	Symbols
The **absolute value** of a real number x, written $\lvert x \rvert$, is its distance from zero on the number line.	$\lvert 4 \rvert = 4$ $\lvert -4 \rvert = 4$	$\lvert x \rvert = x$, if $x \geq 0$ $\lvert x \rvert = -x$, if $x < 0$

An absolute value equation has a variable within the absolute value sign. For example, $\lvert x \rvert = 5$. Here, the value of x can be 5 or −5 since $\lvert 5 \rvert$ and $\lvert -5 \rvert$ both equal 5.


Both 5 and −5 are 5 units from 0.

Problem 1 Solving an Absolute Value Equation

Think

How is solving this equation different from solving a linear equation?
In the absolute value equation, $2x - 1$ can represent two opposite quantities.

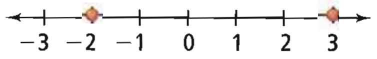
What is the solution of $|2x - 1| = 5$? Graph the solution.

$|2x - 1| = 5$

$2x - 1 = 5$ or $2x - 1 = -5$ Rewrite as two equations. $2x - 1$ could be 5 or -5.

$2x = 6$ $\quad 2x = -4$ Add 1 to each side of both equations.

$x = 3$ or $x = -2$ Divide each side of both equations by 2.



Check $|2(3) - 1| \overset{?}{=} 5$ $\qquad |2(-2) - 1| \overset{?}{=} 5$

$|6 - 1| \overset{?}{=} 5$ $\qquad |-4 - 1| \overset{?}{=} 5$

$|5| = 5$ ✓ $\qquad |-5| = 5$ ✓

Got It? **1.** What is the solution of $|3x + 2| = 4$? Graph the solution.

Problem 2 Solving a Multi-Step Absolute Value Equation

Plan

Is there a simpler way to think of this problem?
Solving $3|x + 2| - 1 = 8$ is similar to solving $3y - 1 = 8$.

What is the solution of $3|x + 2| - 1 = 8$? Graph the solution.

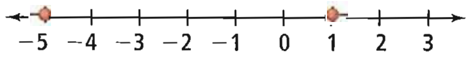
$3|x + 2| - 1 = 8$

$3|x + 2| = 9$ Add 1 to each side.

$|x + 2| = 3$ Divide each side by 3.

$x + 2 = 3$ or $x + 2 = -3$ Rewrite as two equations.

$x = 1$ or $x = -5$ Subtract 2 from each side of both equations.



Check $3|(1) + 2| - 1 \overset{?}{=} 8$ $\qquad 3|(-5) + 2| - 1 \overset{?}{=} 8$

$3|3| - 1 \overset{?}{=} 8$ $\qquad 3|-3| - 1 \overset{?}{=} 8$

$8 = 8$ ✓ $\qquad 8 = 8$ ✓

Got It? **2.** What is the solution of $2|x + 9| + 3 = 7$? Graph the solution.

Distance from 0 on the number line cannot be negative. Therefore, some absolute value equations, such as $|x| = -5$, have no solution. It is important to check the possible solutions of an absolute value equation. One or more of the possible solutions may be *extraneous*.

An **extraneous solution** is a solution derived from an original equation that is *not* a solution of the original equation.

Problem 3 Checking for Extraneous Solutions

Think

Can you solve this the same way as you solved Problem 1?
Yes, let $3x + 2$ equal $4x + 5$ and $-(4x + 5)$.

What is the solution of $|3x + 2| = 4x + 5$? Check for extraneous solutions.

$$|3x + 2| = 4x + 5$$

$3x + 2 = 4x + 5$ or $3x + 2 = -(4x + 5)$ Rewrite as two equations.

$-x = 3$ | $3x + 2 = -4x - 5$ Solve each equation.

$7x = -7$

$x = -3$ or $x = -1$

Check $|3(-3) + 2| \stackrel{?}{=} 4(-3) + 5$ $\quad$ $|3(-1) + 2| \stackrel{?}{=} 4(-1) + 5$

$|-9 + 2| \stackrel{?}{=} -12 + 5$ $\quad$ $|-3 + 2| \stackrel{?}{=} -4 + 5$

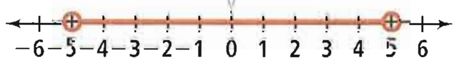
$|-7| \neq -7$ ✗ $\quad$ $|-1| = 1$ ✓

Since $x = -3$ does not satisfy the orginal equation, -3 is an extraneous solution. The only solution to the equation is $x = -1$.

Got It? 3. What is the solution of $|5x - 2| = 7x + 14$? Check for extraneous solutions.

The solutions of the absolute value inequality $|x| < 5$ include values greater than -5 *and* less than 5. This is the compound inequality $x > -5$ *and* $x < 5$, which you can write as $-5 < x < 5$. So, $|x| < 5$ means x is between -5 and 5.

The graph of $|x| < 5$ is all values of x between -5 and 5.



Essential Understanding You can write an absolute value inequality as a compound inequality without absolute value symbols.

Problem 4 Solving the Absolute Value Inequality $|A| < b$

Plan

Is this an *and* problem or an *or* problem?
$2x - 1$ is less than 5 and greater than -5. It is an *and* problem.

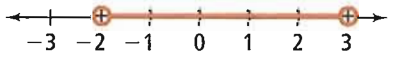
What is the solution of $|2x - 1| < 5$? Graph the solution.

$|2x - 1| < 5$

$-5 < 2x - 1 < 5$ $\quad$ $2x - 1$ is between -5 and 5.

$-4 < 2x < 6$ $\quad$ Add 1 to each part.

$-2 < x < 3$ $\quad$ Divide each part by 2.



Got It? 4. What is the solution of $|3x - 4| \leq 8$? Graph the solution.

$|x| < 5$ means x is between -5 and 5. So, $|x| > 5$ means x is outside the interval from -5 to 5. You can say $x < -5$ *or* $x > 5$.

Problem 5 Solving the Absolute Value Inequality $|A| \geq b$

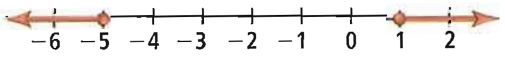
What is the solution of $|2x + 4| \geq 6$? Graph the solution.

$|2x + 4| \geq 6$

$2x + 4 \leq -6$ or $2x + 4 \geq 6$ Rewrite as a compound inequality.

$2x \leq -10$ | $2x \geq 2$ Subtract 4 from each side of both inequalities.

$x \leq -5$ or $x \geq 1$ Divide each side of both inequalities by 2.



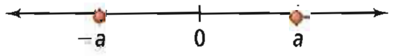
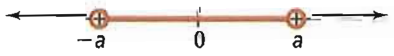
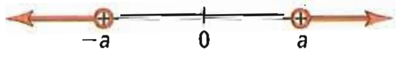
Think

How do you determine the boundary points?
To find the boundary points, find the solutions of the related equation.

Got It? **5. a.** What is the solution of $|5x + 10| > 15$? Graph the solution.

b. Reasoning Without solving $|x - 3| \geq 2$, describe the graph of its solution.

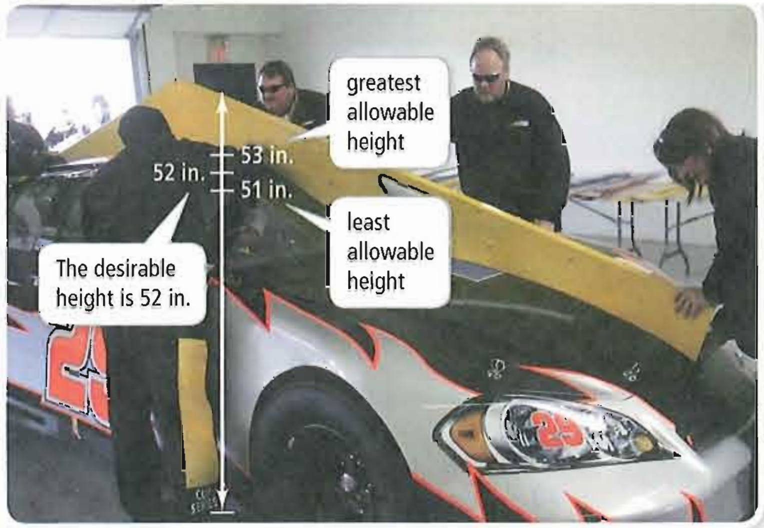
take note Concept Summary Solutions of Absolute Value Statements

Symbols	Definition	Graph
$\lvert x\rvert = a$	The distance from x to 0 is a units.	 $x = -a$ or $x = a$
$\lvert x\rvert < a$ $(\lvert x\rvert \leq a)$	The distance from x to 0 is less than a units.	 $-a < x < a$ $x > -a$ and $x < a$
$\lvert x\rvert > a$ $(\lvert x\rvert \geq a)$	The distance from x to 0 is greater than a units.	 $x < -a$ or $x > a$

A manufactured item's actual measurements and its target measurements can differ by a certain amount, called *tolerance*. Tolerance is one half the difference of the maximum and minimum acceptable values. You can use absolute value inequalities to describe tolerance.

Problem 6 Using an Absolute Value Inequality

Car Racing In car racing, a car must meet specific dimensions to enter a race. Officials use a template to ensure these specifications are met. What absolute value inequality describes heights of the model of race car shown within the indicated tolerance?



Plan

How does *tolerance* relate to an inequality?
Tolerance allows the height to differ from a desired height by no less and no more than a small amount.

$\frac{53 - 51}{2} = \frac{2}{2} = 1$ Find the tolerance.

$-1 \leq h - 52 \leq 1$ Use h for the height of the race car. Write a compound inequality.

$|h - 52| \leq 1$ Rewrite as an absolute value inequality.

Got It? **6.** Suppose the least allowable height of the race car in Problem 6 was 52 in. and the desirable height was 52.5 in. What absolute value inequality describes heights of the model of race car shown within the indicated tolerance?

Lesson Check

Do you know HOW?

Solve each equation. Check your answers.

1. $|-6x| = 24$

2. $|2x + 8| - 4 = 12$

3. $|x - 2| = 4x + 8$

Solve each inequality. Graph the solution.

4. $|2x + 2| - 5 < 15$

5. $|4x - 6| \geq 10$

Do you UNDERSTAND?

6. **Vocabulary** Explain what it means for a solution of an equation to be extraneous.

7. **Reasoning** When is the absolute value of a number equal to the number itself?

8. Give an example of a compound inequality that has no solution.

9. **Compare and Contrast** Describe how absolute value equations and inequalities are like linear equations and inequalities and how they differ.

Practice and Problem-Solving Exercises

A Practice

Solve each equation. Check your answers. **See Problems 1 and 2.**

10. $|3x| = 18$
11. $|-4x| = 32$
12. $|x - 3| = 9$
13. $2|3x - 2| = 14$
14. $|3x + 4| = -3$
15. $|2x - 3| = -1$
16. $|x + 4| + 3 = 17$
17. $|y - 5| - 2 = 10$
18. $|4 - z| - 10 = 1$

Solve each equation. Check for extraneous solutions. **See Problem 3.**

19. $|x - 1| = 5x + 10$
20. $|2z - 3| = 4z - 1$
21. $|3x + 5| = 5x + 2$
22. $|2y - 4| = 12$
23. $3|4w - 1| - 5 = 10$
24. $|2x + 5| = 3x + 4$

Solve each inequality. Graph the solution. **See Problem 4.**

25. $3|y - 9| < 27$
26. $|6y - 2| + 4 < 22$
27. $|3x - 6| + 3 < 15$
28. $\frac{1}{4}|x - 3| + 2 < 1$
29. $4|2w + 3| - 7 \leq 9$
30. $3|5t - 1| + 9 \leq 23$

Solve each inequality. Graph the solution. **See Problem 5.**

31. $|x + 3| > 9$
32. $|x - 5| \geq 8$
33. $|y - 3| \geq 12$
34. $|2x + 1| \geq -9$
35. $3|2x - 1| \geq 21$
36. $|3z| - 4 > 8$

Write each compound inequality as an absolute value inequality. **See Problem 6.**

37. $1.3 \leq h \leq 1.5$
38. $50 \leq k \leq 51$
39. $27.25 \leq C \leq 27.75$
40. $50 \leq b \leq 55$
41. $1200 \leq m \leq 1300$
42. $0.1187 \leq d \leq 0.1190$

B Apply

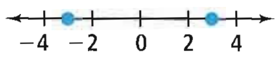
Solve each equation.

43. $-|4 - 8b| = 12$
44. $4|3x + 4| = 4x + 8$
45. $|3x - 1| + 10 = 25$
46. $\frac{1}{2}|3c + 5| = 6c + 4$
47. $5|6 - 5x| = 15x - 35$
48. $7|8 - 3h| = 21h - 49$
49. $2|3x - 7| = 10x - 8$
50. $6|2x + 5| = 6x + 24$
51. $\frac{1}{4}|4x + 7| = 8x + 16$
52. $\frac{2}{3}|3x - 6| = 4(x - 2)$

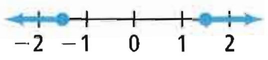
53. **Think About a Plan** The circumference of a basketball for college women must be from 28.5 in. to 29.0 in. What absolute value inequality represents the circumference of the ball?

- What is the tolerance?
- What is the inequality without using absolute value?

Write an absolute value equation or inequality to describe each graph.

54.


55.


56.


Solve each inequality. Graph the solutions.

57. $|3x - 4| + 5 \leq 27$

58. $|2x + 3| - 6 \geq 7$

59. $-2|x + 4| < 22$

60. $2|4t - 1| + 6 > 20$

61. $|3z + 15| \geq 0$

62. $|-2x + 1| > 2$

63. $\frac{1}{9}|5x - 3| - 3 \geq 2$

64. $\frac{1}{11}|2x - 4| + 10 \leq 11$

65. $\left|\frac{x - 3}{2}\right| + 2 < 6$

66. $\left|\frac{x + 5}{3}\right| - 3 > 6$

67. **Writing** Describe the differences in the graphs of $|x| < a$ and $|x| > a$, where a is a positive real number.

68. **Open-Ended** Write an absolute value inequality for which every real number is a solution. Write an absolute value inequality that has no solution.

Write an absolute value inequality to represent each situation.

69. **Cooking** Suppose you used an oven thermometer while baking and discovered that the oven temperature varied between +5 and −5 degrees from the setting. If your oven is set to 350°, let t be the actual temperature.

70. **Time** Workers at a hardware store take their morning break no earlier than 10 A.M. and no later than noon. Let c represent the time the workers take their break.

71. **Climate** A friend is planning a trip to Alaska. He purchased a coat that is recommended for outdoor temperatures from −15°F to 45°F. Let t represent the temperature for which the coat is intended.

Write an absolute value inequality and a compound inequality for each length x with the given tolerance.

72. a length of 36.80 mm with a tolerance of 0.05 mm

73. a length of 9.55 mm with a tolerance of 0.02 mm

74. a length of 100 yd with a tolerance of 4 in.

Is the absolute value inequality or equation *always*, *sometimes*, or *never* true? Explain.

75. $|x| = -6$

76. $-8 > |x|$

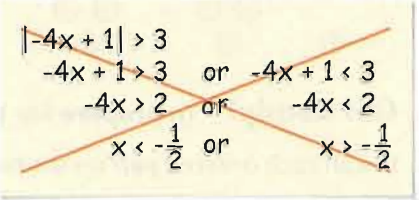
77. $|x| = x$

78. $|x| + |x| = 2x$

79. $|x + 2| = x + 2$

80. $(|x|)^2 < x^2$

81. **Error Analysis** A classmate wrote the solution to the inequality $|-4x + 1| > 3$ as shown. Describe and correct the error.



$|-4x + 1| > 3$

$-4x + 1 > 3$ or $-4x + 1 < 3$

$-4x > 2$ or $-4x < 2$

$x < -\frac{1}{2}$ or $x > -\frac{1}{2}$

Challenge

Solve each equation for x.

82. $|ax| - b = c$

83. $|cx - d| = ab$

84. $a|bx - c| = d$

Graph each solution.

85. $|x| \geq 5$ and $|x| \leq 6$

86. $|x| \geq 6$ or $|x| < 5$

87. $|x - 5| \leq x$

88. **Writing** Describe the difference between solving $|x + 3| > 4$ and $|x + 3| < 4$.

89. **Reasoning** How can you determine whether an absolute value inequality is equivalent to a compound inequality joined by the word *and* or one joined by the word *or*?

Sunshine State Standards Practice

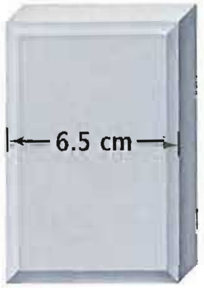
GRIDDED RESPONSE

MA.912.A.3.6 **90.** What is the positive solution of $|3x + 8| = 19$?

MA.912.A.3.6 **91.** If p is an integer, what is the least possible value of p in the following inequality?

$$|3p - 5| \leq 7$$

MA.912.A.3.6 **92.** In wood shop, you have to drill a hole that is 2 inches deep into a wood panel. The tolerance for drilling a hole is described by the inequality $|t - 2| \leq 0.125$. What is the shallowest hole allowed?



MA.912.A.3.6 **93.** The normal thickness of a metal structure is shown. It expands to 6.54 centimeters when heated and shrinks to 6.46 centimeters when cooled down. What is the maximum amount in cm that the thickness of the structure can deviate from its normal thickness?

Mixed Review

Solve each inequality. Graph the solution. See Lesson 1-5.

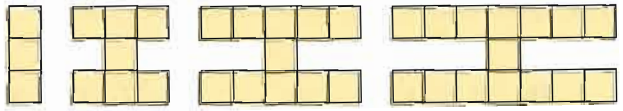
94. $5y - 10 < 20$

95. $15(4s + 1) < 23$

96. $4a + 6 > 2a + 14$

Describe each pattern using words. Draw the next figure in each pattern. See Lesson 1-1.

97.



98.

Get Ready! **To prepare for Lesson 2-1, do Exercises 99–102.** See p. 679.

Graph each ordered pair on the coordinate plane.

99. $(-4, -8)$

100. $(3, 6)$

101. $(0, 0)$

102. $(-1, 3)$

1 Pull It All Together

BIG idea Variable

You can use variables to represent variable quantities in real-world situations and in patterns.

BIG idea Properties

You can use the properties of real numbers to simplify algebraic expressions.

Task 1

The tables below show the same set of inputs and outputs using two different process columns.

Input	Process Column	Output
1	2 (1 − 1) + 5	5
2	2 (2 − 1) + 5	7
3	2 (3 − 1) + 5	9
4	■	■
⋮	⋮	⋮
n	■	■

Input	Process Column	Output
1	2 (1 + 1) + 1	5
2	2 (2 + 1) + 1	7
3	2 (3 + 1) + 1	9
4	■	■
⋮	⋮	⋮
n	■	■

a. Copy and complete each table. Write an algebraic expression for each rule using n.
b. Show that the two algebraic expressions are equivalent.
c. Describe a pattern using a third rule.

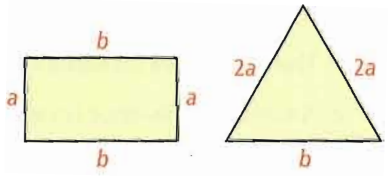
BIG idea Solving Equations and Inequalities

You can use properties of numbers and equality to solve an equation by finding increasingly simpler equations that have the same solution as the original equation.

Task 2

Find all possible values of a and b in the figures below, given the following conditions.

- The perimeters are equal.
- The rectangle has an area between 80 and 100 square units.
- The values of a and b are integers.



1 Chapter Review

Connecting BIG ideas and Answering Essential Questions

1 Variable
You can use variables to represent variable quantities in real-world situations and in patterns.

Patterns and Expressions (Lesson 1-1)

$6, 12, 18, 24, \ldots 6n$

$8, 9, 10, 11, \ldots (n + 7)$

2 Properties
The properties that apply to real numbers also apply to variables that represent them.

Properties of Real Numbers (Lesson 1-2)

$4(6 - 1) = 4(6) - 4(1)$

$23 + 0 = 23$

$5 + 12 = 12 + 5$

Algebraic Expressions (Lesson 1-3)

$6n$

$n + 7$

$5x - x = (5 - 1)x$

$a + 0 = a$

$h + k = k + h$

3 Solving Equations and Inequalities
You can use properties of numbers and equality (or inequality) to solve an equation (or inequality) by finding increasingly simpler equations (or inequalities) which have the same solution as the original equation (or inequality).

Solving Equations and Inequalities (Lessons 1-4 and 1-5)

$$\begin{aligned} 4x - 1 &= 5 \\ 4x &= 6 \\ x &= \tfrac{3}{2} \end{aligned} \qquad \begin{aligned} 7 &> -3h - 2 \\ 9 &> -3h \\ -3 &< h \end{aligned}$$

Absolute Value Equations and Inequalities (Lesson 1-6)

$$|2b + 7| = 15$$

$$\begin{aligned} 2b + 7 &= 15 \text{ or } & 2b + 7 &= -15 \\ 2b &= 8 & 2b &= -22 \\ b &= 4 \text{ or } & b &= -11 \end{aligned}$$

Chapter Vocabulary

- absolute value (p. 41)
- additive inverse (p. 14)
- algebraic expression (p. 5)
- coefficient (p. 20)
- compound inequality (p. 36)
- constant (p. 5)
- constant term (p. 20)
- equation (p. 26)
- evaluate (p. 19)
- extraneous solution (p. 42)
- identity (p. 28)
- inverse operations (p. 27)
- like terms (p. 21)
- literal equation (p. 29)
- multiplicative inverse (p. 14)
- numerical expression (p. 5)
- opposite (p. 14)
- reciprocal (p. 14)
- solution of an equation (p. 27)
- term (p. 20)
- variable (p. 5)
- variable quantity (p. 5)

Choose the correct term to complete each sentence.

1. The _?_ makes an equation true.

2. A number's distance from zero on the number line is its _?_.

3. _?_ is another name for the multiplicative inverse of a number.

4. A pair of inequalities joined by *and* or *or* are called a _?_.

1-1 Patterns and Expressions

Quick Review

You can represent patterns using words, diagrams, numbers, and **algebraic expressions**. You can identify a pattern by looking for the same type of change between consecutive figures or numbers. It often helps to make a table.

Example

Identify a pattern by making a table of inputs and outputs. Include a process column. 7, 14, 21, 28, 35, . . .

Input	Process Column	Output
1	$1 \cdot 7$	7
2	$2 \cdot 7$	14
3	$3 \cdot 7$	21
⋮	⋮	⋮
n	$n \cdot 7$	$7n$

The nth output is $7n$.

Exercises

Identify a pattern and find the next three numbers in the pattern.

5. 5, 10, 15, 20, . . .

6. 3, 4, 5, 6, . . .

Copy and complete the table. Then find the output when the input is n.

7.

Input	Output
1	9
2	10
3	11
4	■
⋮	⋮
n	■

8.

Input	Output
1	19
2	38
3	57
4	■
⋮	⋮
n	■

9. Finance If you put \$20 in your savings account each week, how much have you saved after n weeks?

1-2 Properties of Real Numbers

Quick Review

The natural numbers, whole numbers, integers, rational numbers, and irrational numbers are all subsets of the real numbers.

You can use properties such as the ones listed below to simplify and evaluate expressions.

Commutative Properties	$-3 + 5 = 5 + (-3)$ $2 \times 9 = 9 \times 2$
Associative Properties	$3 + (5 + 7) = (3 + 5) + 7$ $4 \times (8 \times 11) = (4 \times 8) \times 11$
Distributive Property	$5(7 + 9) = 5(7) + 5(9)$

Example

Identify the property illustrated by the equation.

$4 \cdot x = x \cdot 4$ Commutative Property of Multiplication

Exercises

Name the subset(s) of real numbers to which each number belongs.

10. 8.1π

11. -79

12. $\sqrt{121}$

13. $12\frac{7}{8}$

Compare the two numbers. Use < or >.

14. $-\sqrt{60}, -8$

15. $5, \sqrt{32}$

Name the property of real numbers illustrated by each equation.

16. $\frac{9}{4} \cdot \frac{4}{9} = 1$

17. $\left(8 \cdot \frac{1}{3}\right) \cdot 12 = 8 \cdot \left(\frac{1}{3} \cdot 12\right)$

1-3, 1-4, and 1-5 Expressions, Equations, and Inequalities

Quick Review

You **evaluate** an algebraic expression by substituting numbers for the variables. You simplify an algebraic expression by combining **like terms**. To find the **solution of an equation** or inequality, use the properties of equality or inequality. Some **equations** and inequalities are true for all real numbers, and some have no solution.

Example

Evaluate $3(x-4)+2x-x^2$ for $x=6$.

$3(6-4)+2(6)-6^2$	Substitute.
$=3(2)+2(6)-6^2$	Simplify inside parentheses.
$=6+12-36$	Multiply.
$=18-36$	Add.
$=-18$	Subtract.

Exercises

18. Evaluate $3t(t+2)-3t^2$ for $t=19$.

19. Simplify $-(3a-2b)-3(-a-b)$.

Solve each equation. Check your answer.

20. $2x-5=17$

21. $3(x+1)=9+2x$

Solve each inequality. Graph the solution.

22. $4-5z \geq 2$

23. $2(5-3x) < x-4(3-x)$

Solve each compound inequality. Graph the solution.

24. $10 \geq 7+3x$ and
$9-4x \leq 1$

25. $3 \geq 2x$ or
$x-4 > 2$

Write an equation to solve the problem.

26. **Geometry** The length and width of a rectangle are in the ratio 5 : 3. The perimeter of the rectangle is 32 cm. Find the length and width.

1-6 Absolute Value Equations and Inequalities

Quick Review

To rewrite an equation or inequality that involves the **absolute value** of an algebraic expression, you must consider both cases of the definition of absolute value.

Example

Solve $|3x-5|=4+2x$. Check for extraneous solutions.

$3x-5=4+2x$ or $3x-5=-(4+2x)$

$3x-5=-4-2x$

$5x=1$

$x=9$ or $x=\frac{1}{5}$

Check $|3(9)-5| \overset{?}{=} 4+2(9)$ $\quad$ $\left|3\left(\frac{1}{5}\right)-5\right| \overset{?}{=} 4+2\left(\frac{1}{5}\right)$

$|27-5| \overset{?}{=} 22$ $\quad$ $\left|\frac{3}{5}-5\right| \overset{?}{=} 4+\frac{2}{5}$

$|22|=22$ ✔ $\quad$ $\left|-\frac{22}{5}\right|=\frac{22}{5}$ ✔

Exercises

Solve each equation. Check for extraneous solutions.

27. $|2x+8|=3x+7$

28. $|x-4|+3=1$

29. $3|x+10|=6$

30. $2|x-7|=x-8$

Solve each inequality. Graph the solution.

31. $|3x-2|+4 \leq 7$

32. $4|y-9| > 36$

33. $|7x|+3 \leq 21$

34. $\frac{1}{2}|x+2| > 6$

35. The specification for a length x is 43.6 cm with a tolerance of 0.1 cm. Write the specification as an absolute value inequality.

1 Chapter Test

Do you know HOW?

Evaluate each expression for $x = 5$.

1. $\frac{5}{3}(3x - 6) - (6 - 4x)$
2. $3(x^2 - 4) + 7(x - 2)$
3. $x - 2x + 3x - 4x + 5x$

Simplify each expression.

4. $a^2 + a + a^2$
5. $2x + 3y - 5x + 2y$
6. $5(a - 2b) - 3(a - 2b)$
7. $3[2(x - 3) + 2] + 5(x - 3)$

Solve each equation.

8. $4y - 6 = 2y + 8$
9. $3(2z + 1) = 35$
10. $5(3w - 2) - 7 = 23$
11. $t - 2(3 - 2t) = 2t + 9$
12. $5(s - 12) - 24 = 3(s + 2)$

13. The lateral surface area of a cylinder is given by the formula $S = 2\pi rh$. Solve the equation for r.

14. **Savings** Briana and her sister Molly both want to buy the same model bicycle. Briana needs \$73 more before she can afford the bike. Molly needs \$65 more. If they combine their money, they will have just enough to buy one bicycle that they could share. What is the cost of the bicycle?

15. **Musical** There is only one freshman in the cast of a high school musical. There are 6 sophomores and 11 juniors. One third of the cast are seniors. How many seniors are in the musical?

Determine whether each equation is *always*, *sometimes*, or *never* true.

16. $2x + 7 - x = 3 + x + 4$
17. $5a - 1 - 3a = 2a + 1$

Solve each equation or inequality. Graph the solution.

18. $3x + 17 \geq 5$
19. $25 - 2x < 11$
20. $\frac{3}{8}x < -6$ or $5x > 2$
21. $2 < 10 - 4d < 6$
22. $4 - x = |2 - 3x|$
23. $5|3w + 2| - 3 > 7$

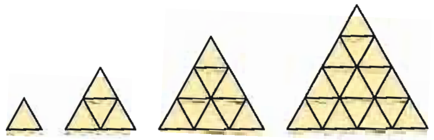
Do you UNDERSTAND?

24. **Writing** Describe the relationships among these sets of numbers: natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers.

25. **Reasoning** Justify each step by identifying the property used.

$$\begin{aligned} t + 5(t + 1) &= t + (5t + 5) \\ &= (t + 5t) + 5 \\ &= (1t + 5t) + 5 \\ &= (1 + 5)t + 5 \\ &= 6t + 5 \end{aligned}$$

26. **Reasoning** The first four figures of a pattern are shown.



Describe the tenth figure in the pattern.

1 Sunshine State Standards Practice

TIPS FOR SUCCESS

Some test questions ask you to enter a numerical answer on a grid like this one.

TIP 1

Fill in
⊖ for a negative number
⊘ for a fraction bar
⊙ for a decimal point

What is the next term? $-\frac{1}{4}, \frac{1}{3}, -\frac{1}{2}, \frac{3}{4}, \dots$

Solution

Using 12 as denominator, the pattern suggests $-\frac{3}{12}, \frac{4}{12}, -\frac{6}{12}, \frac{9}{12}, -\frac{13}{12}, \dots$

Do not leave any space between numbers.

TIP 2

Do not record a fraction with absolute value greater than 1 as a mixed number. If you record $-\frac{13}{12}$ as $-1\frac{1}{12}$ like this −1 1/2, the computer will read your answer as $-\frac{11}{12}$ or $-\frac{101}{12}$.

Think It Through

Using 12 as a common denominator, the increase in each numerator is 1 greater than the previous increase.

Vocabulary Builder

As you solve test items, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. inequality

B. compound inequality

C. extraneous solution

D. expression

E. equation

I. a mathematical sentence that contains >, <, ≥, ≤, or ≠

II. a solution of an equation derived from an original equation but not a solution of the original equation

III. a pair of inequalities joined by *and* or *or*

IV. a mathematical sentence that contains an equals sign

V. a mathematical phrase that uses numbers, variables, and operational symbols

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

1. Which equation represents the data in the table?

x	−2	−1	1	3	5
y	2	0	0	4	8

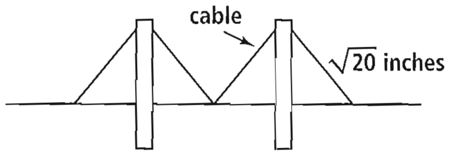
Ⓐ $y = -2x - 2$

Ⓑ $y = |x| + 2$

Ⓒ $y = |x| - 1$

Ⓓ $y = 2|x| - 2$

2. A model of a suspension bridge is built as shown.



Which of the following is closest to the length of the cable?

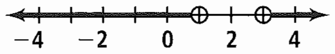
Ⓕ 2 inches

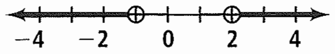
Ⓖ 4 inches

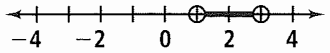
Ⓗ 10 inches

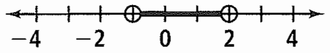
Ⓘ 20 inches

3. Which is the graph of the inequality $|x - 2| - 3 > -2$?

(A)


(B)


(C)


(D)


4. A worker is taking boxes of nails on an elevator. Each box weighs 54 lb, and the worker weighs 170 lb. The elevator has a weight limit of 2500 lb. Which inequality describes the number of boxes *b* that he can safely take on each trip?

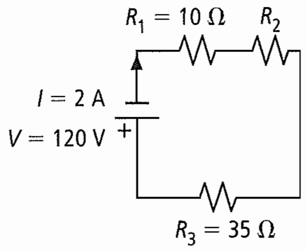
(F) $54b - 170 \leq 2500$
(H) $54(b - 170) \leq 2500$
(G) $54b + 170 \leq 2500$
(I) $54(b + 170) \leq 2500$

5. Which expression is equivalent to $-8(a - 3b) + 2(-a + 4b + 1)$?

(A) $-10a - 16b + 2$
(C) $-10a + 5b + 2$
(B) $-10a + 32b + 2$
(D) $-10a + b + 1$

6. An electrical circuit is connected in series as shown. The total voltage *V* can be calculated by using the equation shown, where *I* is the total current and *R* is the resistance across the circuit. $\left(\textit{Hint: } 1\text{A} = 1\frac{\text{volt}}{\text{ohm}}\right)$

$V = I(R_1 + R_2 + R_3)$



A = amperes
V = volts
Ω = ohms

What is the value of R_2?

(F) 15 Ω
(G) 45 Ω
(H) 60 Ω
(I) 75 Ω

7. Which value is a solution to $2|3x - 6| \leq 6$?

(A) −3
(B) −2
(C) −1
(D) 2

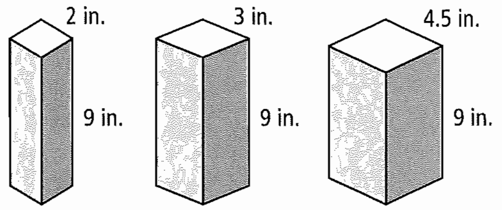
8. For which value of *a* does $4 = a + |x - 4|$ have no solution?

(F) −6
(G) 0
(H) 4
(I) 6

9. A rectangular solid has a volume of 81 cubic feet. If the length, width, and height are all changed to $\frac{1}{3}$ of their original size, what will be the volume of the new rectangular solid?

(A) 54 cubic feet
(C) 9 cubic feet
(B) 27 cubic feet
(D) 3 cubic feet

10. A company makes gift boxes in different sizes following the pattern shown below. What is the volume of the fourth gift box to the nearest cubic inch?

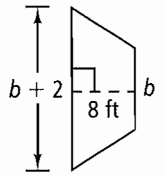


(F) 324 in.^{3}
(G) 352 in.^{3}
(H) 376 in.^{3}
(I) 410 in.^{3}

11. Solve $3(x - 2) + 4 \geq -3x + 1$.

(A) $x \geq -3$
(C) $x \geq \frac{3}{2}$
(B) $x \geq \frac{1}{2}$
(D) $x \leq 3$

12. A trapezoidal deck has dimensions as shown.



What is the longer base length if the area is 88 square feet (ft^2)?

(F) 20 ft
(G) 12 ft
(H) 10 ft
(I) 9 ft

13. What is the solution to the compound inequality $2x > -6$ and $6x < 18$?

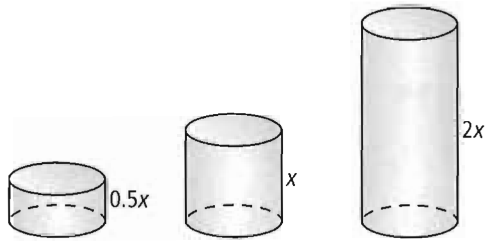
(A) $-3 < x < 2$

(B) $-3 < x < 3$

(C) $-3 < x \leq 3$

(D) $-3 \leq x \leq 3$

14. The cylinders shown below have identical bases. What is the height of the fourth cylinder in the pattern of cylinders shown below?



(F) $2.5x$ (H) $4x$

(G) $3x$ (I) $8x$

15. You used an oven thermometer while baking and found out that the oven temperature varied between +7 degrees and −7 degrees from the setting. If your oven is set to 325°F, let t be the actual temperature. What is the absolute value inequality that represents this situation?

(A) $|t - 325| \geq 7$

(B) $|t - 325| < 7$

(C) $|t - 7| \leq 325$

(D) $|t - 325| \leq 7$

16. A designer is designing a handbag. The height of the handbag must be between 16 in. and 18 in. The desirable height is 17 in. What absolute value inequality represents the height of the handbag?

(F) $|h - 16| \leq 1$ (H) $|h - 17| \geq 2$

(G) $|h - 17| \leq 1$ (I) $|h - 18| \geq 2$

17. Order the numbers $\frac{11}{2}$, 3, and $\sqrt{5}$ from least to greatest.

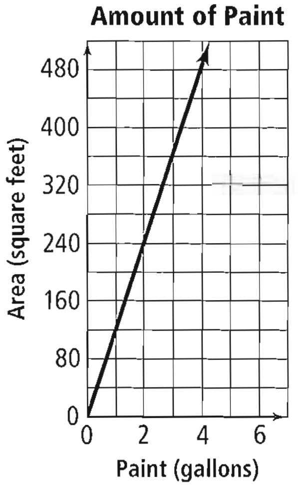
(A) $3 < \sqrt{5} < \frac{11}{2}$ (C) $\sqrt{5} < 3 < \frac{11}{2}$

(B) $\frac{11}{2} < \sqrt{5} < 3$ (D) $\frac{11}{2} < 3 < \sqrt{5}$

GRIDDED RESPONSE

Record your answers in a grid.

18. The graph shows the amount of paint needed (in gallons) to paint the walls (in square feet) of an office building.


Amount of Paint

If the trend continues, how many gallons of paint will be needed to paint 900 square feet? Express your answer as a decimal. Round to the nearest tenth.

19. What is the next term? 1, 4, 9, 16, . . .

20. What is the sum of the solutions of $|2x + 4| - 6 = 8$?

21. What is the value of $4x^2 + 2x - 1$ when $x = \frac{3}{4}$? Express the answer as a decimal.

22. The cost for taking a taxi is $1.80 plus $.10 per eighth of a mile. What is the cost in dollars of a ride that is 4.5 miles long?

23. A new 10-lb dumbbell will pass inspection if it is between 9.95 lb and 10.05 lb. What is the tolerance in pounds of the weight of the dumbbell?

CHAPTER 2

Get Ready!

Lesson 1-3

Simplifying Expressions

Simplify by combining like terms.

1. $7s - s$

2. $3a + b + a$

3. $xy - y + x$

4. $0.5g + g$

5. $4t - (t + 3t)$

6. $b - 2(1 + c - b)$

7. $5f - (5d - f)$

8. $2(h + 2g) - (g - h)$

9. $-(3z - 5) + z$

10. $(2 - d)g - 3d(4 + g)$

11. $5v - 3(2 - v)$

12. $7t - 3s(2 + t) + s$

Lesson 1-4

Solving Equations

Solve each equation.

13. $4 + x = -52$

14. $-y + 13 = -67$

15. $12 = 2 - k$

16. $3x = -72$

17. $\frac{h}{5} = 215$

18. $64 = 4 + 12g$

19. $5 - 4t = 12$

20. $7x - 9 = x$

21. $3(w - 8) = 36$

22. $-10p = 2(p - 12)$

23. $3(2 - c) = -(c + 4)$

24. $7 + b = 11(b - 3)$

Lesson 1-6

Solving Absolute Value Inequalities

Solve each absolute value inequality. Graph the solution.

25. $|x - 3| < 5$

26. $|2a - 1| \geq 2a + 1$

27. $|3x + 4| > -4x - 3$

28. $|3x + 1| + 1 > 12$

29. $3|d - 4| \leq 13 - d$

30. $-\frac{1}{3}|f + 3| + 2 \geq -5$

Looking Ahead Vocabulary

31. A person's field of study is often called that person's *domain*. If your domain is American history, what topics might you be interested in?

32. When is a person's height likely to show a greater *rate of change,* from 1 to 2 years of age or from 30 to 31 years of age? Explain.

33. When you look in the mirror, you see your *reflection*. How does the image in the mirror differ from the way other people see you? How is it the same?

34. The boundaries of a country determine the limit of the country's land. How does an inequality form a *boundary* on a number line?

CHAPTER 2

Functions, Equations, and Graphs

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VIDEO

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VOCABULARY

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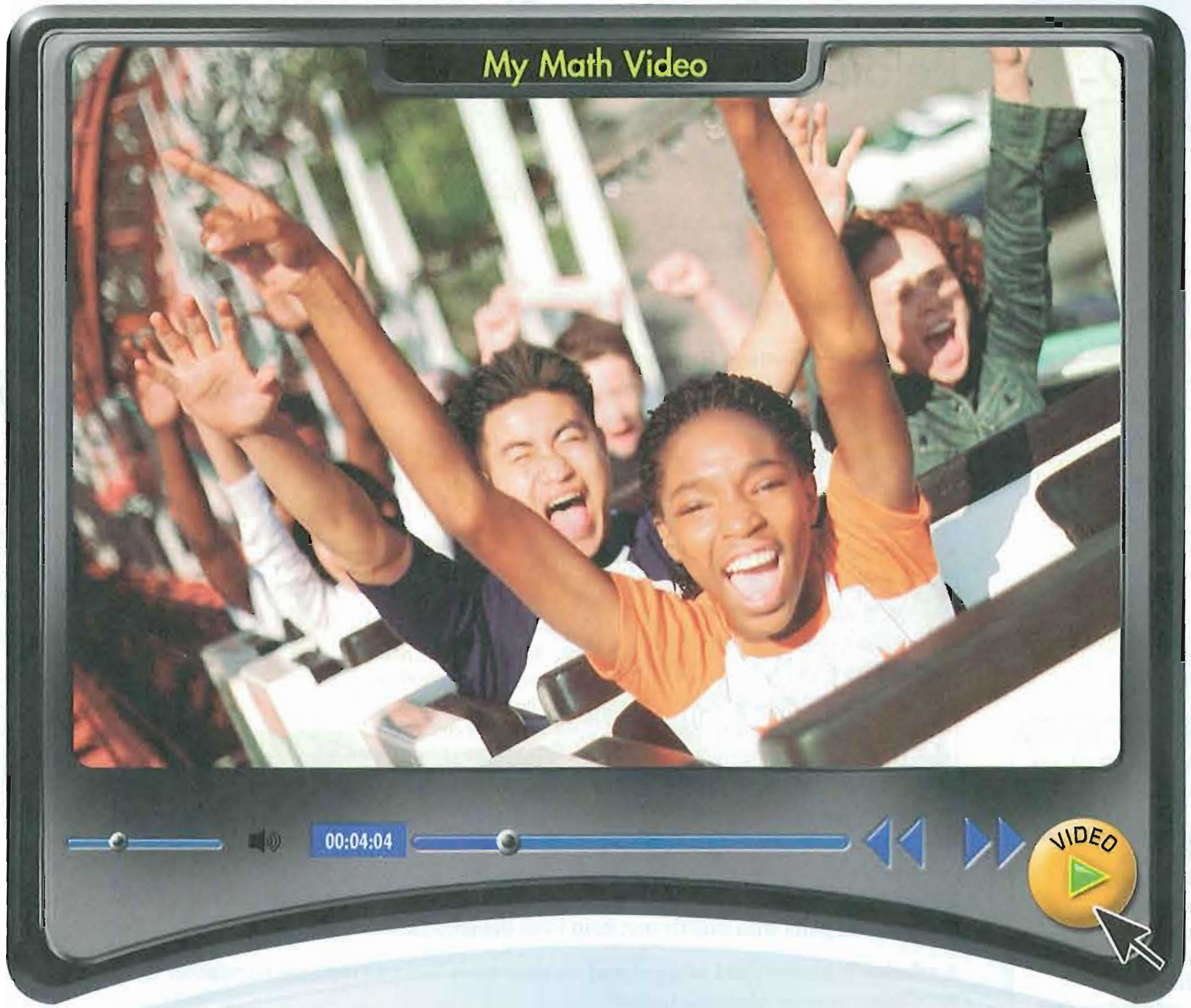
MathXL FOR SCHOOL

You can use functions to model all kinds of real-world situations. A function can model something as simple as a line between two points or as complex as the curves of a roller coaster. You will learn how to work with functions in this chapter.

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
correlation, *p. 92*	correlación
direct variation, *p. 68*	variación directa
domain, *p. 61*	dominio
function, *p. 62*	función
linear equation, *p. 75*	ecuación lineal
range, *p. 61*	rango
relation, *p. 60*	relación
slope, *p. 74*	pendiente



BIG ideas

1 **Equivalence**

Essential Question Does it matter which form of a linear equation you use?

2 **Function**

Essential Question How do you use transformations to help graph absolute value functions?

3. **Modeling**

Essential Question How can you model data with a linear function?

Chapter Preview

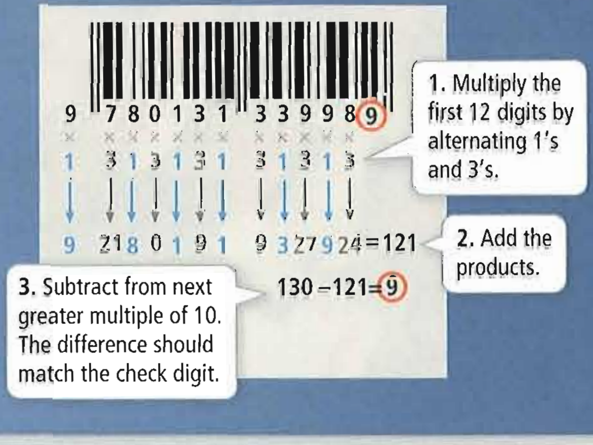
2-1 Relations and Functions

Sunshine State Standard
Prepares for MA.912.A.2.6 Identify and graph common functions.

Objectives To graph relations
To identify functions

SOLVE IT! **Getting Ready!**

The last digit in a 13-digit bar code is a check digit. Steps 1–3 show how the check digit checks the first 12 digits. Is it possible for 12 digits to generate two different check digits? Can two different sets of 12 digits have the same check digit? Explain.



1. Multiply the first 12 digits by alternating 1's and 3's.

2. Add the products.

3. Subtract from next greater multiple of 10. The difference should match the check digit.

Dynamic Activity
Function Explorer

Lesson Vocabulary
- relation
- domain
- range
- function
- vertical-line test
- function rule
- function notation
- independent variable
- dependent variable

You can use mappings to describe relationships between sets of numbers.

Essential Understanding A pairing of items from two sets is special if each item from one set pairs with exactly one item from the second set.

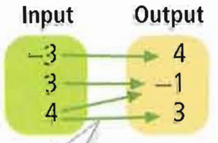
A **relation** is a set of pairs of input and output values. You can represent a relation in four different ways as shown below.

Key Concept Four Ways to Represent Relations

Ordered Pairs
(input, output)

(x, y)
$(-3, 4)$
$(3, -1)$
$(4, -1)$
$(4, 3)$

Mapping Diagram

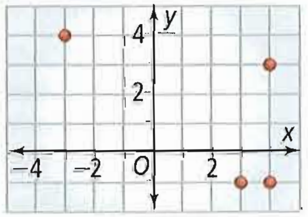


Arrows show how to pair each input with an output.

Table of Values

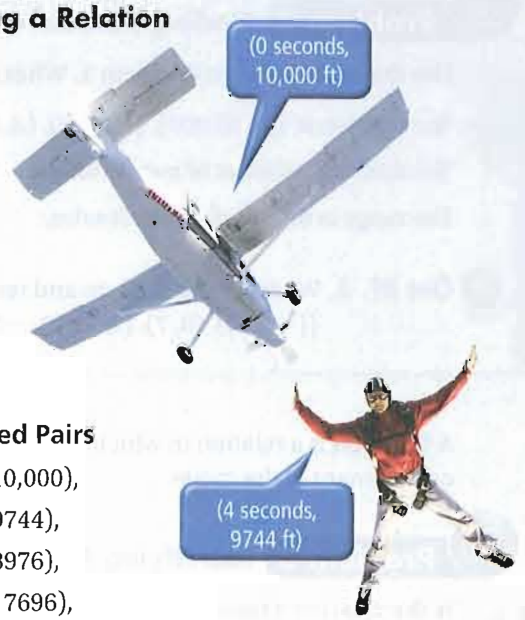
x Input	y Output
−3	4
3	−1
4	−1
4	3

Graph



Problem 1 Representing a Relation

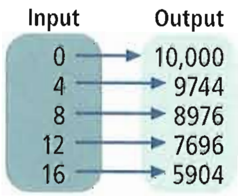
Skydiving When skydivers jump out of an airplane, they experience free fall. The photos show various heights of a skydiver at different times during free fall, ignoring air resistance. How can you represent this relation in four different ways?



Think

What is the input? The output?
The input is the time. The output is the height above the ground.

Mapping Diagram



Ordered Pairs

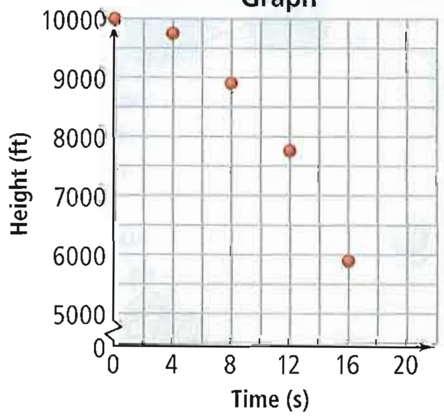
{(0, 10,000), (4, 9744), (8, 8976), (12, 7696), (16, 5904)}

Table of Values

Time (s)	Height (ft)
0	10,000
4	9744
8	8976
12	7696
16	5904

Each time value represents an input, which is paired with its corresponding output value (height).

Graph



Got It? **1.** The monthly average water temperature of the Gulf of Mexico in Key West, Florida varies during the year. In January, the average water temperature is 69°F, in February, 70°F, in March, 75°F, and in April, 78°F. How can you represent this relation in four different ways?

The **domain** of a relation is the set of inputs, also called x-coordinates, of the ordered pairs. The **range** is the set of outputs, also called y-coordinates, of the ordered pairs.

Think

How could you use the mapping diagram in Problem 1 to find the domain and range?
The *input* corresponds to the domain of the relation. The *output* corresponds to the range.

Problem 2 Finding Domain and Range

Use the relation from Problem 1. What are the domain and range of the relation?

The relation is $\{(0, 10{,}000), (4, 9744), (8, 8976), (12, 7696), (16, 5904)\}$.

The domain is the set of x-coordinates.	$\{0, 4, 8, 12, 16\}$
The range is the set of y-coordinates.	$\{10{,}000, 9744, 8976, 7696, 5904\}$

Got It? **2.** What are the domain and range of this relation?
$\{(-3, 14), (0, 7), (2, 0), (9, -18), (23, -99)\}$

A **function** is a relation in which each element of the domain corresponds with exactly one element of the range.

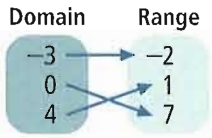
Plan

How can you use a mapping diagram to determine whether a relation is a function?
A function has only one arrow from each element of the domain.

Problem 3 Identifying Functions

Is the relation a function?

A



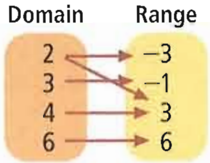
Each element in the domain corresponds with exactly one element in the range. This relation is a function.

B $\{(4, -1), (8, 6), (1, -1), (6, 6), (4, 1)\}$

Each x-coordinate must correspond to only one y-coordinate. The x-coordinate 4 corresponds to -1 and 1. The relation is *not* a function.

Got It? **3.** Is the relation a function?

a.

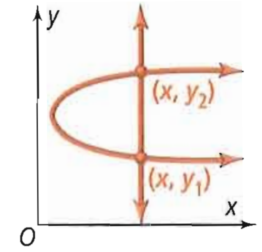


b. $\{(-7, 14), (9, -7), (14, 7), (7, 14)\}$

c. **Reasoning** How does a mapping diagram of a relation that is not a function differ from a mapping diagram of a function?

You can use the *vertical-line test* to determine whether a relation is a function. The **vertical-line test** states that if a vertical line passes through more than one point on the graph of a relation, then the relation is *not* a function.

Here's Why It Works If a vertical line passes through a graph at more than one point, there is more than one value in the range that corresponds to one value in the domain.



Problem 4 Using the Vertical-Line Test

Use the vertical-line test. Which graph(s) represent functions?

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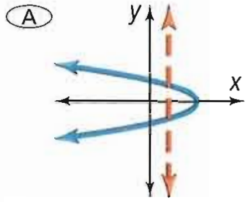

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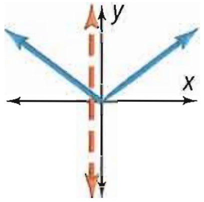

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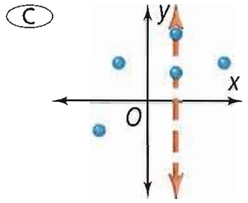
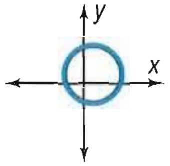


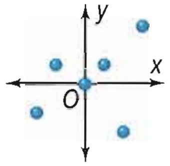
Think

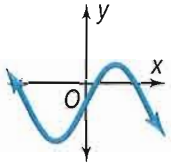
Is a relation a function if it passes through the y-axis twice?
No; the y-axis is a vertical line so the relation fails the vertical-line test.

Graphs A and C fail the vertical-line test because for each graph, a vertical line passes through more than one point. They do not represent functions. Graph B does not fail the vertical-line test so it represents a function.

Got It? **4.** Use the vertical-line test. Which graph(s) represent functions?

a.


b.


c.


A **function rule** is an equation that represents an output value in terms of an input value. You can write a function rule in **function notation**. Shown below are examples of function rules.

$y = 3x + 2$ $\quad f(x) = 3x + 2$ $\quad f(1) = 3(1) + 2$

Output | Input | Read as "f of x" or "function f of x." | "f of 1" is the output when 1 is the input.

The **independent variable**, x, represents the input of the function. The **dependent variable**, $f(x)$, represents the output of the function. It is called the dependent variable because its value depends on the input value.

Problem 5 Using Function Notation

Plan

How do you find the output?
Substitute the input into the function rule and simplify.

For $f(x) = -2x + 5$, what is the output for the inputs, -3, 0, and $\frac{1}{4}$?

x Input	Function Rule $f(x) = -2x + 5$	$f(x)$ Output
-3	$f(-3) = -2(-3) + 5$	11
0	$f(0) = -2(0) + 5$	5
$\frac{1}{4}$	$f\left(\frac{1}{4}\right) = -2\left(\frac{1}{4}\right) + 5$	$4\frac{1}{2}$

Got It? **5.** For $f(x) = -4x + 1$, what is the output for the given input?

a. -2 **b.** 0 **c.** 5

To model a real-world situation using a function rule, you need to identify the dependent and independent quantities. One way to describe the dependence of a variable quantity is to use a phrase such as, "distance is a function of time." This means that distance *depends* on time.

Problem 6 Writing and Evaluating a Function

Ticket Price **Tickets to a concert are available online for $35 each plus a handling fee of $2.50. The total cost is a function of the number of tickets bought. What function rule models the cost of the concert tickets? Evaluate the function for 4 tickets.**

Think

Why is cost the dependent quantity?
Cost is dependent because the cost depends on the number of tickets bought.

Cost is the dependent quantity and the number of tickets is the independent quantity.

Relate	Total cost	is	cost per ticket	times	number of tickets bought	plus	handling fee

Define Let t = number of tickets bought.

Let $C(t)$ = the total cost.

Write	$C(t)$	$=$	35	$\cdot$	t	$+$	2.50

$C(t) = 35t + 2.50$

$C(4) = 35 \cdot 4 + 2.50$ Substitute 4 for t.

$= 142.50$ Simplify.

The cost of 4 tickets is $142.50.

Got It? **6.** You are buying bottles of a sports drink for a softball team. Each bottle costs $1.19. What function rule models the total cost of a purchase? Evaluate the function for 15 bottles.

Lesson Check

Do you know HOW?

List the domain and range of each relation.

1. $\{(3, -2), (4, 4), (0, -2), (4, 1), (3, 2)\}$

2. $\{(0, 4), (4, 0), (-3, -4), (-4, -3)\}$

Determine whether each relation is a function.

3. $\{(3, -8), (-9, 1), (3, 2), (-4, 1), (-11, -2)\}$

4. $\{(1, 1), (2, 0), (3, 1), (4, 3), (0, 2)\}$

Do you UNDERSTAND?

5. **Vocabulary** Can you have a relation that is not a function? Can you have a function that is not a relation? Explain.

6. **Error Analysis** Your friend writes, "In a function, every vertical line must intersect the graph in exactly one point." Explain your friend's error and rewrite the statement so that it is correct.

7. **Reasoning** Why is there no horizontal-line test for functions?

Practice and Problem-Solving Exercises

A Practice

Every year, the Rock and Roll Hall of Fame and Museum inducts legendary musicians and musical acts to the Hall. The table shows the number of inductees for each year.

See Problems 1 and 2.

Rock and Roll Hall of Fame Inductees

Year	Number of inductees	Year	Number of Inductees
2001	11	2004	8
2002	8	2005	7
2003	9	2006	6

SOURCE: Rock and Roll Hall of Fame

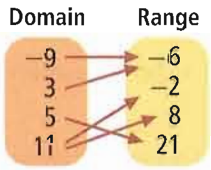
8. Represent the data using each of the following:
- **a.** a mapping diagram
- **b.** ordered pairs
- **c.** a graph on the coordinate plane

9. What are the domain and range of this relation?

Determine whether each relation is a function.

See Problem 3.

10.



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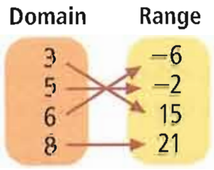

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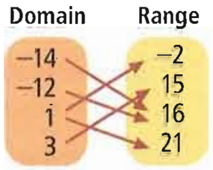
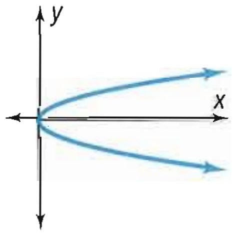


13. $\{(3, -9), (11, 21), (121, 34), (34, 1), (23, 45)\}$

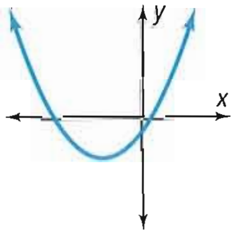
Use the vertical-line test to determine whether each graph represents a function.

See Problem 4.

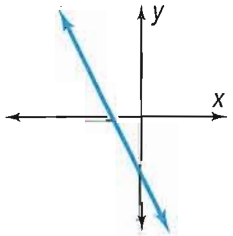
14.



15.



16.



Evaluate each function for the given value of x, and write the input x and output $f(x)$ as an ordered pair.

See Problem 5.

17. $f(x) = 17x + 3$ for $x = 4$

18. $f(x) = -\frac{2x + 1}{3}$ for $x = -5$

19. $f(x) = 2x - 33$ for $x = 9$

20. $f(x) = -9x - 2$ for $x = 7$

21. $f(x) = \frac{7}{3}x - 9$ for $x = 3$

22. $f(x) = -\frac{12x}{5}$ for $x = -1$

23. $f(x) = 11x - 11$ for $x = -11$

24. $f(x) = \frac{2}{9}x - \frac{9}{2}$ for $x = 9$

Write a function rule to model the cost per month of a long-distance cell phone calling plan. Then evaluate the function for the given number of minutes.

See Problem 6.

25. Monthly service fee: $4.52
Rate: $.12 per minute
Minutes used: 250

26. Monthly service fee: $3.12
Rate: $.18 per minute
Minutes used: 175

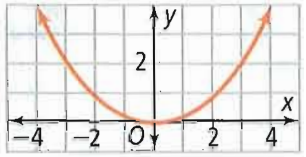
B Apply

27. **Think About a Plan** A cube is a solid figure with six square faces. If the edges of a cube have length 1.5 cm, what is the surface area of the cube?
    - What is the relationship between the length of the edges and the area of each face?
    - What is the relationship between the area of one face and the surface area of the whole cube?

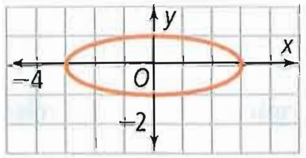
28. **Geometry** Suppose you have a box with a 4 × 4-in. square base and variable height h. The surface area of this box is a function of its height. Write a function to represent the surface area. Evaluate the function for $h = 6.5$ in.

Find the domain and range of each relation, and determine whether it is a function.

29.



30.



31. **Geometry** The volume of a sphere is a function of its radius, $V = \frac{4}{3}\pi r^3$. Evaluate the function for the volume of a volleyball with radius 10.5 cm.

32. **Car Rental** You are considering renting a car from two different rental companies. Proxy car rental company charges $.32 per mile plus an $18 surcharge. YourPal rental company charges $.36 per mile plus a $12 surcharge.
    - **a.** Write a function that shows the cost of renting a car from Proxy.
    - **b.** Write a function that shows the cost of renting a car from YourPal.
    - **c.** Which company offers the better deal for an 820-mile trip?

33. **Temperature** The relation between degrees Fahrenheit F and degrees Celsius C is described by the function $F = \frac{9}{5}C + 32$. In the following ordered pairs, the first element is degrees Celsius and the second element is its equivalent in degrees Fahrenheit. Find the unknown measure in each ordered pair.

    a. $(43, m)$ **b.** $(-12, n)$ **c.** $(p, 12)$ **d.** $(q, 19)$

34. **Reasoning** Suppose a function pairs items from set A with items from set B. You can say that the function maps *into* set B. If the function uses every item from set B, the function maps *onto* set B. Does each function below map the set of whole numbers *into* or *onto* the set of whole numbers?
    - **a.** Function f doubles every number.
    - **b.** Function g maps every number to 1 more than that number.
    - **c.** Function h maps every number to itself.
    - **d.** Function j maps every number to its square.

C Challenge

35. **Reasoning** Given the functions $f(x) = 3x - 21$ and $g(x) = 3x + 21$, show that the function $f(x) - g(x)$ is a constant for all the values of x.

Determine whether y is a function of x. Explain.

36. $y = \frac{3}{x} - 11$ 37. $y^2 = 3x - 7$ 38. $x^2 = 3y + y$

39. **Chemistry** The time required for a certain chemical reaction is related to the amount of catalyst present during the reaction. The domain of the relation is the number of grains of catalyst, and the range is the number of seconds required for a fixed amount of the chemical to react. The table shows the data from several reactions.

a. Is the relation a function?

b. If the domain and range were interchanged, would the relation be a function? Explain.

Catalyst and Reaction Time

Number of Grains	Number of Seconds
2.0	180
2.5	6
2.7	0.05
2.9	0.001
3.0	6
3.1	15
3.2	37
3.3	176

Sunshine State Standards Practice

MA.912.A.2.7 **40.** If $f(x) = -3x + 7$ and $g(x) = -7x + 3$, what is the value of $f(-3) - g(3)$?

Ⓐ 40 Ⓑ 34 Ⓒ 8 Ⓓ -8

MA.912.A.3.3 **41.** What is the formula for the volume of a cylinder, $V = \pi r^2 h$, solved for h?

Ⓕ $h = \frac{r^2}{\pi V}$ Ⓖ $h = \frac{\pi V}{r^2}$ Ⓗ $h = \frac{V}{\pi r^2}$ Ⓘ $h = \frac{\pi r^2}{V}$

MA.912.A.10.3 **42.** Which of the following statements are true?

I. $-(-6) = 6$ and $-(-4) > -4$

II. $-(-4) < 4$ or $-10 > 10 - 10$

III. $5 + 6 = 11$ or $9 - 2 = 11$

IV. $17 > 2$ or $6 < 9$

Ⓐ I and II only Ⓑ I, II, and III only Ⓒ I, III, and IV only Ⓓ III and IV only

MA.912.A.1.2 **43.** **Short Response** What are the numbers 1.9, $\frac{5}{4}$, -1.2, and $\sqrt{3}$ in order from greatest to least?

Mixed Review

Solve each equation or inequality. See Lessons 1-5 and 1-6.

44. $|3x + 9| = 11$

45. $19 + |x - 1| = 33$

46. $2 - 3x < 11$

47. $5x - 3 \leq 12 - 5x$

48. $|2x| + 4 < 7$

49. $4x + 6 \geq -6$

Get Ready! **To prepare for Lesson 2-2, do Exercises 50–52.** See Lesson 1-4.

Solve each equation for *y*.

50. $12y = 3x$

51. $-10y = 5x$

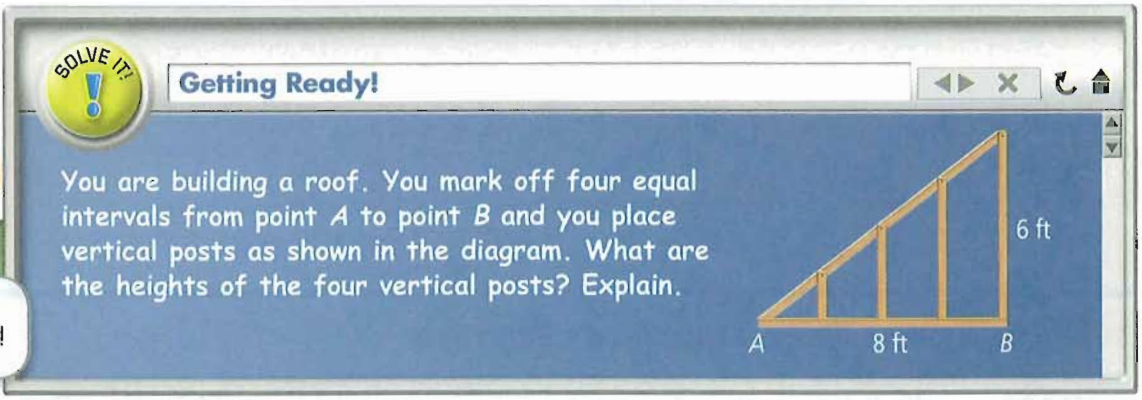
52. $\frac{3}{4}y = 15x$

2-2 Direct Variation

Sunshine State Standard

MA.912.A.2.12 Solve problems using direct variations.

Objective To write and interpret direct variation equations



You are building a roof. You mark off four equal intervals from point A to point B and you place vertical posts as shown in the diagram. What are the heights of the four vertical posts? Explain.

Lesson Vocabulary
- direct variation
- constant of variation

The post heights in the Solve It satisfy a relationship called *direct variation.*

Essential Understanding Some quantities are in a relationship where the ratio of corresponding values is constant.

You can write a formula for a **direct variation** function as $y = kx$, or $\frac{y}{x} = k$, where $k \neq 0$. x represents input values, and y represents output values. The formula $\frac{y}{x} = k$ says that, except for (0, 0), the ratio of all output-input pairs equals the constant k, the **constant of variation**.

Problem 1 Identifying Direct Variation From Tables

For each function, determine whether y varies directly with x. If so, what is the constant of variation and the function rule?

A

x	y
1	2
3	6
4	8

$\frac{y}{x} = \frac{2}{1} = \frac{6}{3} = \frac{8}{4} = 2,$

so y varies directly with x.

The constant of variation is 2.
The function rule is $y = 2x$.

B

x	y
1	4
2	8
3	11

$\frac{y}{x} = \frac{4}{1} = \frac{8}{2} \neq \frac{11}{3}$

so $\frac{y}{x}$ is *not* constant.

y does *not* vary directly with x.

Think

How do you find the constant of variation?

The constant of variation is the ratio of any y-value to the corresponding x-value.

Got It? **1.** For each function, determine whether y varies directly with x. If so, what are the constant of variation and the function rule?

a.

x	3	2	1
y	-21	-14	-7

b.

x	2	3	6
y	5	7	13

Problem 2 Identifying Direct Variation From Equations

For each function, determine whether y varies directly with x. If so, what is the constant of variation?

A $3y = 7x$

Divide each side of the equation $3y = 7x$ by 3 to get $y = \frac{7}{3}x$. Since you can write the equation in the form $y = kx$, y varies directly with x. The constant of variation is $\frac{7}{3}$.

B $7y = 14x + 7$

Divide each side of the equation $7y = 14x + 7$ by 7 to get $y = 2x + 1$. Since you cannot write the equation in the form $y = kx$, y does not vary directly with x.

Think

How is the form of this function different from the function in part (A)?
This function includes a nonzero constant term.

Got It? **2.** For each function, determine whether y varies directly with x. If so, what is the constant of variation?

a. $5x + 3y = 0$

b. $y = \frac{x}{9}$

In a direct variation, $\frac{y}{x}$ is the same for all pairs of data where $x \neq 0$. So, $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ is true for the ordered pairs (x_1, y_1) and (x_2, y_2), where neither x_1 nor x_2 is zero.

Problem 3 Using a Proportion to Solve a Direct Variation

Suppose y varies directly with x, and $y = 9$ when $x = -15$. What is y when $x = 21$?

Know	Need	Plan
y varies directly with x. $\frac{y}{x}$ is constant.	The value of y when x is 21.	Use two forms of $\frac{y}{x}$ in a proportion.

$\frac{9}{-15} = \frac{y}{21}$	In a direct variation, $\frac{y}{x}$ is constant.
$9(21) = -15(y)$	Write the cross products.
$\frac{9(21)}{-15} = \frac{-15y}{-15}$	Divide each side by -15.
$-12.6 = y$	Simplify.

So y is -12.6 when x is 21.

Got It? **3.** Suppose y varies directly with x, and $y = 15$ when $x = 3$. What is y when $x = 12$?

Think

Could you use the method in Problem 3 to solve this problem?
Yes; you could solve this problem by using the proportion $\frac{c_1}{s_1} = \frac{c_2}{s_2}$.

Problem 4 Using Direct Variation to Solve a Problem

A salesperson's commission varies directly with sales. For $1000 in sales, the commission is $85. What is the commission for $2300 in sales?

Step 1 Use $y = kx$ to find k.

Let c = commission.
Let s = sales.

$c = k(s)$

$85 = k(1000)$

Commission varies directly with sales, so it is the dependent variable.

$0.085 = k$

Step 2 Write the direct variation for the situation and find the commission when sales = $2300.

$c = k(s)$

$c = 0.085(s)$ Write the direct variation using k.

$c = 0.085(2300)$ Substitute 2300 for s.

$c = 195.5$ Simplify.

The commission for $2300 in sales is $195.50.

Got It? **4. a.** The number of Calories varies directly with the mass of cheese. If 50 grams of cheese contain 200 Calories, how many Calories are in 70 grams of cheese?

b. Reasoning If y^2 varies directly with x^2, does that mean y must vary directly with x? Explain.

The graph of a direct variation function is always a line through the origin.

Think

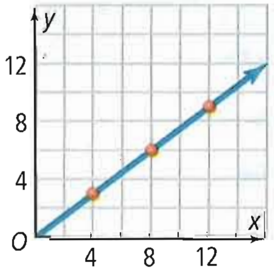
What x-values should you use to make a table of values?
The constant of variation is a fraction. Use multiples of the denominator for x. This ensures integer values for y.

Problem 5 Graphing Direct Variation Equations

What is the graph of each direct variation equation?

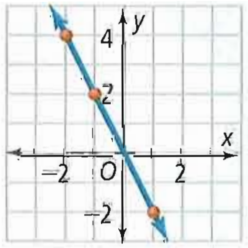
A $y = \frac{3}{4}x$

x	y
4	3
8	6
12	9



B $y = -2x$

x	y
−2	4
−1	2
1	−2



Got It? **5.** What is the graph of each direct variation equation?

a. $y = -\frac{2}{3}x$

b. $y = 3x$

Lesson Check

Do you know HOW?

1. Write a function rule for the direct variation in the table.

x	y
2	-1
4	-2
6	-3

Identify the constant of variation.

2. $y = \frac{3}{2}x$

3. $4y - 5x = 0$

Do you UNDERSTAND?

4. **Vocabulary** Explain what it means for two variables to be directly related.

5. **Reasoning** Explain why the graph of a direct variation function always passes through the origin.

6. Give an example of a function that represents a direct variation.

Practice and Problem-Solving Exercises

A Practice

For each function, determine whether y varies directly with x. If so, find the constant of variation and write the function rule.

See Problem 1.

7.

x	y
2	14
3	21
5	35

8.

x	y
27	9
30	10
60	20

9.

x	y
11	22
16	32
7	42

10.

x	y
3	9
4	10
5	11

Determine whether y varies directly with x. If so, find the constant of variation.

See Problem 2.

11. $y = 12x$

12. $y = 6x$

13. $y = -2x$

14. $y = 4x + 1$

15. $y = 4x - 3$

16. $y = -5x$

17. $y - 6x = 0$

18. $y + 3 = -3x$

For Exercises 19–24, y varies directly with x.

See Problem 3.

19. If $y = 4$ when $x = -2$, find x when $y = 6$.

20. If $y = 6$ when $x = 2$, find x when $y = 12$.

21. If $y = 7$ when $x = 2$, find x when $y = 3$.

22. If $y = 5$ when $x = -3$, find x when $y = -1$.

23. If $y = -7$ when $x = -3$, find y when $x = 9$.

24. If $y = 25$ when $x = 15$, find y when $x = 6$.

See Problem 4.

25. **Distance** For a given speed, the distance traveled varies directly with the time. Kate's school is 5 miles away from her home and it takes her 10 minutes to reach the school. If Josh lives 2 miles from school and travels at the same speed as Kate, how long will it take him to reach the school?

26. **Conservation** A dripping faucet wastes a cup of water if it drips for three minutes. The amount of water wasted varies directly with the amount of time the faucet drips. How long will it take for the faucet to waste $4\frac{1}{2}$ cups of water?

Make a table of x- and y-values and use it to graph the direct variation equation.

See Problem 5.

27. $x = \left(-\frac{1}{3}\right)y$

28. $y = -9x$

29. $x = y$

B Apply

Determine whether y varies directly with x. If so, find the constant of variation and write the function rule.

30.

x	y
1	−2
3	−8
5	14

31.

x	y
9	6
12	8
15	10

32.

x	y
4	1
6	2
8	3

33.

x	y
23	24
55	56
66	67

34. **Think About a Plan** Suppose you make a 4-minute local call using a calling card and are charged 7.6 cents. The cost of a local call varies directly with the length of the call. How much more will it cost to make a 30-minute local call?

- Which quantity is the dependent quantity?
- How does the word "more" affect the method needed to solve the problem?

Write and graph a direct variation equation that passes through each point.

35. (1, 2)

36. (−3, −7)

37. (2, −9)

38. (−0.1, 50)

39. (−5, −3)

40. (9, −1)

41. (7, 2)

42. (−3, 14)

For Exercises 43–46, y varies directly with x.

43. If $y = \frac{1}{2}$ when $x = 4$, find y when $x = 5$.

44. If $y = \frac{3}{4}$ when $x = \frac{1}{2}$, find y when $x = 3$.

45. If $y = \frac{5}{3}$ when $x = \frac{3}{4}$, find x when $y = \frac{1}{2}$.

46. If $y = -\frac{5}{8}$ when $x = \frac{3}{2}$, find x when $y = \frac{2}{5}$.

47. **Reasoning** Explain why you cannot answer the following question.

If $y = 0$ when $x = 0$, what is x when $y = 13$?

Open-Ended **Choose a value of k within the given range. Then write and graph a direct variation function using your value for k.**

48. $0 < k < 1$

49. $3 < k < 4.5$

50. $-1 < k < -\frac{1}{2}$

51. **Error Analysis** Identify the error in the statement shown at the right.

~~If y varies directly with x^2, and y = 2 when x = 4, then y = 3 when x = 9.~~

52. **Sports** The number of rotations of a bicycle wheel varies directly with the number of pedal strokes. Suppose that in the bicycle's lowest gear, 6 pedal strokes move the cyclist about 357 in. In the same gear, how many pedal strokes are needed to move 100 ft?

53. **Writing** Suppose you use the origin to test whether a linear equation is a direct variation function. Does this method work? Support your answer with an example.

Challenge **In Exercises 54–55, y varies directly with x. Explain your answer.**

54. If x is doubled, what happens to y?

55. If x is divided by 7, what happens to y?

56. If z varies directly with the product of x and y ($z = kxy$), then z is said to vary jointly with x and y.

- **a.** **Geometry** The area of a triangle varies jointly with its base and height. What is the constant of variation?
- **b.** Suppose q varies jointly with v and s, and $q = 24$ when $v = 2$ and $s = 3$. Find q when $v = 4$ and $s = 2$.
- **c.** **Reasoning** Suppose z varies jointly with x and y, and x varies directly with w. Show that z varies jointly with w and y.

Sunshine State Standards Practice

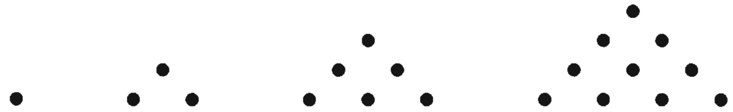
GRIDDED RESPONSE

MA.912.A.2.12 **57.** A speed of 75 mi/h is equal to a speed of 110 ft/s. To the nearest mile per hour, what is the speed of an aircraft traveling at a speed of 1600 ft/s?

MA.912.A.3.6 **58.** What number is a solution to both $|x - 3| = 2$ and $|9 - x| = 8$?

MA.912.A.2.7 **59.** If $f(x) = 7 - 3x$ and $g(x) = 3x - 7$, what is the value of $f(1) + g(1)$?

MA.912.D.11.3 **60.** Look at the pattern. How many circles are in the 6th figure of this pattern?



MA.912.A.3.1 **61.** What is the solution of $4(x - 5) + x = 8x - 10 - x$?

Mixed Review

Graph each relation. Find the domain and range. **See Lesson 2-1.**

62. $\{(0, 1), (1, -3), (-2, -3), (3, -3)\}$

63. $\{(4, 0), (7, 0), (4, -1), (7, -1)\}$

64. $\{(1, -2), (2, -1), (4, 1), (5, 2)\}$

65. $\{(1, 7), (2, 8), (3, 9), (4, 10)\}$

Identify a pattern and find the next three numbers in the pattern. **See Lesson 1-1.**

66. 8, 16, 24, 32, . . .

67. 5, 3, 1, −1, . . .

68. 144, 132, 120, 108, . . .

69. 30, 45, 60, 75, . . .

Get Ready! **To prepare for Lesson 2-3, do Exercises 70–73.**

Evaluate each expression for $x = -2, 0, 1$, and 4. **See Lesson 1-3.**

70. $\frac{2}{3}x + 7$

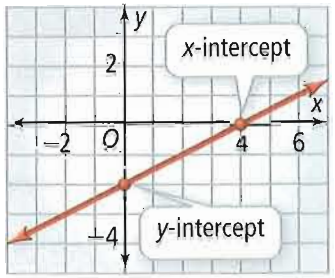
71. $\frac{3}{5}x - 2$

72. $3x + 1$

73. $\frac{1}{2}x - 8$

A special form of a linear equation is called *slope-intercept form.*

An *intercept* of a line is a point where a line crosses an axis. The **y-intercept** of a nonvertical line is the point at which the line crosses the y-axis. The **x-intercept** of a nonhorizontal line is the point at which the line crosses the x-axis.



take note

Key Concept Slope-Intercept Form

The **slope-intercept form** of an equation of a line is $y = mx + b$, where m is the slope of the line and $(0, b)$ is the y-intercept.

Problem 2 Writing Linear Equations

What is an equation of each line?

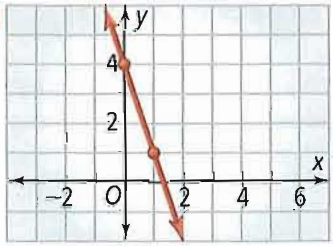
A $m = \frac{1}{5}$ **and the** y**-intercept is** $(0, -3)$

$y = mx + b$	Use the slope-intercept form.
$y = \frac{1}{5}x + (-3)$	Substitute $m = \frac{1}{5}$ and $b = -3$.
$y = \frac{1}{5}x - 3$	Simplify.

Plan

What information do you need from the graph?
You need the y-intercept and another point to find the slope.

B



Look at the line shown in the graph. The y-intercept is the point where the line crosses the y-axis, $(0, 4)$, so $b = 4$.

Use the second point $(1, 1)$ to find the slope.

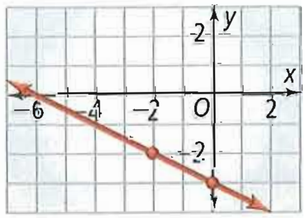
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 4}{1 - 0} = -3.$$

So, $y = -3x + 4$.

Got It? **2.** What is an equation of each line?

a. $m = 6$, y-intercept is $(0, 5)$

b.



c. Reasoning Using the graph from part (b), do you get a different equation if you use $(-6, 0)$ and the y-intercept to find the slope of the line? Explain.

You can rewrite a linear equation in slope-intercept form by solving for y.

Problem 3 Writing Equations in Slope-Intercept Form

Write the equation in slope-intercept form. What are the slope and y-intercept?

A $5x - 4y = 16$

$$-4y = -5x + 16 \qquad \text{Subtract } 5x \text{ from each side.}$$

$$\frac{-4y}{-4} = \frac{-5x}{-4} + \frac{16}{-4} \qquad \text{Divide each side by } -4.$$

$$y = \frac{5}{4}x - 4 \qquad \text{Compare the equation with } y = mx + b$$

The slope $m = \frac{5}{4}$. The y-intercept is $(0, -4)$.

Think

Is there another way to solve this problem?
Yes; clear the fractions by multiplying all terms by 4, the LCM of the denominators.

B $-\frac{3}{4}x + \frac{1}{2}y = -1$

$$\frac{1}{2}y = \frac{3}{4}x - 1 \qquad \text{Add } \tfrac{3}{4}x \text{ to each side.}$$

$$y = \frac{3}{2}x - 2 \qquad \text{Multiply each side by 2.}$$

The slope $m = \frac{3}{2}$. The y-intercept is $(0, -2)$.

Got It? **3.** Write the equation in slope-intercept form. What are the slope and y-intercept?

a. $3x + 2y = 18$ **b.** $-7x - 5y = 35$

Problem 4 Graphing a Linear Equation

What is the graph of $-2x + y = 1$?

Know
- The equation of a line

Need
Two points to draw the line

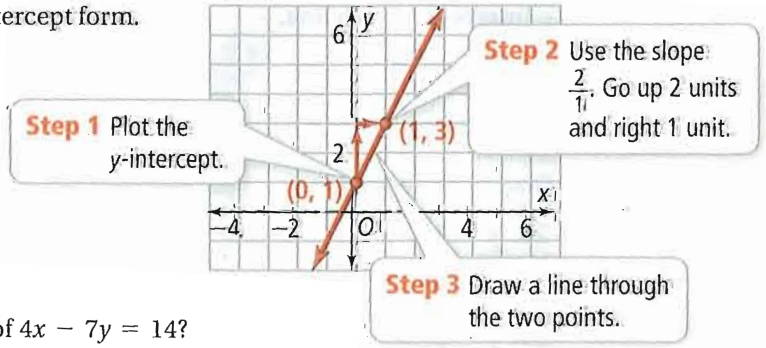
Plan
- Write the equation in slope-intercept form.
- Plot the y-intercept.
- Use the slope to find a second point.
- Draw a line through the two points.

Write the equation in slope-intercept form.

$$-2x + y = 1$$

$$y = 2x + 1$$

The slope is 2 and the y-intercept is $(0, 1)$.



Step 1 Plot the y-intercept.

Step 2 Use the slope $\frac{2}{1}$. Go up 2 units and right 1 unit.

Step 3 Draw a line through the two points.

Got It? **4.** What is the graph of $4x - 7y = 14$?

Lesson Check

Do you know HOW?

Write each equation in slope-intercept form.

1. $x - 2y + 3 = 1$

2. $-4x + 3y = 1$

What is the slope of the line passing through the following points?

3. $(2, 4)$ and $(4, 2)$

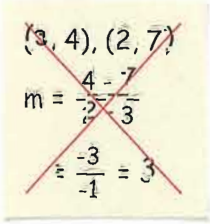
4. $(-1, -3)$ and $(3, 1)$

Do you UNDERSTAND?

5. **Vocabulary** What is a y-intercept? How is a y-intercept different from an x-intercept?

6. Explain why the slope of a vertical line is called "undefined."

7. **Error Analysis** A classmate found the slope between two points. What error did she make?



$$\xcancel{\begin{array}{l}(3, 4), (2, 7)\\ m = \frac{4-7}{2-3}\\ = \frac{-3}{-1} = 3\end{array}}$$

Practice and Problem-Solving Exercises

A Practice

Find the slope of the line through each pair of points.

See Problem 1.

8. $(1, 6)$ and $(8, -1)$

9. $(-3, 9)$ and $(0, 3)$

10. $(0, 0)$ and $(2, 6)$

11. $(-4, -3)$ and $(7, 1)$

12. $(-2, -1)$ and $(8, -3)$

13. $(1, 2)$ and $(2, 3)$

14. $(2, 7)$ and $(-3, 11)$

15. $(-3, 5)$ and $(4, 5)$

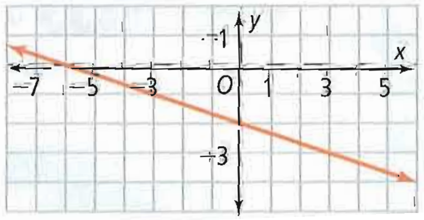
16. $(-5, -7)$ and $(0, 10)$

Write an equation for each line.

See Problem 2.

17. $m = 3$ and the y-intercept is $(0, 2)$.

18.



19. $m = \frac{5}{6}$ and the y-intercept is $(0, 12)$.

20. $m = 0$ and the y-intercept is $(0, -2)$.

21. $m = -5$ and the y-intercept is $(0, -7)$.

Write each equation in slope-intercept form. Then find the slope and y-intercept of each line.

See Problem 3.

22. $5x + y = 4$

23. $-3x + 2y = 7$

24. $-\frac{1}{2}x - y = \frac{3}{4}$

25. $8x + 6y = 5$

26. $9x - 2y = 10$

27. $y = 7$

Graph each equation.

See Problem 4.

28. $y = 2x$

29. $y = -3x - 1$

30. $y = 3x - 2$

31. $y = -4x + 5$

32. $5x - 2y = -4$

33. $-2x + 5y = -10$

34. $y - 3 = -2x$

35. $y + 4 = -3x$

36. $-y + 5 = -2x$

B Apply

Graph each equation.

37. $y = -\frac{1}{2}x - \frac{3}{2}$

38. $y = -2x + 3$

39. $y = -x + 7$

40. $3y - 2x = -12$

41. $4x + 5y = 20$

42. $4x - 3y = -6$

43. $\frac{2}{3}x + \frac{y}{3} = -\frac{1}{3}$

44. $x = 5$

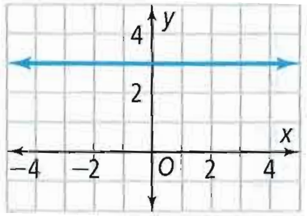
45. $2.4 = -3.6x - 0.4y$

46. **Think About a Plan** Suppose the equation $y = 12 + 10x$ models the amount of money in your wallet, where y is the total in dollars and x is the number of weeks from today. If you graphed this equation, what would the slope represent in the situation? Explain.

- Is the equation in slope-intercept form?
- What units make sense for the slope?

Find the slope and y-intercept of each line.

47.



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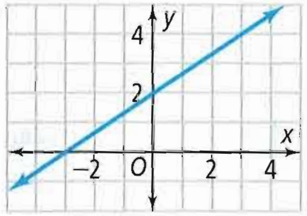

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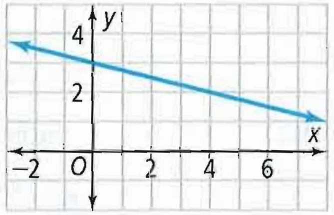


Find the slope and y-intercept of each line.

50. $y = 0.4 - 0.8x$

51. $x = -3$

52. $y = 0$

53. $-\frac{1}{3}x - \frac{2}{3}y = \frac{5}{3}$

54. $-Ax + By = -C$

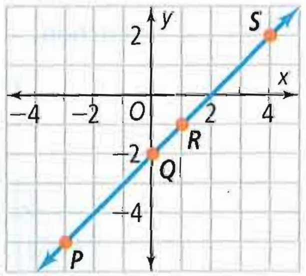
55. $\frac{A}{D}x + \frac{B}{D}y = \frac{C}{D}$

56. The equation $d = 4 - \frac{1}{15}t$ represents your distance from home d for each minute you walk t.

a. If you graphed this equation, what would the slope represent? Explain.

b. Are you walking towards or away from your home? Explain.

57. **Reasoning** Use the graph to find the slope between the following points on the line.



a. P and Q

b. Q and S

c. S and P

d. R and Q

e. Make a conjecture based on your answers to parts (a)–(d).

58. **Error Analysis** A classmate says that the graph of $3y - 2x = 5$ has a slope of 2. What mistake did he make?

Find the slope of the line through each pair of points.

59. $\left(\frac{3}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{2}{3}, \frac{1}{3}\right)$

60. $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and $(-3, -4)$

61. $\left(0, -\frac{1}{2}\right)$ and $\left(\frac{7}{5}, 10\right)$

Challenge **62.** You can find the equation of a line through two points even if one point is not the y-intercept.

- Find the slope m of the line passing through the two points.
- Using either point, substitute for x, y, and m into $y = mx + b$.
- Solve for b and rewrite $y = mx + b$ for the values of m and b.

Write an equation in slope-intercept form for the line passing through each pair of points.

a. (2, 5) and (6, 7) **b.** (−4, 16) and (3, −5) **c.** (−2, 17) and (2, 1)

Sunshine State Standards Practice

MA.912.A.2.12 **63.** Which equation does NOT represent a direct variation?

Ⓐ $y - 3x = 0$ Ⓑ $y + 2 = \frac{1}{2}x$ Ⓒ $\frac{y}{x} = \frac{2}{3}$ Ⓓ $y = \frac{x}{17}$

MA.912.A.2.3 **64.** Which equation models the data in the table?

Ⓕ $y = x^2 - 1$ Ⓗ $y = -x^2 + 3$

Ⓖ $y = x^2 + 3$ Ⓘ $y = x^2 + 1$

x	y
1	4
2	7
3	12
4	19

MA.912.A.3.3 **65.** In the formula $V = \frac{1}{3}\pi r^2 h$, which expression is equal to h?

Ⓐ $\frac{V}{3\pi r^2}$ Ⓒ $\frac{3V}{\pi r^2}$

Ⓑ $V - \frac{1}{3}\pi r^2$ Ⓓ $\frac{3V\pi}{r^2}$

MA.912.A.2.4 **66. Short Response** Graph the relation {(−2, 1), (0, 2), (−1, −1), (−2, −2)}. What are the domain and range?

Mixed Review

Find the domain and range of each relation, and determine whether it is a function. **See Lesson 2-1.**

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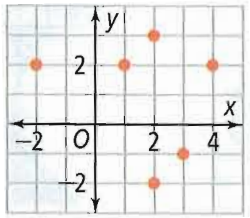

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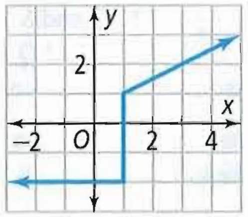

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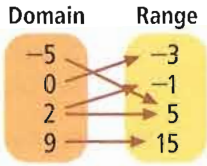


Get Ready! **To prepare for Lesson 2-4, do Exercises 70–72.**

Evaluate each expression for $x = 0$. **See Lesson 1-3.**

70. $5x + 2$ **71.** $(x - 4) + 12$ **72.** $13 - 6.5x$

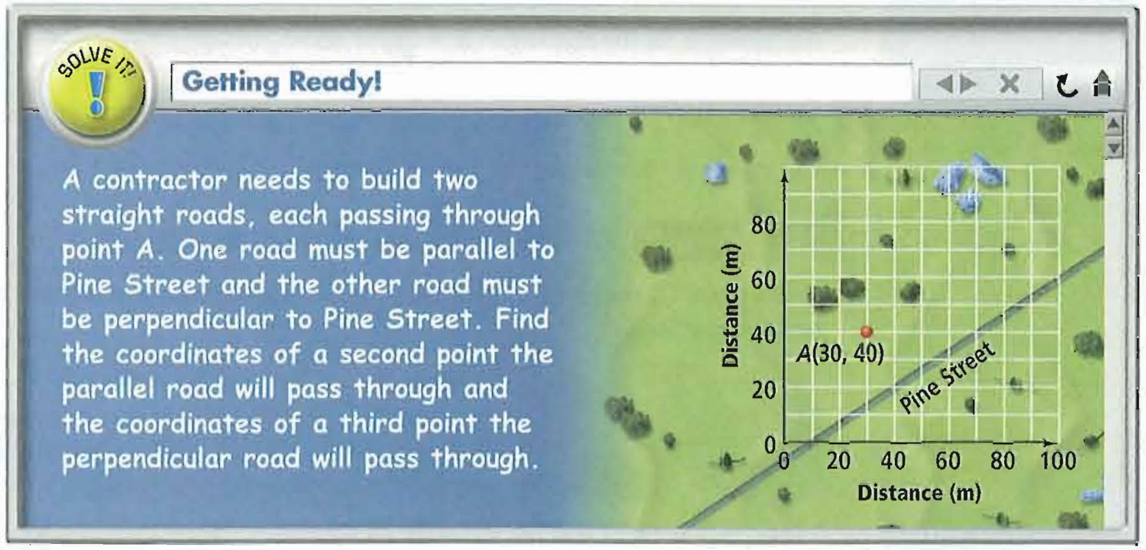
2-4 More About Linear Equations

Sunshine State Standards

MA.912.A.3.10 Write an equation of a line given two points on the line or its slope and one point on the line. Find an equation of a line parallel or perpendicular to a given line, through a given point.

MA.912.A.2.6 Graph linear functions.

Objective To write an equation of a line given its slope and a point on the line



A contractor needs to build two straight roads, each passing through point A. One road must be parallel to Pine Street and the other road must be perpendicular to Pine Street. Find the coordinates of a second point the parallel road will pass through and the coordinates of a third point the perpendicular road will pass through.

Lesson Vocabulary
- point-slope form
- standard form of a linear equation
- parallel lines
- perpendicular lines

If you travel along a line that is parallel to a given line, you will stay the same distance from the given line. If you travel along a line that is perpendicular to a given line, you will travel either toward or away from the given line along the most direct path.

Essential Understanding The slopes of two lines in the same plane indicate how the lines are related.

Given the slope and y-intercept, you can write the equation of a line in slope-intercept form. You can also write the equation of a line in *point-slope form.*

take note

Key Concept **Point-Slope Form**

The equation of a line in **point-slope form** through point (x_1, y_1) with slope m:

$$y - y_1 = m(x - x_1)$$

Here's Why It Works By substituting the general point (x, y), for (x_2, y_2) in the slope formula, you can rewrite the slope formula in point-slope form.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$

$$m(x - x_1) = \frac{y - y_1}{x - x_1}(x - x_1)$$

$$m(x - x_1) = y - y_1$$

$$y - y_1 = m(x - x_1)$$

Problem 1 Writing an Equation Given a Point and the Slope

Plan

Is slope-intercept or point-slope form more helpful for writing this equation?
Since the slope and a point (not the y-intercept) are given, point-slope form is more helpful.

A line passes through (−5, 2) with slope $\frac{3}{5}$. What is an equation of the line?

$y - y_1 = m(x - x_1)$ Use point-slope form.

$y - 2 = \frac{3}{5}[x - (-5)]$ Substitute $m = \frac{3}{5}$ and $(x_1, y_1) = (-5, 2)$.

$y - 2 = \frac{3}{5}(x + 5)$ Simplify.

An equation for the line is $y - 2 = \frac{3}{5}(x + 5)$.

Got It? **1.** What is an equation of the line through (7, −1) with slope −3?

Problem 2 Writing an Equation Given Two Points

A line passes through (3, 2) and (5, 8). What is an equation of the line in point-slope form?

Know	Need	Plan
Two points	An equation written in point-slope form	Substitute the slope and either point in the point-slope form.

Think

Does it matter which point you substitute into point-slope form?
No; you can choose either point as (x_1, y_1).

Let $(x_1, y_1) = (3, 2)$ and $(x_2, y_2) = (5, 8)$.

$m = \frac{8 - 2}{5 - 3} = \frac{6}{2} = 3$ Substitute into the slope formula and simplify.

$y - 2 = 3(x - 3)$ Substitute into point-slope form.

Got It? **2. a.** A line passes through (−5, 0) and (0, 7). What is an equation of the line in point-slope form?

b. Reasoning What is another equation in point-slope form of the line through the points (−5, 0) and (0, 7)? Explain.

Another form of the equation of a line is *standard form*, in which the sum of the x and y terms are set equal to a constant. When possible, you write the coefficients of x and y and the constant term as integers.

take note

Key Concept Standard Form of a Linear Equation

A **standard form of a linear equation** is $Ax + By = C$, where A, B, and C are real numbers and A and B are not *both* zero.

Think

How can you rewrite the equation using only integer values? Multiply each side of the equation by the least common denominator of all fraction coefficients.

Problem 3 Writing an Equation in Standard Form

What is an equation of the line $y = \frac{3}{4}x - 5$ in standard form? Use integer coefficients.

$$y = \frac{3}{4}x - 5$$

$-\frac{3}{4}x + y = -5$ Subtract $\frac{3}{4}x$ from each side.

$-3x + 4y = -20$ Multiply each side by 4.

Got It? 3. What is an equation of the line $y = 9.1x + 3.6$ in standard form?

take note

Concept Summary Writing Equations of Lines

Slope-Intercept Form	Point-Slope Form	Standard Form
$y = mx + b$	$y - y_1 = m(x - x_1)$	$Ax + By = C$
Use this form when you know the slope and the y-intercept.	Use this form when you know the slope and a point, or when you know two points.	A, B, and C are real numbers. A and B cannot both be zero.

You can graph an equation in standard form quickly by determining the x- and y-intercepts and then drawing the line through them.

Think

Why set $x = 0$ to find the y-intercept? The x-coordinate of any point on the y-axis is zero.

Problem 4 Graphing an Equation Using Intercepts

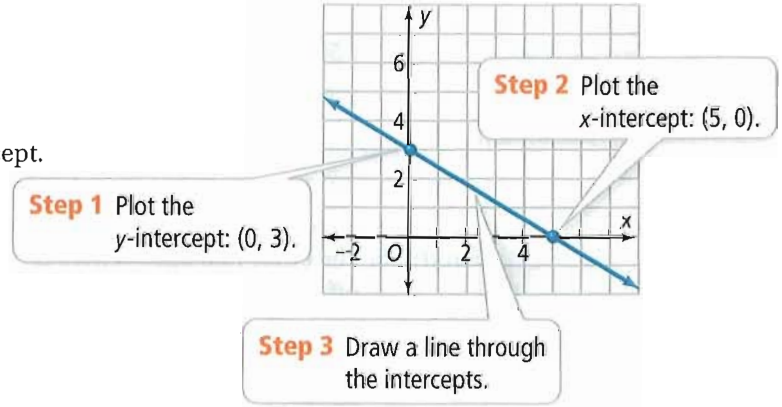
What are the intercepts of $3x + 5y = 15$? Graph the equation.

Set $x = 0$ to find the y-intercept.

$$3(0) + 5y = 15$$
$$5y = 15$$
$$y = 3$$

Set $y = 0$ to find the x-intercept.

$$3x + 5(0) = 15$$
$$3x = 15$$
$$x = 5$$



Step 1 Plot the y-intercept: (0, 3).

Step 2 Plot the x-intercept: (5, 0).

Step 3 Draw a line through the intercepts.

Got It? 4. What are the intercepts of $2x - 4y = 8$? Graph the equation.

Problem 5 Drawing and Interpreting a Linear Graph

Biology The number of times a cricket chirps per minute depends on the temperature. The number of chirps in 2 seconds for two temperatures are shown at the bottom right.

Think

How can you find the number of chirps in a minute given the number of chirps in 2 seconds?

There are 60 seconds in 1 minute. Multiply the number of chirps in 2 seconds by $\frac{60}{2}$ or 30.

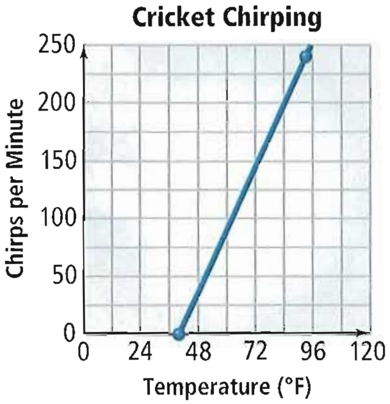
A What graph models the situation?

First, find the number of chirps per minute.

40°F: $30(0) = 0$
93°F: $30(8) = 240$

Let $x =$ temperature in degrees Fahrenheit.
Let $y =$ number of times a cricket chirps.

Plot (40,0) and (93, 240).
Draw a line through the points.


Cricket Chirping

B What is an equation of the line in standard form?

$m = \frac{240 - 0}{93 - 40}$ Use the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

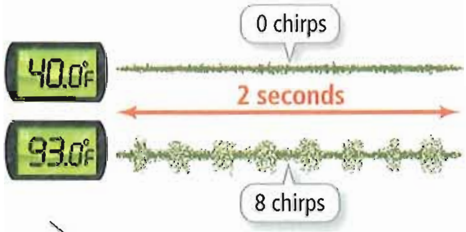
$= \frac{240}{53} \approx 4.5$ Subtract and simplify.

$y - y_1 = m(x - x_1)$ Use point-slope form.

$y - 0 = 4.5(x - 40)$ Substitute one of the points: (40, 0).

$y = 4.5x - 180$ Simplify.

$4.5x - y = 180$ Write in standard form.



C If the temperature is 70°F, how many times would a cricket be expected to chirp in one minute?

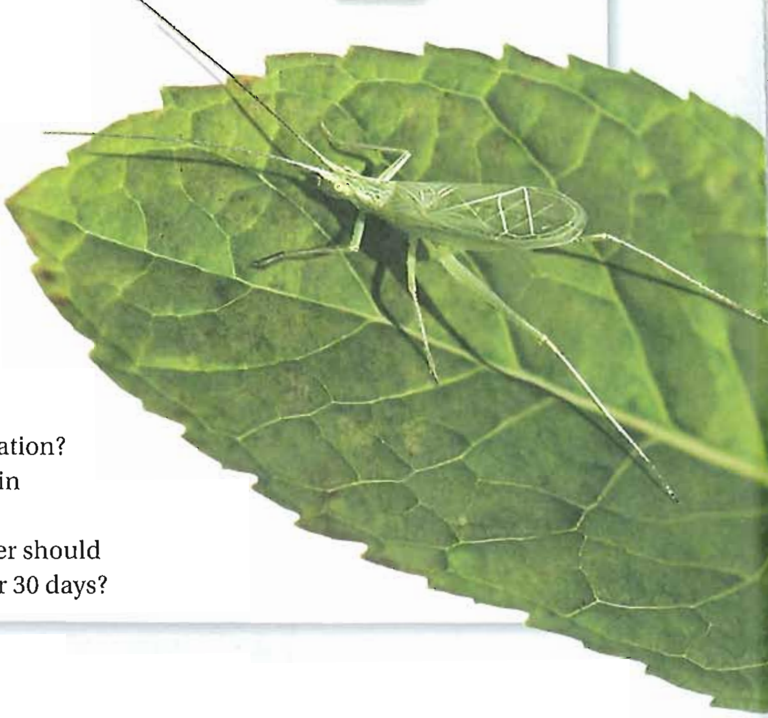
Let $x = 70$.

$y = 4.5x - 180$ Use an equation from part (b).

$y = 4.5(70) - 180$ Substitute.

$y = 135$ Simplify.

If the temperature is 70°F, the cricket would be expected to chirp 135 times in one minute.



Got It? **5.** The office manager of a small office ordered 140 packs of printer paper. Based on average daily use, she knows that the paper will last about 80 days.

a. What graph represents this situation?
b. What is the equation of the line in standard form?
c. How many packs of printer paper should the manager expect to have after 30 days?

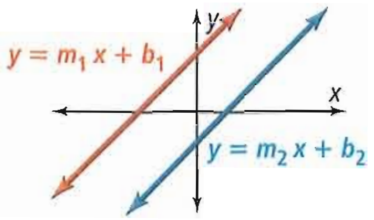
take note

Key Concepts Parallel and Perpendicular Lines

The slopes of **parallel lines** are equal.

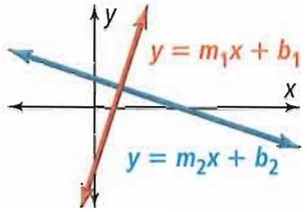
No line can be vertical.

The slopes of **perpendicular lines** are negative reciprocals of each other.



$m_1 = m_2$

$b_1 \neq b_2$



$m_1 \cdot m_2 = -1$

$m_1 = -\frac{1}{m_2}$

$m_2 = -\frac{1}{m_1}$

m_1 and m_2 are negative reciprocals of each other.

Problem 6 Writing Equations of Parallel and Perpendicular Lines

What is the equation of each line in slope-intercept form?

A the line parallel to $y = 6x - 2$ through $(1, -3)$

Identify the slope, use point-slope form, and rewrite in slope-intercept form.

$m = 6$	Parallel lines have the same slope. The slope of the line with equation $y = 6x - 2$ is 6.
$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-3) = 6(x - 1)$	Substitute 6 for m and $(1, -3)$ for (x_1, y_1).
$y + 3 = 6x - 6$	Distributive Property
$y = 6x - 9$	Write in slope-intercept form.

B the line perpendicular to $y = -4x + \frac{2}{3}$ through $(8, 5)$

Identify the slope, use point-slope form, and rewrite in slope-intercept form.

$m = -\frac{1}{-4} = \frac{1}{4}$	The slopes of perpendicular lines are negative reciprocals.
$y - y_1 = m(x - x_1)$	Point-slope form
$y - 5 = \frac{1}{4}(x - 8)$	Substitute $\frac{1}{4}$ for m and $(8, 5)$ for (x_1, y_1).
$y - 5 = \frac{1}{4}x - 2$	Distributive Property
$y = \frac{1}{4}x + 3$	Write in slope-intercept form.

Plan

How can you find the slope of a perpendicular line? Slopes of perpendicular lines are negative reciprocals, so use the equation $m_1 = -\frac{1}{m_2}$.

Got It? 6. What is the equation of each line in slope-intercept form?

a. the line parallel to $4x + 2y = 7$ through $(4, -2)$

b. the line perpendicular to $y = \frac{2}{3}x - 1$ through $(0, 6)$

Lesson Check

Do you know HOW?

Write an equation of each line in slope-intercept form.

1. slope -3; through $(1, -4)$

2. slope $\frac{1}{2}$; through $(2, 3)$

3. What are the intercepts of $3x + y = 6$? Graph the equation.

Write an equation of each line in standard form.

4. the line parallel to $y = -3x + 4$ through $(0, -1)$

5. the line perpendicular to $-2x + 3y = 9$ through $(-1, -3)$

Do you UNDERSTAND?

6. **Vocabulary** Tell whether each equation is in slope-intercept, point-slope, or standard form.

a. $y + 2 = -2(x - 1)$ **b.** $y = -\frac{1}{4}x + 9$

c. $-x - 2y = 1$ **d.** $y - 3 = 4x$

7. Which form would you use to write the equation of a line if you knew its slope and x-intercept? Explain.

8. If the intercepts of a line are $(a, 0)$ and $(0, b)$, what is the slope of the line?

9. **Error Analysis** Your friend says the line $y = -2x + 3$ is perpendicular to the line $x + 2y = 8$. Do you agree? Explain.

Practice and Problem-Solving Exercises

A Practice

Write an equation of each line.

See Problem 1.

10. slope $= 3$; through $(1, 5)$

11. slope $= \frac{5}{6}$; through $(22, 12)$

12. slope $= -\frac{3}{5}$; through $(-4, 0)$

13. slope $= 0$; through $(4, -2)$

14. slope $= -1$; through $(-3, 5)$

15. slope $= 5$; through $(0, 2)$

Write in point-slope form an equation of the line through each pair of points.

See Problem 2.

16. $(-10, 3)$ and $(-2, -5)$

17. $(1, 0)$ and $(5, 5)$

18. $(-4, 10)$ and $(-6, 15)$

19. $(0, -1)$ and $(3, -5)$

20. $(7, 11)$ and $(13, 17)$

21. $(1, 9)$ and $(6, 2)$

Write an equation of each line in standard form with integer coefficients.

See Problem 3.

22. $y = \frac{1}{2}x - 2$

23. $y = -7x - 9$

24. $y = -\frac{3}{5}x + 3$

25. $y = 4.2x + 7.9$

Find the intercepts and graph each line.

See Problem 4.

26. $x - 4y = -4$

27. $2x + 5y = -10$

28. $-3x + 2y = 6$

29. $5x + 7y = 14$

Write and graph an equation to represent each situation.

See Problem 5.

30. You put 15 gallons of gasoline in your car. You know that this amount of gasoline will allow you to drive about 450 miles.

31. A meal plan lets students buy \$20 meal cards. Each meal card lasts about 8 days.

Write the equation of the line through each point. Use slope-intercept form.

See Problem 6.

32. $(1, -1)$; parallel to $y = \frac{2}{5}x - 3$

33. $(-3, 1)$; perpendicular to $y = -\frac{2}{5}x - 4$

34. $(-7, 10)$; parallel to $2x - 3y = -3$

35. $(-2, 1)$; perpendicular to $3x + y = 1$

B Apply

Graph each equation.

36. $3x + 5y = 12$

37. $2x + y = 3$

38. $6y - 4x = -24$

39. $3y - x = -6$

40. $-20x - 45y = 48$

41. $2x - \frac{3}{2}y = -3$

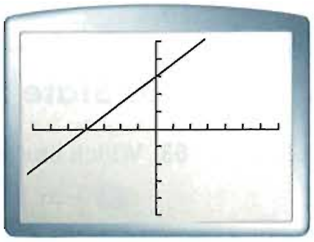
Write an equation of the line through each pair of points. Use point-slope form.

42. $\left(\frac{3}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{2}{3}, \frac{1}{3}\right)$

43. $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and $(-3, -4)$

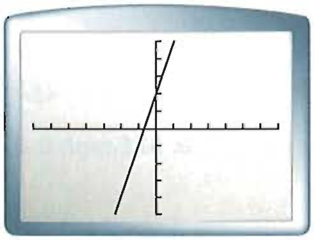
44. $\left(0, \frac{1}{2}\right)$ and $\left(\frac{5}{7}, 0\right)$

45. **Think About a Plan** Write an equation for the line shown here. Each interval is 1 unit.
- What do you know from the graph?
- Which form of the equation of a line could you use with the information you have?



Write an equation for each line. Each interval is 1 unit.

46.



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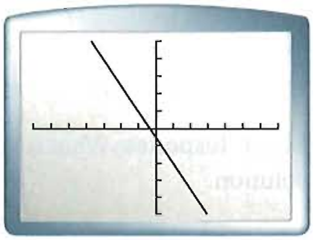


Find the slope, if any, and the intercepts, if any, of each line.

48. $f(x) = \frac{2}{3}x + 4$

49. $y = -x + 1000$

50. $y + 0.8x = 0.4$

51. $g(x) = 54x - 1$

52. $x + 3 = 0$

53. $y + 3 = 3$

54. **a.** Write the point-slope form of the line that passes through $A(-3, 12)$ and $B(9, -4)$. Use point A in the equation.

b. Write the point-slope form of the same line using point B in the equation.

c. Rewrite each equation in standard form. What do you notice?

Write an equation for each line. Then graph the line.

55. $m = 0$, through $(5, -1)$

56. $m = \frac{5}{6}$, through $(-4, 0)$

57. $m = -\frac{3}{2}$, through $(0, -1)$

58. **Reasoning** Suppose lines ℓ_1 and ℓ_2 intersect at the origin. Also, ℓ_1 has slope $\frac{y}{x}$ $(x > 0, y > 0)$ and ℓ_2 has slope $-\frac{x}{y}$. Then ℓ_1 contains (x, y) and ℓ_2 contains $(-y, x)$.

a. Explain why the two right triangles are congruent.

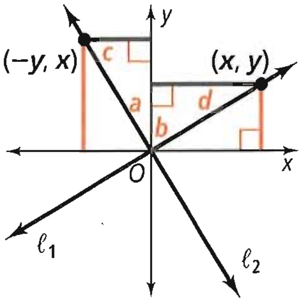
b. Complete each equation about the angle measures a, b, c, and d.

$a = $ ■ $\quad c = $ ■

$a + c = $ ■ $\quad b + d = $ ■

c. What must be true about $a + b$? Why?

d. What must be true about ℓ_1 and ℓ_2? Why?



Challenge

Points that are on the same line are *collinear*. Use the definition of slope to determine whether the given points are collinear.

59. $(-2, 6), (0, 2), (1, 0)$

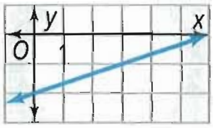
60. $(3, -5), (-3, 3), (0, 2)$

61. Geometry Prove that the triangle with vertices $(3, 5)$, $(-2, 6)$, and $(1, 3)$ is a right triangle.

62. Geometry Prove that the quadrilateral with vertices $(2, 5)$, $(4, 8)$, $(7, 6)$, and $(5, 3)$ is a rectangle.

Sunshine State Standards Practice

MA.912.A.3.10 **63.** Which line is perpendicular to the line shown in the graph?



Ⓐ $-3x + y = 5$

Ⓑ $3x + y = -1$

Ⓒ $x - 3y = 0$

Ⓓ $x + 3y = -3$

MA.912.A.3.10 **64.** Which line is parallel to the line $5x + 6y = 30$?

Ⓕ $5x + 9y = 30$

Ⓖ $9x + 6y = 30$

Ⓗ $5x + 6y = 9$

Ⓘ $6x + 5y = 30$

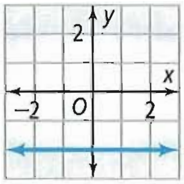
MA.912.A.3.6 **65. Short Response** What is the solution of the inequality $|x - 3| \geq 5$? Graph the solution.

Mixed Review

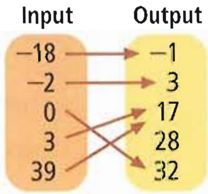
Find the domain and range of each relation and determine whether it is a function. **See Lesson 2-1.**

66. $\{(-3, 4), (-1, 2), (0, -2), (1, 0), (2, 2)\}$

67.



68.



Name the property of real numbers illustrated by each equation. **See Lesson 1-2.**

69. $\frac{2}{5} + \frac{27}{5} \cdot \frac{5}{27} = \frac{2}{5} + 1$

70. $97(7) = 100(7) - 3(7)$

71. $21 + 19.7 - 19.7 = 21$

Get Ready! **To prepare for Lesson 2-5, do Exercises 72–75.**

Write the equation for each line. Use slope-intercept form. **See Lesson 2-3.**

72. $m = 3$ and the y-intercept is $(0, -5)$.

73. $m = \frac{1}{2}$ and the y-intercept is $(0, 0)$.

74. $m = \frac{4}{5}$ and the y-intercept is $(0, 7)$.

75. $m = -\frac{3}{8}$ and the y-intercept is $(0, 12)$.

2 Mid-Chapter Quiz

Do you know HOW?

Determine whether each relation is a function.

1.

x	y
3	7
4	2
3	2
5	1

2.

x	y
1	4
2	3
3	3
4	4

Find the *x*- and *y*-intercepts of each line.

3. $x - 3y = 9$

4. $y = 7x + 5$

5. $y = 6x$

6. $-4x + y = 10$

Write the equation of each line in slope-intercept form and identify the slope.

7. $2x - y = 9$

8. $4x = 2 + y$

9. $5y = -3x - 10$

10. $4x + 6y = 12$

Write an equation of each line in standard form with integer coefficients.

11. the line through (2, 3) and (4, 5)

12. the line through (−4, 6) and (2, −2)

13. the line through (−4, 2) with slope 3

14. the line through (1, 2) with slope $\frac{4}{5}$

15. a line through (3, 1) with slope 0

16. a line with slope of $\frac{2}{3}$ and *y*-intercept (0, 5)

17. $2y = -4x - 12$

18. $\frac{2}{3}x + 3 = 6y - 15$

Write an equation of each line in point-slope form.

19. (−4, 2) and (−3, 5)

20. (0, 0) and (−4, −5)

21. (−4, −3) and (2, 7)

Graph each equation.

22. $2y = 4x + 8$

23. $2x - 3y = 6$

24. $4y - x = 16$

For each function, determine whether *y* varies directly with *x*. If so, identify the constant of variation.

25. $2y = 3x$

26. $4y - 7x = 0$

27. $y + \frac{3}{4}x = 12$

Do you UNDERSTAND?

28. a. A group of friends is going to the movies. Each ticket costs $8.00. Write an equation to model the total cost of the group's tickets.

b. Graph the equation. Explain what the *x*- and *y*-intercepts represent.

c. What would be the cost for 12 tickets?

d. **Writing** Could the domain include fractions? Explain.

29. Which line is perpendicular to $3x + 2y = 6$?

Ⓐ $4x - 6y = 3$ Ⓒ $2x + 3y = 12$

Ⓑ $y = -\frac{3}{2}x + 4$ Ⓓ $y = \frac{3}{2}x + 1$

30. **Reasoning** Why is the slope of a vertical line undefined?

31. Suppose $m = 25 - 0.15n$ describes the amount of money remaining on a $25 phone card m, as a function of the number of minutes of calls you make n. What are a reasonable domain and range?

Concept Byte
For Use With Lesson 2-4

Piecewise Functions

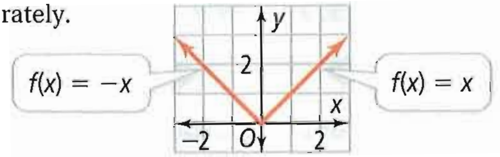
Sunshine State Standard
MA.912.A.2.9 Recognize, interpret, and graph functions defined piece-wise.

Recall from Lesson 1-6 that $|x|$, the absolute value of x, is the distance of x from zero. When $x \geq 0$, $|x| = x$. When $x < 0$, $|x| = -x$. The absolute value function is an example of a *piecewise function*. A **piecewise function** has different rules for different parts of its domain.

Example 1

Graph the absolute value function $f(x) = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$.

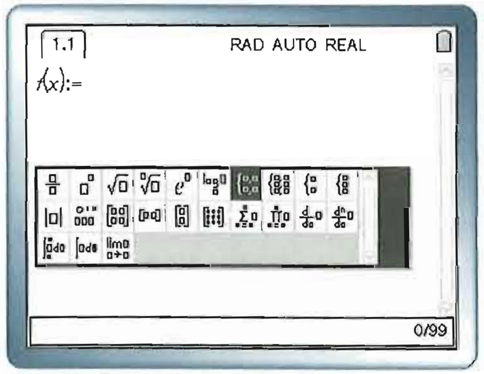
Graph each piece separately.



Example 2

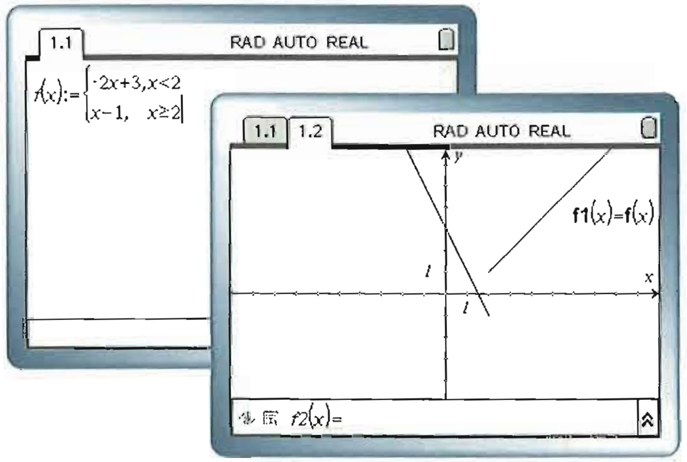
Use a graphing calculator to graph the function $f(x) = \begin{cases} -2x + 3, \text{ if } x < 2 \\ x - 1, \text{ if } x \geq 2 \end{cases}$.

Define the function $f(x)$. Use the brackets from the math expression templates to enter the piecewise function.



f(x):=

Enter the rules for the two branches. Set $f_1(x) = f(x)$ and graph.

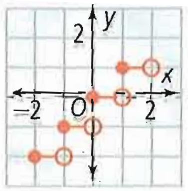


f(x):= {·2x+3,x<2 ; x−1, x≥2

Step functions are piecewise functions. A **step function** pairs every number in an interval with a single value. The graph of a step function can look like the steps of a staircase. One step function is the **greatest integer function** $y = [x]$, where $[x]$ represents the greatest integer less than or equal to x.

Example 3

What is the graph of the function $f(x) = [x]$?



Each piece of the graph is a horizontal segment that is missing its right endpoint. The open circle indicates that the right endpoint is not part of the graph.

Given a piecewise function in one form, you can represent it in each of the other forms, including a table, graph, algebraic function, or verbal statement.

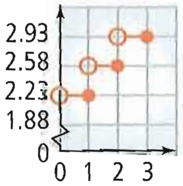
Example 4

Media Postage **You want to mail a book that weighs 2.5 lb. The table lists postage for a book weighing up to 3 lb. Define and graph the media-postage function. How much will you pay in postage?**

Media Postage

Weight (lb)	Price ($)
$x \leq 1$	2.23
$1 < x \leq 2$	2.58
$2 < x \leq 3$	2.93

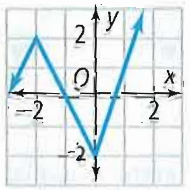
$$f(x) = \begin{cases} 2.23, & \text{for } 0 < x \leq 1 \\ 2.58, & \text{for } 1 < x \leq 2 \\ 2.93, & \text{for } 2 < x \leq 3 \end{cases}$$



Since $2 < 2.5 \leq 3$, you will pay $2.93.

Example 5

What piecewise function represents the graph?



Piece 1 When $x \leq -2$, the rule is $f(x) = 2x + 6$.

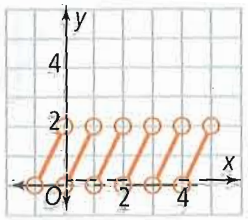
Piece 2 When $-2 < x \leq 0$, the rule is $f(x) = -2x - 2$.

Piece 3 When $x > 0$, the rule is $f(x) = 3x - 2$.

$$f(x) = \begin{cases} 2x + 6, & \text{for } x \leq -2 \\ -2x - 2, & \text{for } -2 < x \leq 0 \\ 3x - 2, & \text{for } x > 0 \end{cases}$$

Exercises

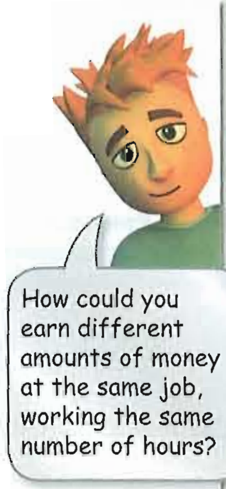
1. Graph the sign function. $f(x) = \begin{cases} -1, & \text{for } x < 0 \\ 0, & \text{for } x = 0. \\ 1, & \text{for } x > 0 \end{cases}$
2. Graph the function $f(x) =$ the least integer greater than x.
3. What piecewise function represents the graph at the right?
4. **Postage** In 2008, first-class letter postage was $.42 for up to one ounce and $.17 for each additional ounce up to 3.5 oz. Graph this postage function.



2-5 Using Linear Models

Sunshine State Standard
MA.912.A.3.10 Write an equation of a line given two points.

Objectives To write linear equations that model real-world data
To make predictions from linear models



Solve It! **Getting Ready!**

The graph shows the number of hours and the amount of money you earned each day last week. How many hours should you work to earn $200? What assumptions did you make to find your answer? Explain.

Lesson Vocabulary
- scatter plot
- correlation
- line of best fit
- correlation coefficient

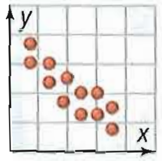
Graphs of data pairs for a real-world situation rarely fall in a line. Their arrangement, however, can suggest a relationship that you can model with a linear function.

Essential Understanding Sometimes it is possible to model data from a real-world situation with a linear equation. You can then use the equation to draw conclusions about the situation.

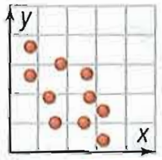
A **scatter plot** is a graph that relates two sets of data by plotting the data as ordered pairs. You can use a scatter plot to determine the strength of the relationship, or **correlation**, between data sets. The closer the data points fall along a line with positive slope,

- the stronger the linear relationship and
- the stronger the positive correlation

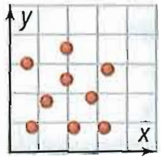
between the two variables.



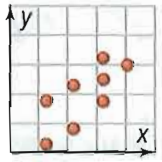
Strong Negative Correlation



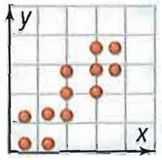
Weak Negative Correlation



No Correlation



Weak Positive Correlation



Strong Positive Correlation

Problem 1 Using a Scatter Plot

Utilities The table lists average monthly temperatures and electricity costs for a Texas home in 2008. The table displays the values rounded to the nearest whole number. Make a scatter plot. How would you describe the correlation?

Average Temperatures and Electricity Costs

Month	Average Temp. (°F)	Electricity Bill ($)
January	61	150
February	58	139
March	67	172
April	75	205
May	79	170
June	83	234
July	84	255
August	85	245
September	81	210
October	76	183
November	65	132
December	58	110

Plan

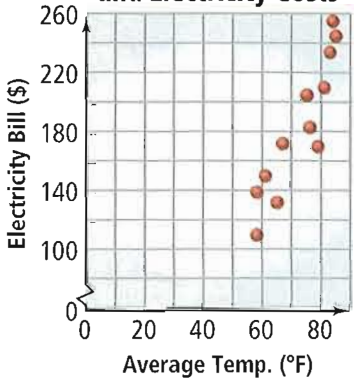
Which variable is the independent variable?
Temperature does not depend on the electric bill, so temperature is the independent variable.

Step 1 Make a scatter plot.

Step 2 Describe the correlation.

As the temperature increases, the electricity cost also increases. The points are relatively tightly clustered around a line. There is a strong positive correlation between temperature and electricity cost.

Average Temperatures and Electricity Costs



Got It? 1. **a.** The table shows the numbers of hours students spent online the day before a test and the scores on the test. Make a scatter plot. How would you describe the correlation?

b. Reasoning Using the graph from Problem 1, how much would you expect to pay for electricity if the average temperature was 70°F? Explain.

Computer Use and Test Scores

Number of Hours Online	0	0	1	1	1.5	1.75	2	2	3	4	4.5	5
Test score	100	94	98	88	92	89	75	70	78	72	57	60

A *trend line* is a line that approximates the relationship between the variables, or data sets, of a scatter plot. You can use a trend line to make predictions from the data. In a previous lesson you learned how to use two points to write the equation of a line to model a real-world problem. You can use this method to write the equation of a trend line.

Problem 2 Writing the Equation of a Trend Line

Finance **The table shows the median home prices in Florida. What is the equation of a trend line that models a relationship between time and home prices? Use the equation to predict the median home price in 2020.**

Florida Median Home Prices

Year	1940	1950	1960	1970	1980	1990	2000
Median Price ($)	23,100	40,100	58,100	57,600	89,300	98,500	105,500

Know	Need	Plan
Seven data points	The equation of a trend line	• Plot the given points. • Find a trend line. • Write the equation.

Step 1 Make a scatter plot. Let $x = 0$ correspond to 1940.

Step 2 Sketch a trend line.

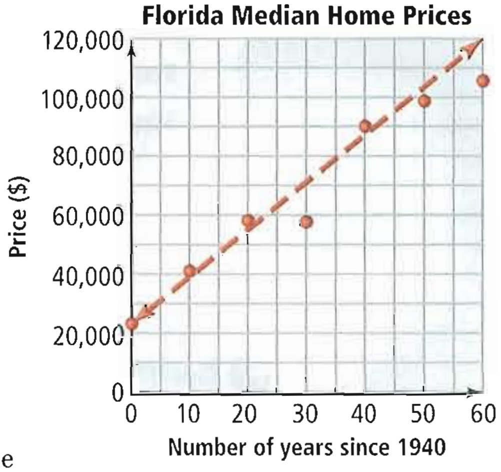
Step 3 Choose two points on the trend line, (10, 40,000) and (55, 110,000). Use slope-intercept form to write an equation for the line.

$y = 1556x + 24{,}440$

Step 4 Use the equation to predict the median home price in 2020.

$y = 1556(80) + 24{,}440 = 148{,}920$

Based on the trend line, the median home price in 2020 will be around $149,000.



Got It? **2.** The table shows median home prices in California. What is an equation for a trend line that models the relationship between time and home prices?

California Median Home Prices

Year	1940	1950	1960	1970	1980	1990	2000
Median Price ($)	36,700	57,900	74,400	88,700	167,300	249,800	211,500

The trend line that gives the most accurate model of related data is the **line of best fit**. One method for finding a line of best fit is *linear regression*. You can use the **LinReg** function on your graphing calculator to find the line of best fit. The **correlation coefficient**, r, indicates the strength of the correlation. The closer r is to 1 or -1, the more closely the data resembles a line and the more accurate your model is likely to be.

Problem 3 Finding the Line of Best Fit

Food You research the average cost of whole milk for several recent years to look for trends. The table shows your data.

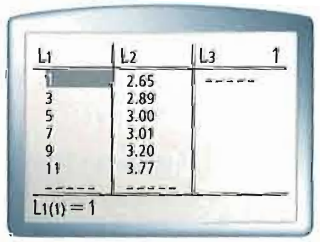
Cost of Whole Milk						
Year	1998	2000	2002	2004	2006	2008
Average cost for one gallon ($)	2.65	2.89	3.00	3.01	3.20	3.77

Source: U.S. Department of Agriculture

A What is the equation for the line of best fit? How accurate is your line of best fit?

Step 1

Use the **STAT** feature to enter the data in your graphing calculator. Enter the x-values (year) in **L1** and the y-values (price) in **L2**. Let 1997 = year 0.

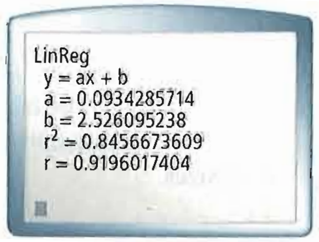


```
L1      |L2      |L3      1
1       | 2.65   | -----
3       | 2.89   |
5       | 3.00   |
7       | 3.01   |
9       | 3.20   |
11      | 3.77   |
-----   | -----  |
L1(1) = 1
```

Step 2

Use **LinReg** to find the linear regression line of best fit for the data.

$y = 0.09x + 2.53$



```
LinReg
 y = ax + b
 a = 0.0934285714
 b = 2.526095238
 r² = 0.8456673609
 r = 0.9196017404
```

The correlation coefficient, r, is approximately 0.92. Since r is close to 1, the line of best fit is quite accurate.

Think

What factors could affect the accuracy of your prediction? Predictions based on strongly correlated data are likely to be more reliable than predictions based on weakly correlated data.

B Based on your linear model, how much would you expect to pay for a gallon of whole milk in 2020?

$y = 0.09x + 2.53$ Use the line of best fit.

$y = 0.09(23) + 2.53$ Substitute 23 for x.

$y = 4.60$

In 2020, you would expect to pay about $4.60 for a gallon of whole milk.

Got It? 3. The table lists the cost of 2% milk. Use a scatter plot to find the equation of the line of best fit. Based on your linear model, how much would you expect to pay for a gallon of 2% milk in 2025?

Cost of 2% Milk						
Year	1998	2000	2002	2004	2006	2008
Average cost for one gallon ($)	2.57	2.83	2.93	2.93	3.10	3.71

Source: U.S. Department of Agriculture

Lesson Check

Do you know HOW?

Make a scatter plot of each set of points and describe the correlation.

1. {(1.2, 1), (2.5, 6), (2.5, 7.5), (4.1, 11), (7.9, 19)}

2. {(1, 55), (2, 38), (3, 54), (4, 37), (5, 53), (6, 40), (7, 53), (8, 36)}

3. Make a scatter plot for the following set of points. Describe the correlation and sketch a trend line. {(2, 58), (6, 105), (8, 88), (8, 118), (12, 117), (16, 137), (20, 157), (20, 169)}

Do you UNDERSTAND?

4. **Writing** How can you determine whether two variables x and y for a real-life situation are correlated?

5. Do you think a trend line on a graph is always the same as the line of best fit? Why or why not?

6. **Compare and Contrast** What is the difference between a positive correlation and a negative correlation? How might you relate positive correlation with direct variation?

Practice and Problem-Solving Exercises

A Practice

Make a scatter plot and describe the correlation.

See Problem 1.

7. {(0, 11), (2, 8), (3, 7), (7, 2), (8, 0)}

8. **Manufacturing** The table shows the numbering system used in Europe and the United States for shoe sizes.

Shoe Sizes

U.S. Size	1	3	5	7	9	11
European Size	31	34	36	39	41	44

Write the equation of a trend line.

See Problem 2.

9. {(−10, 3), (−5, 1), (−1, −4), (3, −7), (12, −12)}

10. {(−15, 8), (−8, 7), (−3, 0), (0, 0), (7, −3)}

11. The table shows the number of hours you studied before your eight math tests and your percent score on each test.

Studying Hours and Test Score

Number of Hours	8	5	12	10	2	9	11	14
Score (%)	75	62	80	85	35	70	82	95

See Problem 3.

12. **a.** **Food Production** The table below shows pork production in China from 2000 to 2007. Use a calculator to find the line of best fit.

b. Use your linear model to predict how many metric tons of pork will be produced in 2025.

c. Use your linear model to predict when production is likely to reach 100,000 metric tons.

Pork Production in China

Year	2000	2001	2002	2003	2004	2005	2006	2007
Production (metric tons)	40,475	42,010	43,413	45,331	47,177	50,254	52,407	54,491

SOURCE: USDA Foreign Agricultural Service GAIN Report

B Apply

13. **Think About a Plan** The table shows the relationship between the production and the export of rice in Vietnam from 1985 to 2005.

Rice Production and Export

Production (1000 tonnes)	15,875	19,225	24,964	32,554	35,600
Export (1000 tonnes)	59	1624	1988	3400	5100

SOURCE: International Rice Research Institute

How much rice would you expect Vietnam to export in 2015 if the production that year is 42,250,000 tonnes?

- How can you use a scatter plot to find a linear model?
- How can you use your model to make a prediction?

14. **Nutrition** The table shows the relationship between Calories and fat in various fast-food hamburgers.

Fast Food Calories

Restaurant	A	B	C	D	E	F	G	H	I
Number of Calories	720	530	510	500	305	410	440	320	598
Grams of fat	46	30	27	26	13	20	25	13	26

a. Find the line of best fit for the relationship between Calories and fat.

b. How much fat would you expect a 330-Calorie hamburger to have?

c. **Error Analysis** Which estimate is *not* reasonable: 10 g of fat for a 200-Calorie hamburger or 36 g of fat for a 660-Calorie hamburger? Explain.

Reasoning **For a strong correlation, people often assume that change in one quantity causes change in the second quantity. This is not always true. For each situation, predict the type of correlation you might find. Do you think that change in the first quantity causes change in the second quantity? Explain.**

15. the number of ice cream cones sold and the temperature

16. the size of a car's engine and the number of passengers it is designed for

17. a person's age and the number of cassette tapes he or she owns

18. **Data Analysis** The table shows population and licensed driver statistics from a recent year.

a. Make a scatter plot.

b. Draw a trend line.

c. The population of Michigan was approximately 10 million that year. About how many licensed drivers lived in Michigan that year?

d. **Writing** Is the correlation between population and number of licensed drivers strong or weak? Explain.

Licensed Drivers

State	Population (millions)	Number of Drivers (millions)
Arkansas	2.8	2.0
Illinois	12.8	8.1
Kansas	2.8	2.0
Massachusetts	6.4	4.7
Pennsylvania	12.4	8.5
Texas	23.5	14.9

Challenge

19. Social Studies The table shows per capita revenues and expenditures for selected states for a recent year.

a. Show the data on a scatter plot. Draw a trend line.

b. If a state collected revenue of \$3000 per capita, how much would you expect it to spend per capita?

c. Ohio spent \$5142 per capita during that year. According to your model, how much did it collect in taxes per capita?

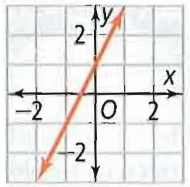
d. In that same year, New Jersey collected \$5825 per capita in taxes and spent \$5348 per capita. Does this information follow the trend? Explain.

Per Capita Revenue and Expenditure

State	Per Capita Revenue (\$)	Per Capita Expenditure (\$)
Arizona	4144	3789
Georgia	3904	3834
Maryland	5109	4557
Mississippi	5292	4871
New Mexico	6205	5793
Nevada	4345	3723
New York	7081	6891
Texas	4030	3442
Utah	5439	4459

Sunshine State Standards Practice

MA.912.A.3.10 **20.** What is the equation of the line shown in the graph?



Ⓐ $y = -2x + 2$

Ⓑ $y = 2x$

Ⓒ $y = 2x + 1$

Ⓓ $y = 2x + 2$

MA.912.A.2.12 **21.** Shauna drove 75 miles in 3 hours at a constant speed. How many miles did she drive in 2 hours?

Ⓕ 25 miles Ⓖ 50 miles Ⓗ 75 miles Ⓘ 100 miles

MA.912.A.2.6 **22.** Which equation does NOT represent a direct variation?

Ⓐ $y = x$

Ⓑ $2x - y = 5$

Ⓒ $2x - y = 0$

Ⓓ $2x - 5y = 0$

MA.912.A.3.14 **23. Short Response** The line $(y - 1) = \frac{2}{3}(x + 1)$ contains point $(a, -3)$. What is the value of a? Show your work.

Mixed Review

Graph the following linear equations. See Lesson 2-3.

24. $y = -7.5x + 11$

25. $-\frac{2}{9}x - \frac{5}{9}y = 10$

26. $5x - 4y = 3$

Write the equation of each line in standard form. See Lesson 2-4.

27. slope = 2; (2, 6)

28. slope = −1; (−3, 3)

29. slope = 0; (0, 2)

Get Ready! **To prepare for Lesson 2-6, do Exercises 30–32.**

Graph each pair of functions on the same coordinate plane. See Lesson 2-3.

30. $y = -x; y = x$

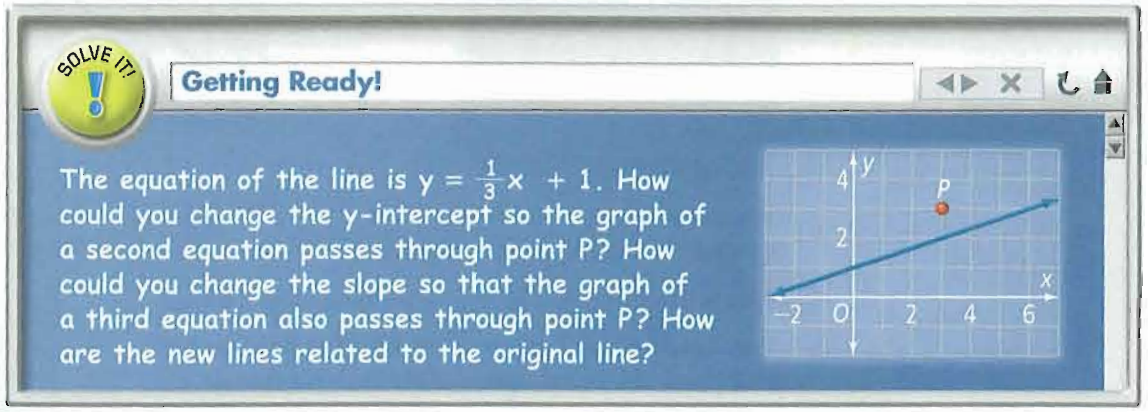
31. $y = x + 1; y = 2x - 1$

32. $y = -\frac{1}{4}x; y = -\frac{1}{4}x + 2$

2-6 Families of Functions

Sunshine State Standard
MA.912.A.2.10 Describe and graph transformations of functions.

Objective To analyze transformations of functions



The equation of the line is $y = \frac{1}{3}x + 1$. How could you change the y-intercept so the graph of a second equation passes through point P? How could you change the slope so that the graph of a third equation also passes through point P? How are the new lines related to the original line?

Dynamic Activity
Translating Functions

Lesson Vocabulary
- parent function
- transformation
- translation
- reflection
- vertical stretch
- vertical compression

Different nonvertical lines have different slopes, or y-intercepts, or both. They are graphs of different linear functions. For two such lines, you can think of one as a *transformation* of the other.

Essential Understanding There are sets of functions, called *families*, in which each function is a transformation of a special function called the *parent*.

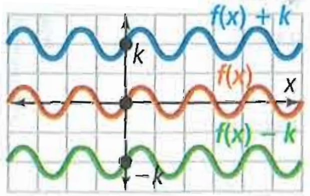
The linear functions form a family of functions. Each linear function is a transformation of the function $y = x$. The function $y = x$ is the *parent* linear function.

A **parent function** is the simplest form in a set of functions that form a family. Each function in the family is a **transformation** of the parent function.

One type of transformation is a **translation**. A translation shifts the graph of the parent function horizontally, vertically, or both without changing shape or orientation. For a positive constant k and a parent function $f(x)$, $f(x) \pm k$ is a vertical translation. For a positive constant h, $f(x \pm h)$ is a horizontal translation.

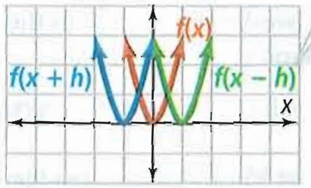
Adding k to the outputs shifts the graph up.

Vertical Translation



Subtracting h from the inputs shifts the graph right.

Horizontal Translation



Problem 1 Vertical Translation

Plan

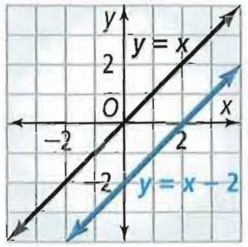
What is one way to compare two functions?
Use a table to compare their values.

A **How are the functions $y = x$ and $y = x - 2$ related? How are their graphs related?**

Make a table of values.

x	$y = x$	$y = x - 2$
−2	−2	−4
−1	−1	−3
0	0	−2
1	1	−1
2	2	0

Draw their graphs.

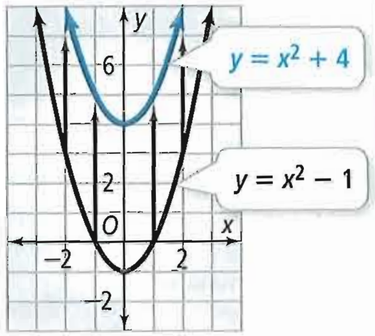


Each output for $y = x - 2$ is two less than the corresponding output for $y = x$.

The graph of $y = x - 2$ is the graph of $y = x$ translated down two units.

B **What is the graph of $y = x^2 - 1$ translated up 5 units?**

Translate the graph of $y = x^2 - 1$ up 5 units to get the blue parabola. The equation of the blue parabola is $y = x^2 + 4$.



Check Every value in the $y = x^2 + 4$ column is 5 greater than the corresponding value in the $y = x^2 - 1$ column.

x	$y = x^2 - 1$	$y = x^2 + 4$
−2	3	8
−1	0	5
0	−1	4
1	0	5
2	3	8

Got It? **1.** **a.** How are the functions $y = 2x$ and $y = 2x - 3$ related? How are their graphs related?

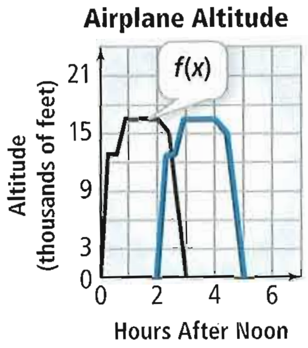
b. What is the graph of $y = 3x$ translated up 2 units?

Problem 2 Horizontal Translation

Think

If the time of the flight is later, why do you subtract rather than add in $f(x - 2)$?
The y-values of the delayed flight correspond to x-values 2 hours *earlier* than the delayed flight.

The graph shows the projected altitude $f(x)$ of an airplane scheduled to depart an airport at noon. If the plane leaves two hours late, what function represents this transformation?



A two-hour delay means the plane leaves at 2 P.M. This shifts the graph to the right 2 units.

The function $f(x - 2)$ represents this transformation.

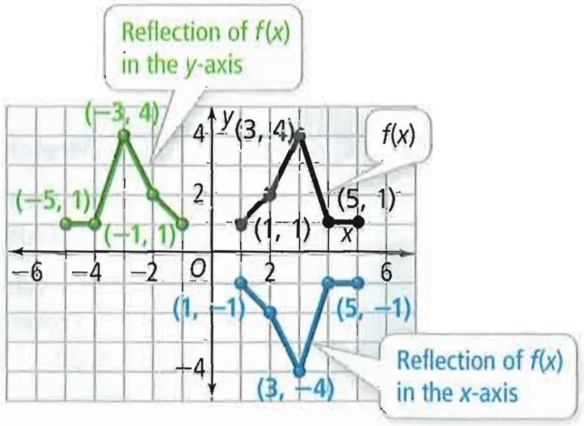
Got It? **2.** Suppose the flight leaves 30 minutes early. What function represents this transformation?

A **reflection** flips the graph of a function across a line, such as the *x*- or *y*-axis. Each point on the graph of the reflected function is the same distance from the line of reflection as its corresponding point on the graph of the original function.

When you reflect a graph in the *y*-axis, the *x*-values change signs and the *y*-values stay the same.

When you reflect a graph in the *x*-axis, the *x*-values stay the same and the *y*-values change signs.

For a function $f(x)$, the reflection in the *y*-axis is $f(-x)$ and the reflection in the *x*-axis is $-f(x)$.



Problem 3 Reflecting a Function Algebraically

Let $g(x)$ be the reflection of $f(x) = 3x + 3$ in the *y*-axis. What is a function rule for $g(x)$?

Think

Write

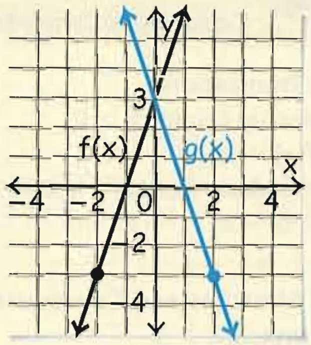
For a reflection in the *y*-axis, change the sign of *x*.

$g(x) = f(-x)$

Evaluate $f(-x)$ and simplify.

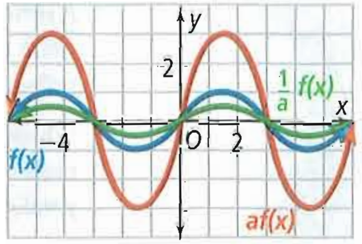
$$g(x) = f(-x)$$
$$= 3(-x) + 3$$
$$g(x) = -3x + 3$$

You can check by graphing $f(x)$ and $g(x)$.



Got It? **3.** Let $h(x)$ be the reflection of $f(x) = 3x + 3$ in the *x*-axis. What is a function rule for $h(x)$?

A **vertical stretch** multiplies all y-values of a function by the same factor greater than 1. A **vertical compression** reduces all y-values of a function by the same factor between 0 and 1. For a function $f(x)$ and a constant a, $y = af(x)$ is a vertical stretch when $a > 1$ and a vertical compression when $0 < \frac{1}{a} < 1$.



Problem 4 Stretching and Compressing a Function

Think

Is this a vertical stretch or compression?
3 is greater than 1, so this is a vertical stretch.

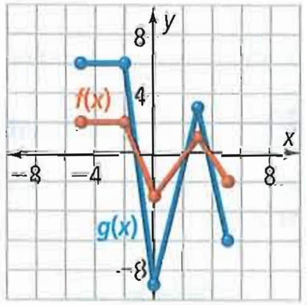
The table at the right represents the function $f(x)$. What are corresponding values of $g(x)$ and possible graphs for the transformation $g(x) = 3f(x)$?

x	$f(x)$
−5	2
−2	2
0	−3
3	1
5	−2

Step 1 Multiply each value of $f(x)$ by 3 to find each corresponding value of $g(x)$.

x	$f(x)$	$3f(x)$	$g(x)$
−5	2	3(2)	6
−2	2	3(2)	6
0	−3	3(−3)	−9
3	1	3(1)	3
5	−2	3(−2)	−6

Step 2 Use the values from the table in Step 1. Draw simple graphs for $f(x)$ and $g(x)$.



Got It? **4. a.** For the function $f(x)$ shown in Problem 4, what are the corresponding table and graph for the transformation $h(x) = \frac{1}{3}f(x)$?

b. Reasoning If several transformations are applied to a graph, will changing the order of transformations change the resulting graph? Explain.

take note

Concept Summary Transformations of $f(x)$

Vertical Translations

Translation up k units, $k > 0$
$y = f(x) + k$
Translation down k units, $k > 0$
$y = f(x) - k$

Horizontal Translations

Translation right h units, $h > 0$
$y = f(x - h)$
Translation left h units, $h > 0$
$y = f(x + h)$

Vertical Stretches and Compressions

Vertical stretch, $a > 1$
$y = af(x)$
Vertical compression, $0 < a < 1$
$y = af(x)$

Reflections

In the x-axis
$y = -f(x)$
In the y-axis
$y = f(-x)$

Problem 5 Combining Transformations

A **The graph of $g(x)$ is the graph of $f(x) = 4x$ compressed vertically by the factor $\frac{1}{2}$ and then reflected in the y-axis. What is a function rule for $g(x)$?**

$\frac{1}{2}(4x) = 2x$ Compress $f(x)$.

$2(-x) = -2x$ Reflect the new function in the y-axis.

The function rule is $g(x) = -2x$.

Think

How can you write $g(x)$ in terms of $f(x)$?
Factor out 3 from the $6x^2$ term.

B **What transformations change the graph of $f(x)$ to the graph of $g(x)$?**

$f(x) = 2x^2 \quad g(x) = 6x^2 - 1$

$$\begin{aligned} g(x) &= 6x^2 - 1 \\ &= 3(2x^2) - 1 \\ &= 3(f(x)) - 1 \end{aligned}$$

The graph of $g(x)$ is the graph of $f(x)$ stretched vertically by a factor of 3 and then translated down 1 unit.

Got It? **5. a.** The graph of $g(x)$ is the graph of $f(x) = x$ stretched vertically by a factor of 2 and then translated down 3 units. What is the function rule for $g(x)$?

b. What transformations change the graph of $f(x) = x^2$ to the graph of $g(x) = (x + 4)^2 - 2$?

Lesson Check

Do you know HOW?

Describe the transformation of the parent function $f(x)$.

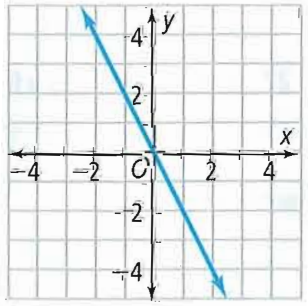
1. $g(x) = f(x) + 6$

2. $h(x) = 0.25f(x)$

3. $j(x) = f(x - 4)$

4. $k(x) = f(-x)$

The graph of $f(x) = -2x$ is shown. Describe and graph each transformation.

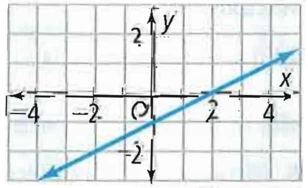


5. $g(x) = f(x + 1) - 2$

6. $h(x) = 2f(x) + 1$

Do you UNDERSTAND?

7. **Compare and Contrast** The graph shows $f(x) = 0.5x - 1$.



Graph $g(x)$ by translating $f(x)$ up 2 units and then stretching it vertically by the factor 2. Graph $h(x)$ by stretching $f(x)$ vertically by the factor 2 and then translating it up 2 units. Compare the graphs of $g(x)$ and $h(x)$.

8. **Reasoning** Can you give an example of a function for which a horizontal translation gives the same resulting graph as a vertical translation? Explain.

9. Find a new function $g(x)$ transformed from $f(x) = -x - 2$ such that $g(x)$ is perpendicular to $f(x)$.

Practice and Problem-Solving Exercises

A Practice

How is each function related to $y = x$? Graph the function by translating the parent function. See Problem 1.

10. $y = x - 3$ **11.** $y = x + 4.5$ **12.** $y = x + 1.5$

Make a table of values for $f(x)$ after the given translation.

13. 3 units up

x	$f(x)$
−2	3
0	1
1	−2
3	−1

14. 1 unit down

x	$f(x)$
−1	1
0	0
2	−4
3	2

15. 4 units up

x	$f(x)$
−3	1
−1	−2
1	0
4	3

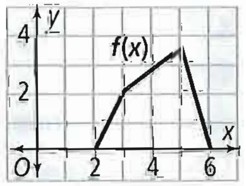
Write an equation for each vertical translation of $y = f(x)$.

16. $\frac{2}{3}$ unit down **17.** 4 units up **18.** 2 units up

For each function, identify the horizontal translation of the parent function, $f(x) = x^2$. Then graph the function. See Problems 2 and 3.

19. $y = (x - 4)^2$ **20.** $y = (x + 1)^2$ **21.** $y = (x + 3)^2$

22. The graph of the function $f(x)$ is shown at the right.
  a. Make a table of values for $f(x)$ and $f(x + 3)$.
  b. Graph $f(x)$ and $f(x + 3)$ on the same coordinate grid.



Write the function rule for each function reflected in the given axis.

23. $f(x) = x + 1$; x-axis **24.** $f(x) = 3x$; y-axis **25.** $f(x) = 2x - 4$; x-axis

Write an equation for each transformation of $y = x$. See Problem 4.

26. vertical stretch by a factor of 4 **27.** vertical stretch by a factor of 2

28. vertical compression by a factor of $\frac{1}{2}$ **29.** vertical compression by a factor of $\frac{1}{4}$

Write the function rule $g(x)$ after the given transformations of the graph of $f(x) = 4x$. See Problem 5.

30. translation up 5 units; reflection in the x-axis

31. reflection in the y-axis; vertical compression by a factor of $\frac{1}{8}$

Describe the transformations of $f(x)$ that produce $g(x)$.

32. $f(x) = \frac{x}{2}$; $g(x) = -2x + 4$ **33.** $f(x) = 3x$; $g(x) = \left(\frac{3x}{4} - 2\right)$

B Apply

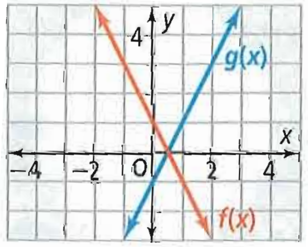
34. Think About a Plan Suppose you are playing with a yo-yo during a school talent show. The string is 3 ft long and you hold your hand 4 ft above the stage. The stage is 3.5 ft above the floor of the auditorium. Make a graph of the yo-yo's distance from the auditorium floor with respect to time during the show.

- How could you graph the position of the yo-yo with respect to the stage, if you let time $t = 0$ when you start your routine?
- How could you transform this graph to show the position with respect to the auditorium floor?

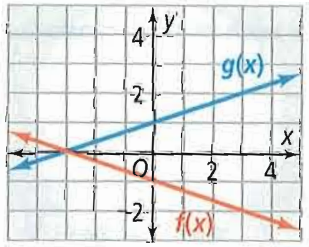
35. If someone started to take a video of your yo-yo routine when you were introduced, 10 seconds before you actually started, what transformation would you have to make to your graph to match their video?

Write the equations for $f(x)$ and $g(x)$. Then identify the reflection that transforms the graph of $f(x)$ to the graph of $g(x)$.

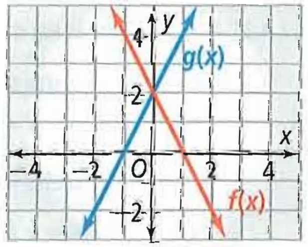
36.



37.



38.



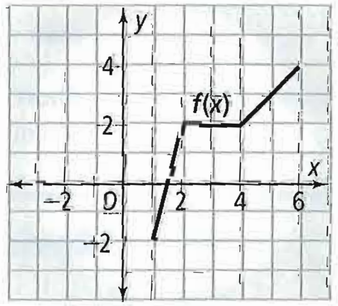
39. Open-Ended Draw a figure in Quadrant I. Use a translation to move your figure into Quadrant III. Describe your translation.

40. Writing The graph of $f(x)$ is shown at the right. Suppose each transformation of $f(x)$ results in the given functions.

i. vertical translation; $g(x)$
ii. reflection in the x-axis; $h(x)$
iii. vertical stretch; $k(x)$,
iv. horizontal translation; $m(x)$

a. Describe how the domain and range of the four new functions compare with the domain and range of $f(x)$.

b. Reasoning Do you think these effects on the domain and range of the original function hold true for all functions? Explain.



Graph each pair of functions on the same coordinate plane. Describe a transformation that changes $f(x)$ to $g(x)$.

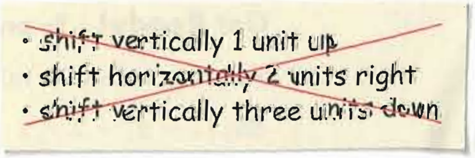
41. $f(x) = x + 1; g(x) = x - 5$

42. $f(x) = -x + 3; g(x) = x - 4$

43. $f(x) = x - 3; g(x) = x + 1$

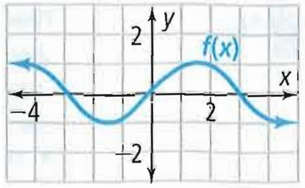
44. $f(x) = -x - 1; g(x) = -x + 2$

45. Error Analysis Your friend wrote the transformations shown to describe how to change the graph of $f(x) = x^2$ to the graph of $g(x) = 2(x + 1)^2 - 3$. Explain the error and give the correct transformations.



- ~~shift vertically 1 unit up~~
- ~~shift horizontally 2 units right~~
- ~~shift vertically three units down~~

Challenge **Using the graph of the function $f(x)$ shown, sketch the graph of each transformed function.**



46. $f(x + 1)$

47. $f(x) - 2$

48. $f(x + 2) + 1$

49. $-2f(x)$

50. Graph all of the following functions in the same viewing window. After you enter each new function, view its graph.

i. $y = x^2$ ii. $y = x^2 + 2$ iii. $y = (x + 2)^2$ iv. $y = (x - 1)^2 - 4$ v. $y = -x^2 - 2$

Based on your results, make a sketch of the graph of $f(x) = -(x + 2)^2 + 1$ and check your prediction on your calculator.

Sunshine State Standards Practice

MA.912.A.2.10

What is an equation for each vertical translation of $y = 2x - 1$?

51. 3 units down

Ⓐ $y = 2x - 7$ Ⓑ $y = 2x + 2$ Ⓒ $y = 2x + 5$ Ⓓ $y = 2x - 4$

52. $\frac{3}{5}$ units up

Ⓕ $y = 2x - \frac{2}{5}$ Ⓖ $y = 2x - \frac{11}{5}$ Ⓗ $y = 2x - \frac{8}{5}$ Ⓘ $y = 2x + \frac{1}{5}$

MA.912.A.3.9

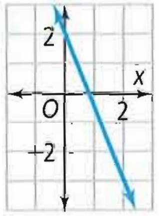
53. What is the slope of the line in the graph at the right?

Ⓐ $-\frac{5}{2}$

Ⓒ $\frac{2}{5}$

Ⓑ $-\frac{2}{5}$

Ⓓ $\frac{5}{2}$



MA.912.A.2.12

54. Short Response The weight of a gold bar varies directly with its volume. If a 40 cm^3 bar weighs 772 grams, how much will a 100 cm^3 bar weigh?

Mixed Review

55. A musician's manager keeps track of the ticket prices and the attendance at recent performances. Use a graphing calculator to determine the equation of the line of best fit for the given data.

See Lesson 2-5.

Ticket Prices($)	41.00	41.50	42.00	43.00	43.50	44.00	44.50	45.00	45.00	47.00
Number Sold	256	276	250	241	210	235	195	194	205	180

Get Ready! **To prepare for Lesson 2-7, do Exercises 56–58.**

Solve each absolute value equation.

See Lesson 1-6.

56. $|x - 3| + 2 = 7$

57. $|2x + 1| - 14 = 9$

58. $\frac{1}{3}|5x - 3| = 6$

2-7

Absolute Value Functions and Graphs

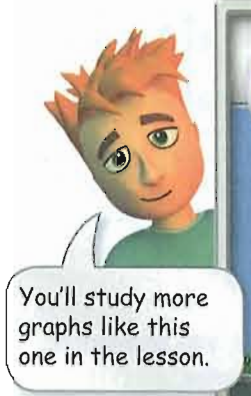
Sunshine State Standards

MA.912.A.2.5 Graph absolute value equations in two variables.

MA.912.A.2.6 Identify and graph absolute value functions.

MA.912.A.2.10 Describe and graph transformations of functions.

Objective To graph absolute value functions



Getting Ready!

You jog at a constant speed. Your jogging route takes you across the county line. Suppose you graph your distance from the county line with respect to time. What would the graph look like? Explain.

Dynamic Activity
Absolute Value with Linear Equations

Lesson Vocabulary
- absolute value function
- axis of symmetry
- vertex

There is a family of functions related to the one you represented in the Solve It.

Essential Understanding Just as the absolute value of x is its distance from 0, the absolute value of $f(x)$, or $|f(x)|$, gives the distance from the line $y = 0$ for each value of $f(x)$.

The simplest example of an **absolute value function** is $f(x) = |x|$. The graph of the absolute value of a linear function in two variables is V-shaped and symmetric about a vertical line called the **axis of symmetry**. Such a graph has either a single maximum point or a single minimum point, called the **vertex**.

take note

Key Concept Absolute Value Parent Function $f(x) = |x|$

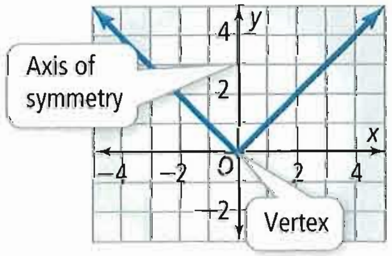
Table

x	$y = \|x\|$
−3	3
−2	2
−1	1
0	0
1	1
2	2
3	3

Function

$$f(x) = \begin{cases} |x| = x, \text{ when } x \geq 0 \\ |x| = -x, \text{ when } x < 0 \end{cases}$$

Graph



Problem 1 Graphing an Absolute Value Function

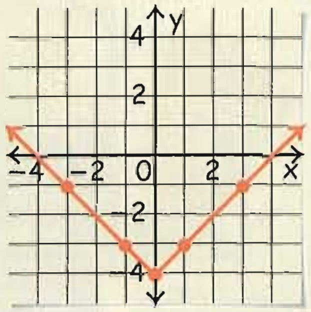
What is the graph of the absolute value function $y = |x| - 4$? How is this graph different from the graph of the parent function $f(x) = |x|$?

Think

Make a table of values and graph the function.

Write

x	y
-3	-1
-1	-3
0	-4
1	-3
3	-1



Use the location of the vertex to see how the graph has been translated. The parent function was not multiplied by a number, so the graph wasn't stretched, compressed, or reflected.

Since the vertex is at (0, −4), you translated the graph of $y = |x|$ down 4 units.

Got It? **1. a.** What is the graph of the function $y = |x| + 2$? How is this graph different from the parent function?

b. Reasoning Do transformations of the form $y = |x| + k$ affect the axis of symmetry? Explain.

The transformations you studied in Lesson 2-6 also apply to absolute value functions.

Key Concept The Family of Absolute Value Functions

Parent Function $y = |x|$

Vertical Translation

Translation up k units, $k > 0$
$y = |x| + k$

Translation down k units, $k > 0$
$y = |x| - k$

Horizontal Translation

Translation right h units, $h > 0$
$y = |x - h|$

Translation left h units, $h > 0$
$y = |x + h|$

Vertical Stretch and Compression

Vertical stretch, $a > 1$
$y = a|x|$

Vertical compression, $0 < a < 1$
$y = a|x|$

Reflection

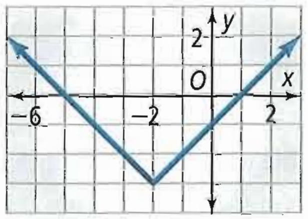
In the x-axis
$y = -|x|$

In the y-axis
$y = |-x|$

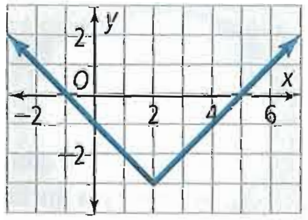
Problem 2 Combining Translations

Multiple Choice Which of the following is the graph of $y = |x + 2| + 3$?

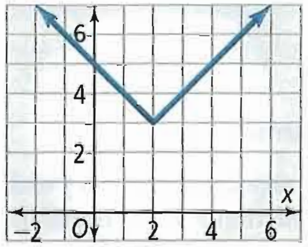
Ⓐ



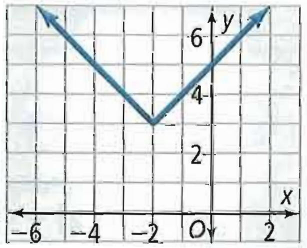
Ⓒ



Ⓑ



Ⓓ



Think

Can you eliminate any answers after this comparison?
Only choices A and D show translations of $y = |x|$ to the left.

Compare $y = |x + 2| + 3$ to each form, $y = |x + h|$ and $y = |x| + k$.

$y = |x + h|$ The parent function, $y = |x|$, is translated left 2 units.

$y = |x| + k$ The parent function, $y = |x|$, is translated up 3 units.

The parent function $y = |x|$ is translated left 2 units and up 3 units. The vertex will be at $(-2, 3)$. The correct choice is D.

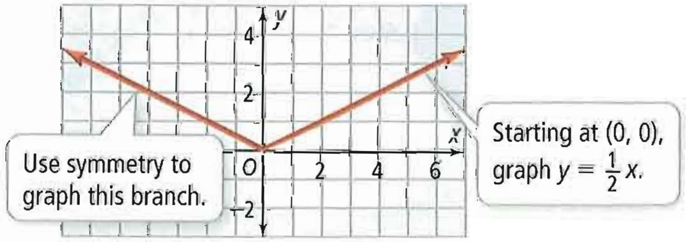
Got It? **2.** What is the graph of the function $y = |x - 2| + 1$?

The right branch of the graph of $y = |x|$ has slope 1. The graph of $y = a|x|, a > 0$, is a stretch or compression of the graph of $y = |x|$. Its right branch has slope a. The graph of $y = -a|x|$ is a reflection of $y = a|x|$ in the x-axis and its right branch has slope $-a$.

Problem 3 Vertical Stretch and Compression

What is the graph of $y = \frac{1}{2}|x|$?

The graph is a vertical compression of the graph of $f(x) = |x|$ by the factor $\frac{1}{2}$. Graph the right branch and use symmetry to graph the left branch.



Use symmetry to graph this branch.

Starting at (0, 0), graph $y = \frac{1}{2}x$.

Got It? **3.** What is the graph of each function?

a. $y = 2|x|$ **b.** $y = -\frac{2}{3}|x|$

You can combine the equations for stretches and compressions with the equations for translations to write a general form for absolute value functions.

take note

Key Concept General Form of the Absolute Value Function

$y = a|x - h| + k$

The stretch or compression factor is $|a|$, the vertex is located at (h, k), and the axis of symmetry is the line $x = h$.

Problem 4 Identifying Transformations

Plan

To what should you compare $y = 3|x - 2| + 4$?
Compare it to the general form, $y = a|x - h| + k$.

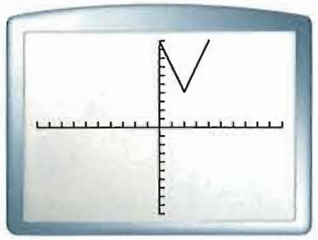
Without graphing, what are the vertex and axis of symmetry of the graph of $y = 3|x - 2| + 4$? How is the parent function $y = |x|$ transformed?

Compare $y = 3|x - 2| + 4$ with the general form $y = a|x - h| + k$.

$a = 3, h = 2$, and $k = 4$.

The vertex is $(2, 4)$ and the axis of symmetry is $x = 2$.

The parent function $y = |x|$ is translated 2 units to the right, vertically stretched by the factor 3, and translated 4 units up.



Check Check by graphing the equation on a graphing calculator.

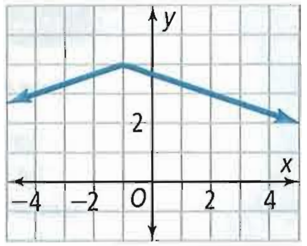
Got It? **4.** What are the vertex and axis of symmetry of $y = -2|x - 1| - 3$? How is $y = |x|$ transformed?

Problem 5 Writing an Absolute Value Function

Think

What does the graph tell you about a?
The upside-down V suggests that $a < 0$.

What is the equation of the absolute value function?



Step 1 Identify the vertex.

The vertex is at $(-1, 4)$, so $h = -1$ and $k = 4$.

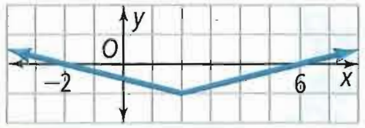
Step 2 Identify a.

The slope of the branch to the right of the vertex is $-\frac{1}{3}$, so $a = -\frac{1}{3}$.

Step 3 Write the equation.

Substitute the values of a, h, and k into the general form $y = a|x - h| + k$. The equation that describes the graph is $y = -\frac{1}{3}|x + 1| + 4$.

Got It? **5.** What is the equation of the absolute value function?



Lesson Check

Do you know HOW?

Find the vertex and the axis of symmetry of the graph of each function.

1. $y = 2|x + 4| - 3$

2. $y = |-x - 3| + 9$

Determine if each function is a vertical stretch or vertical compression of the parent function $y = |x|$.

3. $y = -\frac{7}{2}|x|$

4. $y = \frac{3}{2}|x|$

Do you UNDERSTAND?

5. Is it true that without making a graph of an absolute value function, you can describe its position in the coordinate plane? Explain with an example.

6. Write two absolute value functions such that they have a common vertex in Quadrant III and one is the reflection of the other in a horizontal line.

7. **Compare and Contrast** How is the graph of $y = x$ different from the graph of $y = |x|$?

Practice and Problem-Solving Exercises

A Practice

Make a table of values for each equation. Then graph the equation. **See Problems 1 and 2.**

8. $y = |x| + 1$

9. $y = |x| - 1$

10. $y = |x| - 3$

11. $y = |x + 2|$

12. $y = |x + 4|$

13. $y = |x + 5|$

14. $y = |x - 1| + 3$

15. $y = |x + 6| - 1$

16. $y = |x - 5| + 4$

Graph each equation. Then describe the transformation from the parent function $f(x) = |x|$. **See Problem 3.**

17. $y = 3|x|$

18. $y = -\frac{1}{2}|x|$

19. $y = -2|x|$

20. $y = \frac{1}{3}|x|$

21. $y = \frac{3}{2}|x|$

22. $y = -\frac{3}{4}|x|$

Without graphing, identify the vertex, axis of symmetry, and transformations from the parent function $f(x) = |x|$. **See Problem 4.**

23. $y = |x + 2| - 4$

24. $y = \frac{3}{2}|x - 6|$

25. $y = 3|x + 6|$

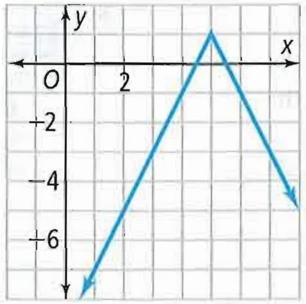
26. $y = 4 - |x + 2|$

27. $y = -|x - 5|$

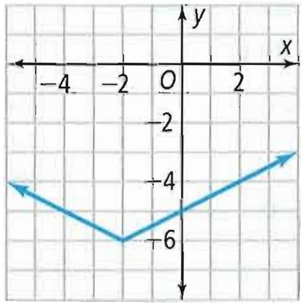
28. $y = |x - 2| - 6$

Write an absolute value equation for each graph. **See Problem 5.**

29.



30.



B Apply

31. **Think About a Plan** Graph $y = -2|x + 3| + 4$. List the x- and y-intercepts, if any.

- What is the vertex?
- What does y equal at the x-intercept(s)? What does x equal at the y-intercept(s)?

32. Graph $y = 4|x - 3| + 1$. List the vertex and the x- and y-intercepts, if any.

33. **Error Analysis** A classmate says that the graphs of $y = -3|x|$ and $y = |-3x|$ are identical. Graph each function and explain why your classmate is not correct.

34. Graph each pair of equations on the same coordinate grid.

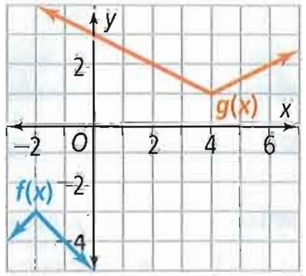
a. $y = 2|x + 1|; y = |2x + 1|$ **b.** $y = 5|x - 2|; y = |5x - 2|$

c. **Reasoning** Explain why each pair of graphs in parts (a) and (b) are different.

35. The graphs of the absolute value functions $f(x)$ and $g(x)$ are given.

a. Describe a series of transformations that you can use to transform $f(x)$ into $g(x)$.

b. **Reasoning** If you change the order of the transformations you found in part(a), could you still transform $f(x)$ into $g(x)$? Explain.



Graph each absolute value equation.

36. $y = \left|-\frac{1}{4}x - 1\right|$ **37.** $y = \left|\frac{5}{2}x - 2\right|$ **38.** $y = \left|\frac{3}{2}x + 2\right|$

39. $y = |3x - 6| + 1$ **40.** $y = -|x - 3|$ **41.** $y = |2x + 6|$

42. $y = 2|x + 2| - 3$ **43.** $y = 6 - |3x|$ **44.** $y = 6 - |3x + 1|$

45. **a.** Graph the equations $y = \left|\frac{1}{2}x - 6\right| + 3$ and $y = -\left|\frac{1}{2}x + 6\right| - 3$ on the same set of axes.

b. **Writing** Describe the similarities and differences in the graphs.

C Challenge

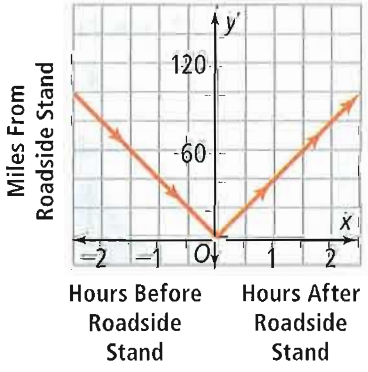
Graph each absolute value equation.

46. $y = |3x| - \frac{x}{3}$ **47.** $y = \frac{1}{2}|x| + 4|x - 1|$ **48.** $y = |2x| + |x - 4|$

49. The graph at the right models the distance between a roadside stand and a car traveling at a constant speed. The x-axis represents time and the y-axis represents distance. Which equation best represents the relation shown in the graph?

Ⓐ $y = |60x|$ Ⓒ $y = |x| + 60$

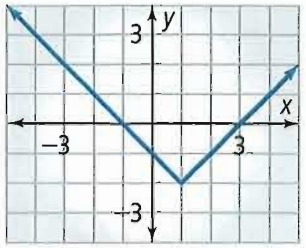
Ⓑ $y = |40x|$ Ⓓ $y = |x| + 40$



50. **a.** **Open-Ended** Find two absolute value equations with graphs that share a vertex.

b. Find two absolute value equations with graphs that share part of a ray.

Sunshine State Standards Practice



MA.912.A.2.5 **51.** The graph shows which equation?

Ⓐ $y = |3x - 1| + 2$ Ⓒ $y = |x - 1| - 2$

Ⓑ $y = |x - 1| + 2$ Ⓓ $y = |3x - 3| - 2$

MA.912.A.2.10 **52.** How are the graphs of $y = 2x$ and $y = 2x + 2$ related?

Ⓕ The graph of $y = 2x + 2$ is the graph of $y = 2x$ translated down two units.

Ⓖ The graph of $y = 2x + 2$ is the graph of $y = 2x$ translated up two units.

Ⓗ The graph of $y = 2x + 2$ is the graph of $y = 2x$ translated to the left two units.

Ⓘ The graph of $y = 2x + 2$ is the graph of $y = 2x$ translated to the right two units.

MA.912.A.3.10 **53.** What is the equation of a line parallel to $y = x$ that passes through the point (0, 1)?

Ⓐ $y = x + 1$ Ⓒ $y = x - 1$

Ⓑ $y = 2x + 2$ Ⓓ $y = -x$

MA.912.A.2.6 **54. Short Response** Is $|y| = x$ a function? Explain.

Mixed Review

Write an equation for each transformation of the graph of $y = x + 2$. **See Lesson 2-6.**

55. 2 units up, 3 units right

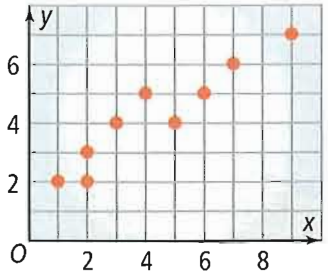
56. vertical compression by a factor of $\frac{1}{2}$, reflection in the y-axis

Write the function rule for each function reflected in the given axis.

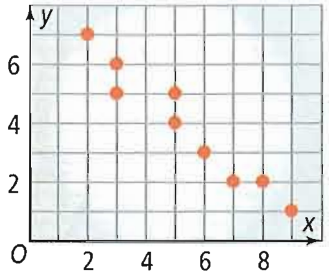
57. $f(x) = x - 7$; y-axis **58.** $f(x) = 2x - 6$; y-axis **59.** $f(x) = 4 + x$; x-axis

Find a trend line for each scatter plot. Write the equation for each trend line. **See Lesson 2-5.**

60.



61.



Get Ready! **To prepare for Lesson 2-8, do Exercises 62–64.**

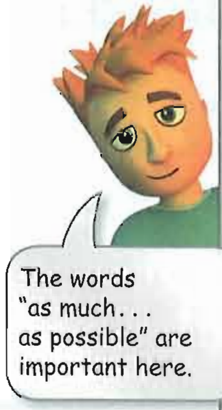
Solve each inequality. Graph the solution on a number line. **See Lesson 1-5.**

62. $12p \le 15$ **63.** $4 + t > 17$ **64.** $5 - 2t \ge 11$

2-8 Two-Variable Inequalities

Sunshine State Standard
MA.912.A.2.5 Graph absolute value inequalities in two variables.

Objective To graph two-variable inequalities



You have a gift card for a store that sells pre-owned CDs and paperback books. You want to spend as much of the gift card as possible. How many of each item can you buy? Explain.

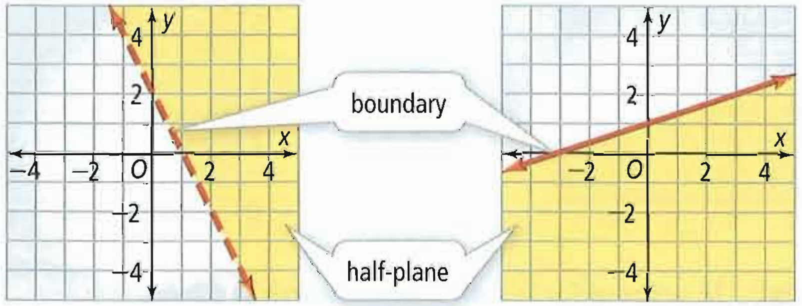
Dynamic Activity
Linear Inequalities

In some situations you need to compare quantities. You can use inequalities for situations that involve these relationships: *less than, less than or equal to, greater than,* and *greater than or equal to.*

Essential Understanding Graphing an inequality in two variables is similar to graphing a line. The graph of a linear inequality contains all points on one side of the line and may or may not include the points on the line.

Lesson Vocabulary
- linear inequality
- boundary
- half-plane
- test point

A **linear inequality** is an inequality in two variables whose graph is a region of the coordinate plane bounded by a line. This line is the **boundary** of the graph. The boundary separates the coordinate plane into two **half-planes**, one of which consists of solutions of the inequality.



To determine which half-plane to shade, pick a **test point** that is *not* on the boundary. Check whether that point satisfies the inequality. If it does, shade the half-plane that includes the test point. If not, shade the other half-plane. The origin, (0, 0), is usually an easy test point as long as it is not on the boundary.

Problem 1 Graphing Linear Inequalities

What is the graph of each inequality?

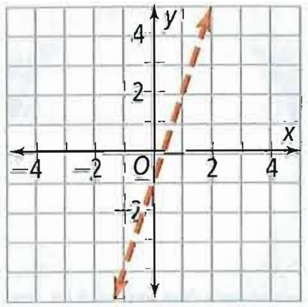
A $y > 3x - 1$

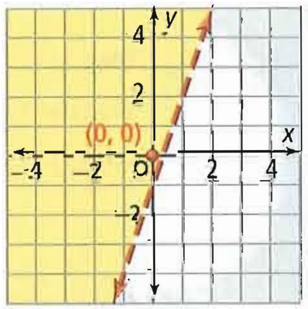
Step 1
Graph the boundary line $y = 3x - 1$. Use a dashed boundary line because the inequality is *greater than*, and the points on the line do not satisfy the inequality.

Step 2
Choose a test point, (0, 0). Substitute $x = 0$ and $y = 0$ into $y > 3x - 1$.

$$0 > 3(0) - 1$$
$$0 > -1$$

Since $0 > -1$ is true, shade the half plane that includes (0, 0).





B $y \leq 3x - 1$

The boundary line is again $y = 3x - 1$, but it is solid because the inequality is less than or *equal to*.

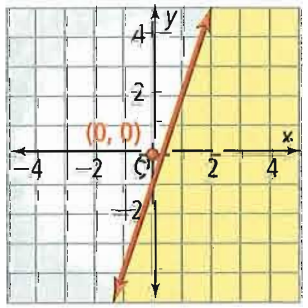
Shade the region opposite the region shaded above (for >) because the inequality is *less than* or equal to.

You can also check the point (0, 0).

$0 \leq 3(0) - 1$

$0 \leq -1$

Since $0 \leq -1$ is false, (0, 0) is not part of the solution.



Plan

Can you use the graph of $y > 3x - 1$ to help graph $y \leq 3x - 1$?
If you shaded above the line for $y > 3x - 1$, then shade below the line for $y \leq 3x - 1$.

Got It? **1.** What is the graph of each inequality?

a. $y \geq -2x + 1$ **b.** $y < -2x + 1$

You can also inspect inequalities solved for *y*, such as $y > mx + b$ to determine which half-plane describes the solution. Since *y* describes vertical position, the solution of $y > mx + b$ will be *above* the boundary line. The solution of $y < mx + b$ will be *below* the boundary line.

Problem 2 Using a Linear Inequality

Entertainment The map shows the number of tickets needed for small or large rides at the fair. You do not want to spend more than $15 on tickets. How many small or large rides can you ride?

Think

What are the unknowns?
The unknowns are the number of small rides and the number of large rides you can get on.

You can buy 60 tickets with $15.

Relate	the number of tickets for small rides	plus	the number of tickets for large rides	is less than or equal to	60
Define	Let x = the number of small rides.				
	Let y = the number of large rides.				
Write	$3x$	$+$	$5y$	$\leq$	60

Step 1

Find the intercepts of the boundary line. Use the intercepts to graph the boundary line.

When $y = 0,\ 3x + 5(0) = 60.$

$3x = 60$

$x = 20$

When $x = 0,\ 3(0) + 5y = 60.$

$5y = 60$

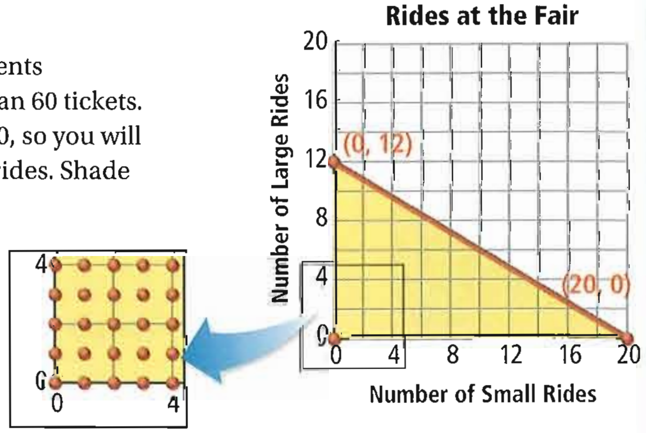
$y = 12$

Graph the line that connects the intercepts (20, 0) and (0, 12). Since the inequality is $\leq$, use a solid boundary line.

Step 2

The region above the boundary line represents combinations of rides that require more than 60 tickets. You purchased a *finite* number of tickets, 60, so you will not be able to go on an infinite number of rides. Shade the region below the boundary line.

The number of small rides x and the number of large rides y are whole numbers. In math, such a situation is called *discrete*. All points with whole number coordinates in the shaded region represent possible combinations of small and large rides.



Got It? **2. a.** Suppose that you decide to spend no more than $30 for tickets. What are the possible combinations of small and large rides that you can ride now? Use a graph to find your answer.

b. **Reasoning** Why did the graph of the solution in Problem 2 only include Quadrant I?

You can graph two-variable absolute value inequalities in the same way that you graph linear inequalities.

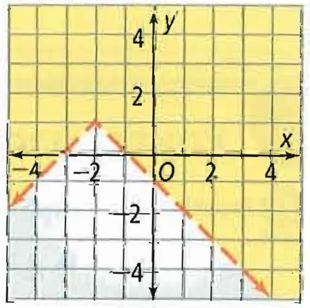
Problem 3 Graphing an Absolute Value Inequality

What is the graph of $1 - y < |x + 2|$?

Know
Absolute value inequality

Need
Boundary

Plan
- Solve the inequality for y.
- Graph the related equation.
- Shade the solution.



$$1 - y < |x + 2|$$

$$-y < |x + 2| - 1 \quad \text{Subtract 1 from each side.}$$

$$y > -|x + 2| + 1 \quad \text{Multiply both sides by } -1.$$

The graph of $y = -|x + 2| + 1$ is the graph of $y = |x|$, reflected in the x-axis and translated left 2 units and up 1 unit.

Since the inequality is solved for y and $y > -|x + 2| + 1$, shade the region above the boundary.

Got It? **3.** What is the graph of $y - 4 \geq 2|x - 1|$?

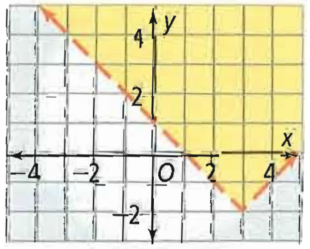
You can use the transformations discussed in previous lessons to help draw the boundary graphs more quickly. You can also use them to write an inequality based on a graph.

Problem 4 Writing an Inequality Based on a Graph

Plan

How can you tell that the graph is not a stretch or compression of the graph of $y = |x|$?
The slopes of the branches are 1 and −1.

What inequality does this graph represent?

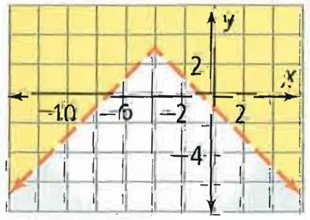


The boundary is the graph of the absolute value function $y = |x|$, translated. The vertex of $y = |x|$ is translated to $(3, -2)$, so the boundary is the graph of $y = |x - 3| - 2$.

The solution is shaded above the boundary, so the inequality is either $>$ or $\geq$. Since the boundary is a dashed line, the correct inequality is $y > |x - 3| - 2$.

Got It? **4. a.** What inequality does this graph represent?

b. Reasoning You can tell from looking at the inequality $y > 5x - 3$ to shade above the boundary line to represent the solution. Can you use the same technique to show the solution of an inequality like $2x - y > 1$? Explain.



Lesson Check

Do you know HOW?

What is the graph of each inequality?

1. $9y \leq 12x$

2. $7x + y \geq 8$

What is the graph of each absolute value inequality?

3. $y \leq |x + 1|$

4. $y \geq |2x - 3|$

Do you UNDERSTAND?

5. Do the points on the boundary line of the graph of an inequality help determine the shaded area of the graph? Explain.

6. **Compare and Contrast** How is graphing a linear inequality in two variables different from graphing a linear equation in two variables?

7. **Reasoning** Is the ordered pair $\left(\frac{3}{4}, 0\right)$ a solution of $3x + y > 3$? Explain.

Practice and Problem-Solving Exercises

A Practice

Graph each inequality. **See Problem 1.**

8. $y > 2x + 1$

9. $y < 3$

10. $x \leq 0$

11. $y \leq x - 5$

12. $2x + 3y \geq 12$

13. $2y \geq 4x - 6$

14. $3x - 2y \leq 9$

15. $-y < 2x + 2$

16. $5 - y \geq x$

17. **Cooking** The time needed to roast a chicken depends on its weight. Allow at least 20 min/lb for a chicken weighing as much as 6 lb. Allow at least 15 min/lb for a chicken weighing more than 6 lb. **See Problem 2.**

a. Write two inequalities to represent the time needed to roast a chicken.

b. Graph the inequalities.

Graph each absolute value inequality. **See Problem 3.**

18. $y \geq |2x - 1|$

19. $y \leq |3x| + 1$

20. $y \leq |4 - x|$

21. $y > |-x + 4| + 1$

22. $y - 7 > |x + 2|$

23. $y + 2 \leq \left|\frac{1}{2}x\right|$

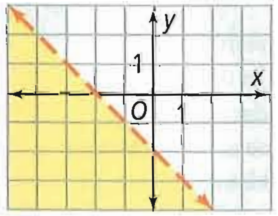
24. $3 - y \geq -|x - 4|$

25. $1 - y < |2x - 3|$

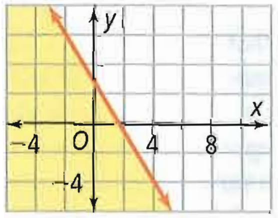
26. $y + 3 \leq |3x| - 1$

Write an inequality for each graph. The equation for the boundary line is given. **See Problem 4.**

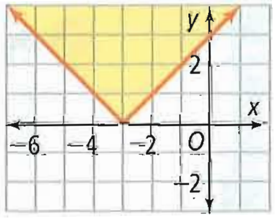
27. $y = -x - 2$



28. $5x + 3y = 9$



29. $2y = |2x + 6|$



B Apply

Graph each inequality on a coordinate plane.

30. $5x - 2y \geq -10$

31. $2x - 5y < -10$

32. $\frac{3}{4}x + \frac{2}{3}y > \frac{5}{2}$

33. $3(x - 2) + 2y \leq 6$

34. $|x - 1| > y + 7$

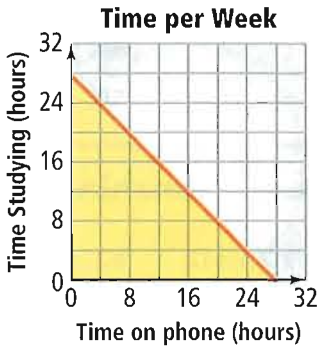
35. $y - |2x| \leq 21$

36. $\frac{2}{3}x + 2 \leq \frac{2}{9}y$

37. $0.25y - 1.5x \geq -4$

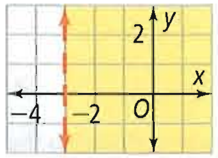
38. Think About a Plan The graph at the right relates the number of hours you spend on the phone to the number of hours you spend studying per week. Describe the domain for this situation. Write an inequality for the graph.

- What is the least amount of time you can spend on the phone per week? What is the most?
- What is the least amount of time you can spend studying per week? What is the most?
- What is the greatest amount of time you can spend either on the phone or studying per week?



Write an inequality for each graph.

39.



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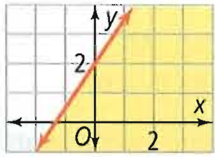

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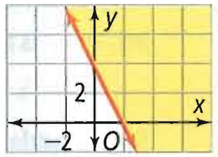

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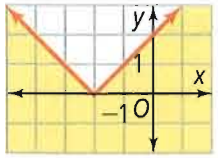

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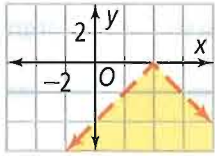

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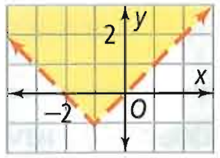
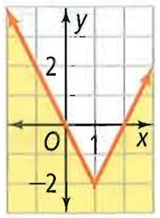


45. Which graph best represents the solution of the inequality $y \geq 2|x - 1| - 2$?

Ⓐ



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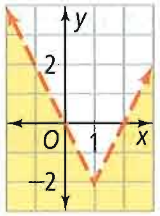

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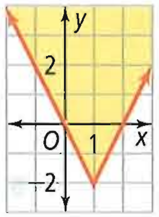

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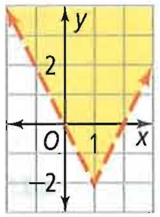


46. The graph at the right relates the amount of gas in the tank of your car to the distance you can drive.

a. Describe the domain for this situation.

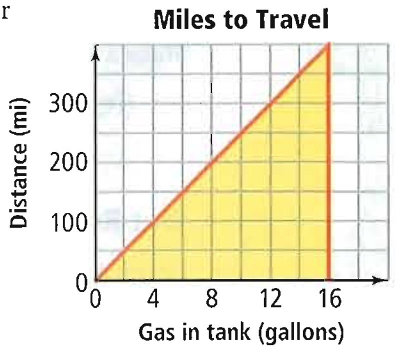
b. Why does the graph stop?

c. Why is only the first quadrant shown?

d. Reasoning Would every point in the solution region be a solution?

e. Write an inequality for the graph.

f. What does the coefficient of x represent?



Challenge

47. **Writing** When you graph an inequality, you can often use the point (0, 0) to test which side of the boundary line to shade. Describe a situation in which you could not use (0, 0) as a test point.

Graphing Calculator **Graph each inequality on a graphing calculator. Then sketch the graph.**

48. $y \leq |x + 1| - |x - 1|$ **49.** $y > |x| + |x + 3|$

50. $y < |x - 3| - |x + 3|$ **51.** $y < 7 - |x - 4| + |x|$

Sunshine State Standards Practice

MA.912.A.2.12 **52.** Suppose y varies directly with x. If x is 30 when y is 10, what is x when y is 9?

Ⓐ 3 Ⓑ 27 Ⓒ 29 Ⓓ $\frac{300}{9}$

MA.912.A.2.6 **53.** Which equation represents a line with slope −2 and y-intercept 3?

Ⓕ $3y = x - 2$ Ⓖ $3y = -2x + 1$ Ⓗ $y = 2x - 3$ Ⓘ $y = -2x + 3$

MA.912.A.2.10 **54.** What is the vertex of $y = |x| - 5$?

Ⓐ (5, 0) Ⓑ (−5, 0) Ⓒ (0, 5) Ⓓ (0, −5)

MA.912.A.2.12 **55. Extended Response** The amount of a commission is directly proportional to the amount of a sale. A realtor received a commission of $48,000 on the sale of an $800,000 house. How much would the commission be on a $650,000 house?

Mixed Review

Graph each function by translating its parent function. **See Lesson 2-7.**

56. $y = |2x + 5|$ **57.** $y = |x| - 3$ **58.** $f(x) = |x + 6|$

59. $f(x) = |x| - 2$ **60.** $y = |x + 2|$ **61.** $y = |x - 1| + 5$

Determine whether y varies directly with x. If so, find the constant of variation. **See Lesson 2-2.**

62. $y = x + 1$ **63.** $y = 100x$ **64.** $5x + y = 0$ **65.** $y - 2 = 2x$

66. $x = \frac{y}{3}$ **67.** $-4 = y - x$ **68.** $y = -10x$ **69.** $xy = 1$

Make a scatter plot and describe the correlation. **See Lesson 2-5.**

70. {(0, 6), (1, 4), (2, 4), (4, 1), (5, 0)}

71. {(−10, 5), (−5, −5), (−2, 0), (0, 3), (5, −2)}

Get Ready! **To prepare for Lesson 3-1, do Exercises 72–74.**

Graph each equation. Use one coordinate plane for all three graphs. **See Lesson 2-3.**

72. $3x - y = 2$ **73.** $3x - y = -2$ **74.** $x + 3y = -2$

2 Pull It All Together

To solve these problems, you will pull together concepts about equivalence, linear functions and modeling. Show your work and justify your reasoning.

BIG idea Equivalence

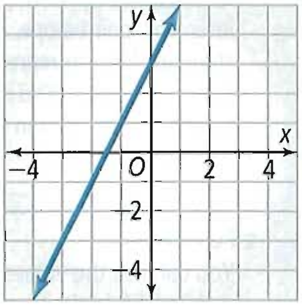
You can represent a function with many different equivalent equations.

Task 1

The graph represents a function.

a. Write the equation for the graph in standard form, point-slope form, and slope-intercept form.

b. Which equation is the easiest to write by looking at the graph? Explain why.



BIG idea Modeling

You can use scatter plots to model, analyze, and make predictions about certain kinds of data.

Task 2

The table at the right shows the boiling point of water at various elevations.

a. Identify the independent and dependent quantities. Explain your choices.

b. Make a scatter plot that models this data.

c. Determine what kind of correlation is shown in your plot.

d. Write an equation of the line of best fit.

e. Use your equation to predict the temperature at which water boils at 8000 feet above sea level.

f. At what elevation would you expect water to boil at 207°F?

Boiling Point of Water

Elevation (ft)	Boiling Point (°F)
sea level	212
1000	210.2
2000	208.4
3000	206.6
4000	204.8
5000	203

Task 3

Consider the following data: (2, 1), (4, 3), (5, 5), (7, 6), (3, 18).

a. Exclude (3, 18) and draw a trend line for the rest of the data.

b. Use a graphing calculator to find the line of best fit for all of the points.

c. Which linear model does a better job predicting most of the values? Explain.

2 Chapter Review

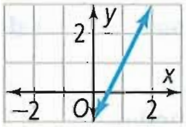
Connecting BIG ideas and Answering the Essential Questions

1 Equivalence
You can use either slope-intercept, point-slope, or standard form to represent linear functions. (You can transform one version to another as needed.)

Slope-Intercept Form (Lesson 2-3)

$y = mx + b$

$y = 2x - 1$



More Linear Equations (Lesson 2-4)

$y - y_1 = m(x - x_1)$ $Ax + By = C$

$y - 5 = 2(x - 3)$ $2x - y = 1$

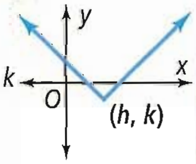
2 Function
You can use the values of a, h, and k in the form $y = a|x - h| + k$ to determine how the parent function $y = |x|$ has been transformed.

Families of Functions (Lesson 2-6)

$f(x) + k$	vertical translation
$f(x - h)$	horizontal translation
$af(x)$	stretch or compression
$-f(x)$	reflection in the x-axis
$f(-x)$	reflection in the y-axis

Absolute Value Functions and Graphs (Lesson 2-7)

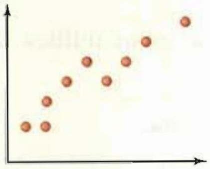
Parent: $y = |x|$
General form:
$y = a|x - h| + k$
vertex: (h, k)



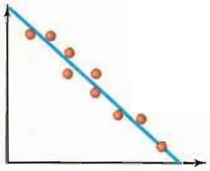
3 Modeling
You can use the equation of a trend line or line of best fit to model data that cluster in a linear pattern.

Using Linear Models (Lesson 2-5)

Positive correlation



Trend Line



Chapter Vocabulary

- absolute value function (p. 107)
- axis of symmetry (p. 107)
- boundary (p. 114)
- constant of variation (p. 68)
- correlation (p. 92)
- correlation coefficient (p. 94)
- dependent variable (p. 63)
- direct variation (p. 68)
- domain (p. 61)
- function (p. 62)
- function notation (p. 63)
- function rule (p. 63)
- half-plane (p. 114)
- independent variable (p. 63)
- line of best fit (p. 94)
- linear equation (p. 75)
- linear function (p. 75)
- linear inequality (p. 114)
- parallel lines (p. 85)
- parent function (p. 99)
- perpendicular lines (p. 85)
- point-slope form (p. 81)
- range (p. 61)
- reflection (p. 101)
- relation (p. 60)
- scatter plot (p. 92)
- slope (p. 74)
- slope-intercept form (p. 76)
- standard form of a linear equation (p. 82)
- test point (p. 115)
- transformation (p. 99)
- translation (p. 99)
- vertex (p. 107)
- vertical compression (p. 102)
- vertical stretch (p. 102)
- vertical-line test (p. 62)
- x-intercept (p. 76)
- y-intercept (p. 76)

Choose the correct term to complete each sentence.

1. The graph of a function is (*always*/*sometimes*) a line.

2. The equation $y - 5 = 3(x + 2)$ is in (*point-slope*/*slope-intercept*) form.

2-1 Relations and Functions

Quick Review

A **relation** is a set of ordered pairs. The **domain** of a relation is the set of x-coordinates. The **range** is the set of y-coordinates. When each element of the domain is paired with exactly one element of the range, the relation is a **function**.

Example

Determine whether the relation is a function. Find the domain and range.

$\{(5, 0), (8, 1), (1, 3), (5, 2), (3, 8)\}$

In this relation, the x-coordinate 5 is paired with both 0 and 2. This relation is not a function.

The domain is the set of x-coordinates, which is $\{5, 8, 1, 3\}$.

The range is the set of y-coordinates, which is $\{0, 1, 3, 2, 8\}$.

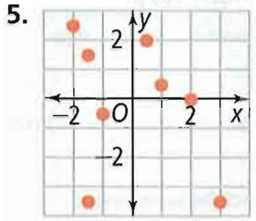
Exercises

Determine whether each relation is a function. Find the domain and range.

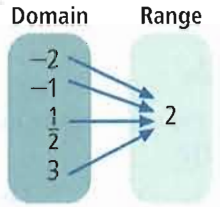
3. $\{(10, 2), (-10, 2), (6, 4), (5, 3), (-6, 7)\}$

4. $\{(4, 5), (1, 5), (3, 8), (4, 6), (10, 12)\}$

5.



6.



For each function, find $f(-2)$, $f(-0.5)$, and $f(3)$.

7. $f(x) = -x + 4$

8. $f(x) = \frac{3}{8}x - 3$

2-2 Direct Variation

Quick Review

A linear equation of the form $y = kx$, $k \neq 0$, represents **direct variation**. The **constant of variation** is k. You can use proportions to solve direct variation problems.

Example

In the table, determine whether y varies directly with x. If so, what is the constant of variation and the function rule?

x	y
2	6
3	9
8	24

$\frac{6}{2} = \frac{9}{3} = \frac{24}{8} = 3$, so y varies directly with x, and the constant of variation is 3.

The function rule is $y = 3x$.

Exercises

For each function, determine whether y varies directly with x. If so, find the constant of variation and write the function rule.

9.

x	y
−2	3
1	4
2	7

10.

x	y
4	5
6	9
10	17

11.

x	y
1	1
2	2
5	5

For each function, y varies directly with x. Find each constant of variation. Then find the value of y when $x = -0.3$.

12. $y = 2$ when $x = -\frac{1}{2}$

13. $y = \frac{2}{3}$ when $x = 0.2$

14. $y = 7$ when $x = 2$

15. $y = 4$ when $x = -3$

2-3 Linear Functions and Slope-Intercept Form

Quick Review

The graph of a **linear function** is a line. You can represent a linear function with a **linear equation**. Given two points on a line, the **slope** of the line is the ratio of the change in the y-coordinates to the change in the corresponding x-coordinates. The slope is the coefficient of x when you write a linear equation in **slope-intercept form**.

Example

What is the slope of the line that passes through (3, 5) and (−1, −2)?

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Find the difference between the coordinates.

$= \frac{5 - (-2)}{3 - (-1)} = \frac{7}{4}$ Simplify.

Exercises

Identify the slope of the line that passes through the given points.

16. (1, 3) and (6, 1)

17. (4, 4) and (−2, −3)

18. (3, 2) and (−3, −2)

19. (5, 2) and (−4, 6)

Write an equation for each line in slope-intercept form.

20. slope = −3 and the y-intercept is (0, 4)

21. slope = $\frac{1}{2}$ and the y-intercept is (0, 6)

Rewrite each equation in slope-intercept form. Graph each line.

22. $4x - 2y = 3$

23. $-4x + 6y = 18$

24. $3y + 3x = 15$

25. $3y + x = 5$

2-4 More About Linear Equations

Quick Review

You write the equation of a line in **point-slope form** when you have a point and the slope or when you have two points. The **standard form** of an equation has both variables and no constants on the left side.

When two lines have the same slope, they are **parallel**. When two lines have slopes that are negative reciprocals of each other, they are **perpendicular**.

Example

Write an equation in standard form for the line with a slope of 2, going through (1, 6).

$y - 6 = 2(x - 1)$ Write the equation in point-slope form, substituting the given point and slope.

$y = 2x - 2 + 6$ Simplify.

$-2x + y = 4$ Write in standard form.

Exercises

Write an equation for each line in point-slope form and then convert it to standard form.

26. slope = −3, through (4, 0)

27. slope = 5, through (1, −1)

28. through (0, 0) and (3, −7)

29. through (2, 3) and (3, 5)

30. **a.** Write an equation of the line parallel to $x + 2y = 6$ through (8, 3).
  b. Write an equation of the line perpendicular to $x + 2y = 6$ through (8, 3).
  c. Graph the three lines on the same coordinate plane.

2-5 Using Linear Models

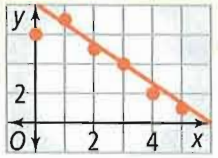
Quick Review

You can use a **scatter plot** to show relationships between data sets. You can make predictions using a trend line, which approximates the relationship between two data sets. The most accurate trend line is a **line of best fit**.

Example

Draw a scatter plot of the data. Is a linear model reasonable? If so, predict the value of y when $x = 9$.

$\{(0, 6), (1, 7), (2, 5), (3, 4), (4, 2), (5, 1)\}$



The points are close to the line $y = -\frac{4}{3}x + 8$, so a linear model is reasonable. When $x = 9$,

$$y = -\frac{4}{3}(9) + 8$$
$$= -4$$

Exercises

Draw a scatter plot of each set of data. Decide whether a linear model is reasonable. If so, describe the correlation. Then draw a trend line and write its equation. Predict the value of y when x is 15.

31. $\{(3, 5), (4, 7), (5, 9), (7, 10), (8, 10), (9, 11), (10, 13)\}$

32. $\{(6, 15.5), (7, 14.0), (8, 13.0), (9, 12.5), (10, 12.0), (11, 11.5), (12, 10.0)\}$

33.

x	0	3	6	9	12
y	17.5	35.4	50.5	60.6	66.3

2-6 Families of Functions

Quick Review

A **parent function** is the simplest form of a function in a family of functions. Each member is a **transformation** of the parent function.

Translations shift the graph horizontally, vertically, or both. A **reflection** flips the graph over a line of symmetry. **Vertical stretches** and **compressions** change the shape of the graph by a factor.

Example

Write the equation of the transformation of the graph of $f(x) = x^2$ translated 3 units up, vertically stretched by a factor of 6, and reflected across the y-axis.

$y = x^2 + 3$ Translated 3 units up.

$y = 6(x^2 + 3)$ Vertically stretched.

$y = 6(-x)^2 + 18$ Reflected across the y-axis.

$y = 6x^2 + 18$

Exercises

Write the equation for the transformation of the graph of $y = f(x)$.

34. translated 2 units left, 7 units down

35. translated 5 units right, reflected across the x-axis

36. translated 3 units up, reflected across the y-axis

Describe the transformation(s) of the parent function $f(x)$.

37. $g(x) = f(x) - 4$

38. $h(x) = 12f(x) + 2$

39. $k(x) = -2f(-x)$

2-7 Absolute Value Functions and Graphs

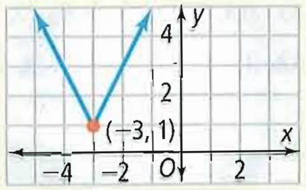
Quick Review

The **absolute value function** $y = |x|$ is the **parent function** for the family of functions of the form $y = a|x - h| + k$. The maximum or minimum point of the graph is the vertex of the graph.

$y = 2|x + 3| + 1$

$a = 2, h = -3, k = 1$

- Vertex is at $(-3, 1)$
- Translated left 3 units
- Stretched by a factor of 2
- Translated up 1 unit



Example

Write an equation for the translation of the graph $y = |x|$ up 5 units.

Because the graph is translated up, k is positive, so the equation of the translated graph is $y = |x| + 5$.

Exercises

Write an equation for each translation of the graph of $y = |x|$.

40. up 4 units, right 2 units

41. vertex $(-3, 0)$

42. vertex $(5, 2)$

43. vertex $(4, 1)$

Graph each function.

44. $f(x) = |x| - 8$

45. $f(x) = 2|x - 5|$

46. $y = -\frac{1}{4}|x - 2| + 3$

47. $y = -2|x + 1| - 1$

Without graphing, identify the vertex and axis of symmetry of each function.

48. $y = 2|x - 4|$

49. $y = -|x| + 2$

2-8 Two Variable Inequalities

Quick Review

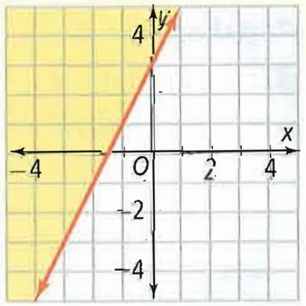
An inequality describes a region of the coordinate plane that has a **boundary**. To graph an inequality involving two variables, first graph the boundary. Then determine which side of the boundary contains the solutions. Points on a dashed boundary are not solutions. Points on a solid boundary are solutions.

Example

Graph the inequality $y \geq 2x + 3$.

Graph the solid boundary line $y = 2x + 3$.

Since y is *greater than* $2x + 3$, shade above the boundary.



Exercises

Graph each inequality.

50. $y \geq -2$

51. $y < 3x + 1$

52. $y < -|x - 5|$

53. $y > |2x + 1|$

54. Transportation An air cargo plane can transport as many as 15 regular shipping containers. One super-size container takes up the space of 3 regular containers.

a. Write an inequality to model the number of regular and super size containers the plane can transport.

b. Describe the domain and range.

c. Graph the inequality you wrote in part (a).

55. Open-Ended Write an absolute value inequality with a solid boundary that only has solutions below the x-axis.

2 Chapter Test

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Do you know HOW?

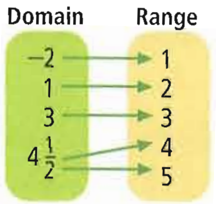
Find the domain and range. Graph each relation.

1. $\{(0, 0), (1, -1), (2, -4), (3, -9), (4, -16)\}$

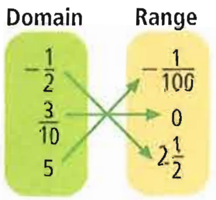
2. $\{(3, 2), (4, 3), (5, 4), (6, 5), (7, 6)\}$

Determine whether each relation is a function.

3.



4.



Suppose $f(x) = 2x - 5$ and $g(x) = |-3x - 1|$. Find each value.

5. $f(3)$ **6.** $f(1) + g(2)$ **7.** $g(0)$

8. $g(2) - f(0)$ **9.** $f(-1) - g(3)$ **10.** $2g(-4)$

Find the slope of each line.

11. through $(3, 5)$, parallel to $y = 5x - 1$

12. through $(-0.5, 0.5)$, perpendicular to $y = -2x - 4$

Write an equation of the line in standard form with the given slope through the given point.

13. slope $= -3$, $(0, 0)$ **14.** slope $= \frac{2}{5}$, $(6, 7)$

15. slope $= 4$, $(-2, -5)$ **16.** slope $= -0.5$, $(0, 6)$

Write an equation of the line in point-slope form through each pair of points.

17. $(0, 0)$ and $(-4, 7)$ **18.** $(-1, -6)$ and $(-2, 10)$

19. $(3, 0)$ and $(-1, -2)$ **20.** $(9, 5)$ and $(8, 2)$

For each direct variation, find the constant of variation. Then find the value of y when $x = -0.5$.

21. $y = 4$ when $x = 0.5$ **22.** $y = 2$ when $x = 3$

Write an equation of the line with the given slope and y-intercept. Use slope-intercept form. Then rewrite each equation in standard form.

23. $m = 3, b = -7$ **24.** $m = -6, b = 9$

25. $m = \frac{1}{4}, b = 11$ **26.** $m = -\frac{1}{2}, b = 4$

Graph each inequality.

27. $y \geq x + 7$ **28.** $y > 2|x + 3| - 3$

29. $4x - 3y < 2$ **30.** $y \leq -\frac{1}{2}|x + 2| - 3$

Do you UNDERSTAND?

31. **Open-Ended** Graph a relation that is *not* a function. Find its domain and range.

32. **Writing** Explain how point-slope form is related to the formula for slope.

Describe each transformation of the parent function $y = |x|$. Then, graph each function.

33. $y = |x| - 4$ **34.** $y = |x - 1| - 5$

35. $y = -|x + 4| + 3$ **36.** $y = 2|x + 1|$

37. **Recreation** The table displays the amounts the Jackson family spent on vacations during the years 2000–2009.

Family Vacations

Year	Cost	Year	Cost
2000	\$1750	2005	\$2750
2001	\$1750	2006	\$3200
2002	\$2000	2007	\$2900
2003	\$2200	2008	\$3100
2004	\$2700	2009	\$3300

a. Make a scatter plot of the data.
b. Draw a trend line. Write its equation.
c. Estimate the amount the Jackson family will spend on vacations in 2015.
d. **Writing** Explain how to use a trend line to make a prediction.

2 Sunshine State Standards Practice

TIPS FOR SUCCESS

Some problems require you to use direct variation to solve for an unknown quantity. Read the question at the right. Then follow the tips to answer the sample question.

A salad dressing recipe calls for $1\frac{1}{3}$ cups of buttermilk, 1 egg, $\frac{1}{2}$ cup of orange juice, and 1 tablespoon of lemon juice. Dan plans to use 2 cups of buttermilk instead. How much orange juice should he use?

- (A) $\frac{3}{8}$ cup
- (B) $\frac{3}{4}$ cup
- (C) $1\frac{1}{6}$ cup
- (D) $1\frac{1}{3}$ cup

TIP 1

Some problems give more information than you need. Decide what information you need to answer the question.

TIP 2

Use some of the information to find k, the constant of variation in $y = kx$.

Think It Through

The ratio that shows how the amount of buttermilk changes is $k = \frac{2}{1\frac{1}{3}}$.

You can simplify this ratio.

$$2 \div \frac{4}{3} = \frac{2}{1} \cdot \frac{3}{4} = \frac{3}{2}$$

Use $k = \frac{3}{2}$ in the direct variation equation $y = \frac{3}{2}x$.
Let x represent the original amount of orange juice.

$$y = \frac{3}{2}x = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

The correct answer is B.

Vocabulary Builder

As you solve test items, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. linear function

B. direct variation

C. range

D. translation

I. the set of all outputs, or y-coordinates, of a relation

II. a transformation that shifts a graph horizontally, vertically, or both

III. a function that can be written in the form $y = mx + b$

IV. a function that can be written in the form $y = kx$, $k \neq 0$

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

1. Which of the following absolute value inequalities has no solutions in Quadrant IV?

- (A) $y + 2 \geq |x - 3|$
- (B) $y > 3 - |5 - x|$
- (C) $y - 1 > |2x + 6|$
- (D) $y \leq |4x| - 7$

2. For which value of b would the equation $3|x - 2| = bx - 6$ have infinitely many solutions?

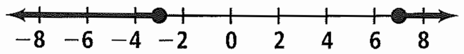
- (F) -6
- (G) 3
- (H) -3
- (I) 6

3. A meteorologist predicts the daily high and low temperatures as 91°F and 69°F. If t represents the temperature, then this situation can be described with the inequality $69 \le t \le 91$. Which of the following absolute value inequalities is an equivalent way of expressing this?

Ⓐ $69 \le |t| \le 91$ Ⓒ $|t - 69| \le 91$

Ⓑ $|t - 80| \le 11$ Ⓓ $|t - 11| \le 80$

4. What is the solution of the inequality $|x - 2| - 3 \le 2$?

Ⓕ 

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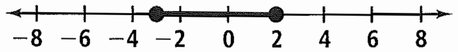
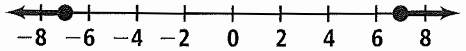
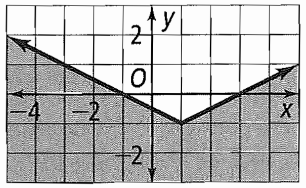
Ⓗ 

Ⓘ 

5. Which inequality best describes the graph?



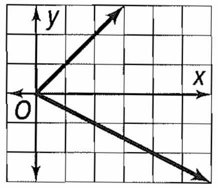
Ⓐ $y \le \frac{1}{2}|x + 1| - 1$

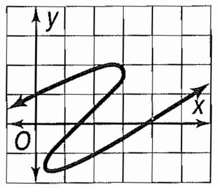
Ⓑ $y \ge \frac{1}{2}|x - 1| - 1$

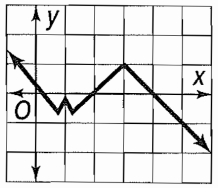
Ⓒ $y \le \frac{1}{2}|x - 1| - 1$

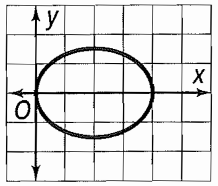
Ⓓ $y \ge \frac{1}{2}|x + 1| - 1$

6. Which relation is a function?

Ⓕ 

Ⓗ 

Ⓖ 

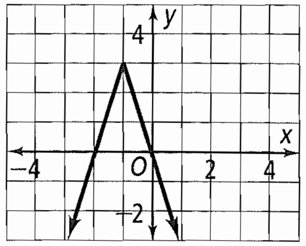
Ⓘ 

7. Which phrase does NOT describe $\sqrt{625}$?

Ⓐ whole number Ⓒ irrational number

Ⓑ integer Ⓓ rational number

8. Which equation is graphed?



Ⓕ $y = -3|x + 1| + 3$

Ⓖ $y = 3|x + 1| + 3$

Ⓗ $y = -3|x - 1| + 3$

Ⓘ $y = 3|x + 1| - 3$

9. Which describes the translation of $y = |x - 3| + 5$?

Ⓐ $y = |x|$ translated 3 units left and 5 units up

Ⓑ $y = |x|$ translated 3 units right and 5 units up

Ⓒ $y = |x|$ translated 5 units left and 3 units up

Ⓓ $y = |x|$ translated 5 units right and 3 units up

10. Which lines are parallel?

I. $y = -2x + 1$ II. $y = x - 4$

III. $y = -x + 5$ IV. $y = 3 - 2x$

Ⓕ I and II Ⓗ I and III

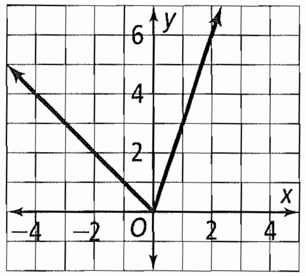
Ⓖ I and IV Ⓘ II and III

11. Which value is in the solution set of $4 < -4x - 2 < 8$ and $3 > 4x + 2 > -10$?

Ⓐ −2 Ⓒ 0

Ⓑ 3 Ⓓ 4

12. Which function has the graph shown?



Ⓕ $f(x) = \begin{cases} -x, & x < 0 \\ 3x, & x \ge 0 \end{cases}$

Ⓖ $f(x) = \begin{cases} x, & x < 0 \\ -3x, & x \ge 0 \end{cases}$

Ⓗ $f(x) = \begin{cases} x, & x > 0 \\ -3x, & x \le 0 \end{cases}$

Ⓘ $f(x) = \begin{cases} -x, & x > 0 \\ 3x, & x \le 0 \end{cases}$

13. Which equation has the same graph as

$$f(x) = \begin{cases} \frac{1}{3}x, & x > 6 \\ -\frac{1}{3}x + 4, & x \le 6 \end{cases}?$$

Ⓐ $f(x) = -\frac{1}{3}|x - 6| + 4$

Ⓑ $f(x) = \frac{1}{3}x$

Ⓒ $f(x) = \frac{1}{3}|x - 6| + 2$

Ⓓ $f(x) = -\frac{1}{3}x + 10$

14. Which line is perpendicular to $y = 2x - 4$?

(F) $y = -2x + 4$
(H) $y = -4 - 2x$
(G) $y = -x - 2$
(I) $y = -\frac{1}{2}x + 4$

15. Which equation is in standard form?

(A) $x - y = 7$
(C) $y = 3x - 1$
(B) $x = 4y + 2$
(D) $10 - 5x = 2y$

16. Which ordered pairs are solutions of $y < 2x + 3$?

I. (0, 2) II. (−1, 1) III. (2, 0)

(F) I only
(H) II only
(G) I and III
(I) II and III

17. Use the Addition Property of Equality to complete the statement: If $p = q$, then $x + p =$ ______.

(A) $p + x$
(C) $x + 0$
(B) $x + q$
(D) $p + q$

18. Which phrase describes $\sqrt{145}$?

(F) whole number
(H) irrational number
(G) integer
(I) rational number

19. If a rate of speed r is constant, then distance, rate, and time are related by the direct variation equation $d = rt$, where d represents distance and t represents time. If $r = 30$ miles per hour, which of the following best describes the graph of $d = rt$?

(A) A straight line through the point (0, 0)
(B) A straight line through the point (0, 30)
(C) A parabola through the point (0, 0)
(D) A parabola through the point (0, 30)

20. The product of an integer and a natural number is always which of the following?

(F) integer
(H) fraction
(G) natural number
(I) irrational number

GRIDDED RESPONSE

21. What is the slope of the line $5y + 3 = \frac{2}{5}x$?

22. A recipe for custard sauce calls for 6 egg yolks, $\frac{2}{3}$ cup of sugar, and $1\frac{1}{2}$ cups of hot milk. Chelsea needs more sauce than the recipe yields. She plans to use 1 cup of sugar. How many cups of hot milk should she use?

23. What is the y-coordinate of the point through which the graph of every direct variation passes?

24. The line $(y - 2) = k(x + 1)$ passes through the points $L(3, 3)$ and $M(7, 4)$. Find k.

25. Matt drove at a steady speed during the first morning of his road trip. The table shows data about his driving.

Time Driving (hours)	Total Distance (miles)
1.5	87
2.25	130.5
3	174

After lunch, Matt drove an additional 232 miles at the same steady speed. How many hours did he drive after lunch?

26. Find $f(-3)$ for $f(x) = \frac{2}{5}x - 2$.

27. What is the greatest integer y-value for which (x, y) is NOT part of the graph of $y < |x - 3|$ for every x?

28. Suppose y varies directly with x, and $y = 2$ when $x = -2$. Find the value of x when $y = 3$.

29. Two pairs of points in the graph of $\{(-3, 2), (-1, 4), (0, 0), (-1, -1)\}$ have the same slope. What is that slope? (Give your answer to the nearest tenth if the slope is not a whole number)?

30. A line through $(-2, 6)$ has slope 2. A line perpendicular to the first line contains $(1, 1)$. What is the x-coordinate of the intersection of the two lines?

31. Line n is the graph of $y = -3x + 4$. Line m contains $(-1, -1)$ and is perpendicular to line n. Suppose you draw lines p and q such that all four lines form a square. Of the slopes of lines p and q, what is the greater slope? Express your answer as a fraction.

CHAPTER 3

Get Ready!

Lesson 1-3

Evaluating Algebraic Expressions

Evaluate each expression for the given values of the variables.

1. $9t + 6(2v - t) - 7v$; $t = 1$ and $v = 5$

2. $11(a + 2b) + 2(a - 2b)$; $a = -3$ and $b = 4$

3. $\frac{3}{5}d + \frac{1}{10}h - \frac{7}{10}d - \frac{4}{5}h$; $d = 5$ and $h = 10$

4. $12\left(\frac{3}{4}x - \frac{1}{2}y\right) - 6\left(\frac{1}{2}x - \frac{3}{4}y\right)$; $x = 2$ and $y = -2$

Lesson 2-3

Writing Linear Equations in Slope-Intercept Form

Write the equation of each line in slope-intercept form.

5. $2x - 4y = 10$

6. $3y + 9 = -6x$

7. $y - 5x = 16$

8. $-7 - y = -3x$

9. $\frac{x}{6} - \frac{5}{12}y = \frac{5}{8}$

10. $4x = y - 11$

11. $2y = -12x - 16$

12. $\frac{y}{9} + \frac{x}{3} = 2$

Lesson 2-3

Graphing Linear Equations

Graph each equation.

13. $3x = y - 1$

14. $x - 5y = 10$

15. $12 + 2y = 3x$

16. $y = 4x$

Lesson 2-8

Graphing Inequalities

Graph each inequality.

17. $4y \leq 24x$

18. $y \geq 2|x - 1.5|$

19. $x + 5y \geq 20$

20. $y > |x + 6| - 2$

Looking Ahead Vocabulary

21. A *system* of mountains is a group of mountains that share similar geographic and geological features. What are some mountain systems in the United States?

22. Two things are *consistent* if they are in agreement with each other. What does it mean for your actions to be consistent with your words?

23. How many books does Jeff own if he has more than 14 books? Describe the number of books that Jeff owns if you add the *constraint* that he owns fewer than his sister, who owns 19 books.

CHAPTER

3

Linear Systems

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The cables in the photo intersect at certain points, just like the graphs of linear equations might intersect.

How can you solve a system of linear equations? How can you use linear programming to solve real-world problems? How can you use a matrix to represent a system of equations? You will learn how in this chapter.

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
dependent system, *p. 137*	sistema dependiente
equivalent systems, *p. 144*	sistemas equivalentes
independent system, *p. 137*	sistema independiente
linear system, *p. 134*	sistema lineal
matrix, *p. 174*	matriz
matrix element, *p. 174*	elemento matricial
row operation, *p. 176*	operación de filas
system of equations, *p. 134*	sistema de ecuaciones

BIG ideas

1 **Function**

Essential Question How does representing functions graphically help you solve a system of equations?

2 **Equivalence**

Essential Question How does writing equivalent equations help you solve a system of equations?

3 **Solving Equations and Inequalities**

Essential Question How are the properties of equality used in the matrix solution of a system of equations?

Chapter Preview

3-1 Solving Systems Using Tables and Graphs

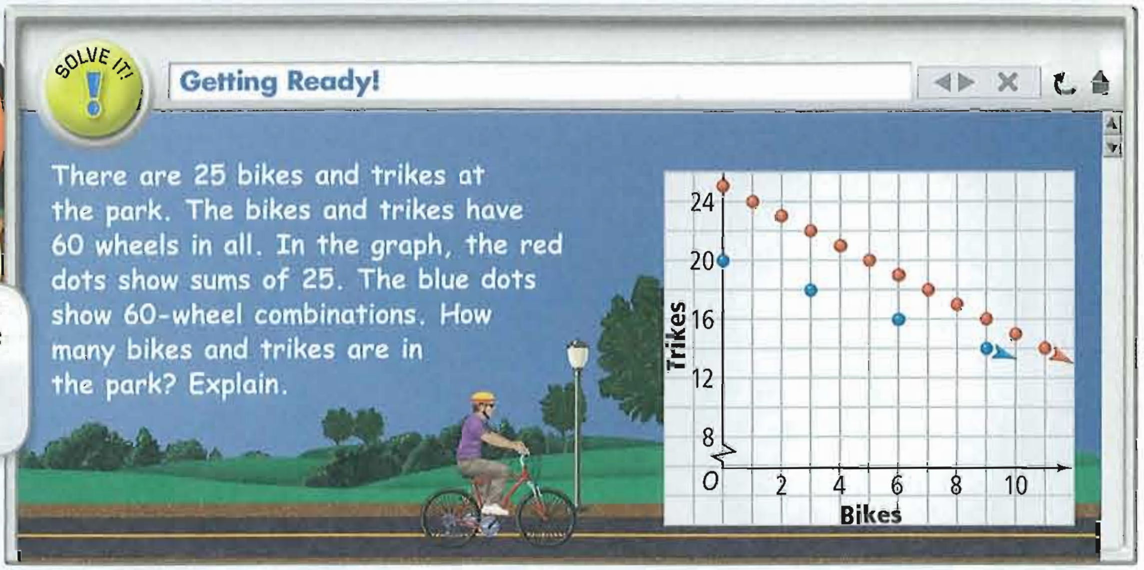
Sunshine State Standards

MA.912.A.3.14 Solve systems of linear equations in two variables using graphical methods.

MA.912.A.3.15 Solve real-world problems involving systems of linear equations in two variables.

Objective To solve a linear system using a graph or a table


There is just one way that 25 bikes and trikes can have a total of 60 wheels.



There are 25 bikes and trikes at the park. The bikes and trikes have 60 wheels in all. In the graph, the red dots show sums of 25. The blue dots show 60-wheel combinations. How many bikes and trikes are in the park? Explain.

Lesson Vocabulary

- system of equations
- linear system
- solution of a system
- inconsistent system
- consistent system
- independent system
- dependent system

When you have two or more related unknowns, you may be able to represent their relationship with a **system of equations**—a set of two or more equations.

Essential Understanding To solve a system of equations, find a set of values that replace the variables in the equations and make each equation true.

A **linear system** consists of linear equations. A **solution of a system** is a set of values for the variables that makes all the equations true. You can solve a system of equations graphically or by using tables.

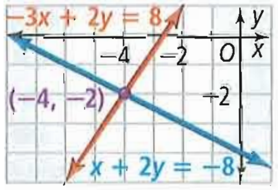
Problem 1 Using a Graph or Table to Solve a System

Think

How can you use a graph to find the solution of a system?

Find the point where the two lines intersect.

What is the solution of the system? $\begin{cases} -3x + 2y = 8 \\ x + 2y = -8 \end{cases}$



Method 1 Graph the equations. The point of intersection appears to be $(-4, -2)$.

Check by substituting the values into both equations.

$$-3x + 2y = 8 \qquad x + 2y = -8$$

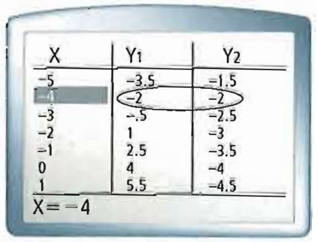
$$-3(-4) + 2(-2) = 8 \checkmark \qquad -4 + 2(-2) = -8 \checkmark$$

Both equations are true so $(-4, -2)$ is the solution of the system.

Dynamic Activity
Systems of Linear Equations

Method 2 Use a table. Write the equations in slope-intercept form.

$$-3x + 2y = 8 \qquad x + 2y = -8$$
$$2y = 3x + 8 \qquad 2y = -x - 8$$
$$y_1 = \frac{3}{2}x + 4 \qquad y_2 = -\frac{1}{2}x - 4$$



```
  X    | Y1   | Y2
 -5    | -3.5 | -1.5
 -4    | -2   | -2
 -3    | -.5  | -2.5
 -2    | 1    | -3
 -1    | 2.5  | -3.5
 0     | 4    | -4
 1     | 5.5  | -4.5
X= -4
```

Enter the equations in the **Y=** screen as **Y1** and **Y2**.
View the table. Adjust the x-values until you see $y_1 = y_2$.

When $x = -4$, both y_1 and y_2 equal -2. So, $(-4, -2)$ is the solution of the system.

Got It? **1.** What is the solution of the system? $\begin{cases} x - 2y = 4 \\ 3x + y = 5 \end{cases}$

Problem 2 Using a Table to Solve a Problem

Biology **The diagrams show the birth lengths and growth rates of two species of shark. If the growth rates stay the same, at what age would a Spiny Dogfish and a Greenland shark be the same length?**

Step 1 Define the variables and write the equation for the length of each shark.

Let x = age in years.
Let y = length in centimeters.

Length of Greenland: $y_1 = 0.75x + 37$
Length of Spiny Dogfish: $y_2 = 1.5x + 22$

Think
How can you use slope-intercept form to write each equation?
Use the growth rate for m and the length at birth for b.

Step 2 Use the table to solve the problem.

List x-values until the corresponding y-values match.

The sharks will be the same length when they are 20 years old.

Shark Length in cm

Age	Greenland	Spiny Dogfish
x	$y_1 = 0.75x + 37$	$y_2 = 1.5x + 22$
15	48.25	44.5
16	49	46
⋮	⋮	⋮
20	52	52

Got It? **2.** **a.** If the growth rates continue, how long will each shark be when it is 25 years old?
b. **Reasoning** Explain why growth rates for these sharks may not continue indefinitely.

Problem 3 Using Linear Regression

Population The table shows the populations of the New York City and Los Angeles metropolitan regions from the census reports for 1950 through 2000. Assuming these linear trends continue, when will the populations of these regions be equal? What will that population be?

Populations of New York City and Los Angeles Metropolitan Regions (1950–2000)

	1950	1960	1970	1980	1990	2000
New York City	12,911,994	14,759,429	16,178,700	16,121,297	18,087,251	21,199,865
Los Angeles	4,367,911	6,742,696	7,032,075	11,497,568	14,531,529	16,373,645

Source: U.S. Census Bureau

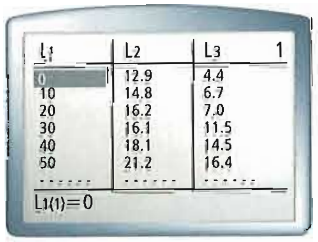
Know
Population data for two regions

Need
The point in time when their populations will be the same

Plan
- Use a calculator to find linear regression models.
- Plot the models.
- Find the point of intersection.

Enter all the numbers as millions, rounded to the nearest hundred thousand. For example, enter 12,911,994 as 12.9.

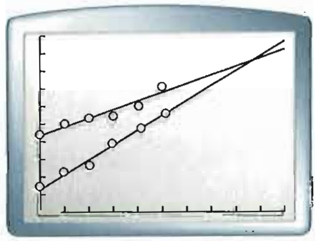
Step 1 Enter the data into lists on your calculator.
L1: number of years since 1950
L2: New York City populations
L3: Los Angeles populations



```
L1      L2      L3      1
0       12.9    4.4
10      14.8    6.7
20      16.2    7.0
30      16.1    11.5
40      18.1    14.5
50      21.2    16.4
------  ------  ------
L1(1)=0
```

Step 2 Use **LinReg(ax + b)** to find lines of best fit.
Use **L1** and **L2** for New York City.
Use **L1** and **L3** for Los Angeles.

Step 3 Graph the linear regression lines.
Use the **Intersect** feature.



The *x*-value of the point of intersection is about 87, which represents the year 2037. The data suggest that the populations of the New York City and Los Angeles metropolitan regions will each be about 25.6 million in 2037.

Think
What does *x* represent?
The *x*-value is the number of years *since* the zero year.

Got It? **3.** The table shows the populations of the San Diego and Detroit metropolitan regions. When were the populations of these regions equal? What was that population?

Populations of San Diego and Detroit Metropolitan Regions (1950–2000)

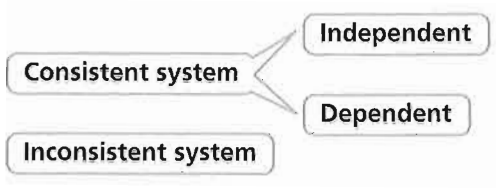
	1950	1960	1970	1980	1990	2000
San Diego	334,387	573,224	696,769	875,538	1,110,549	1,223,400
Detroit	1,849,568	1,670,144	1,511,482	1,203,339	1,027,974	951,270

Source: U.S. Census Bureau

You can classify a system of two linear equations by the number of solutions.

A **consistent system** has at least one solution.

An **inconsistent system** has no solution.



An **independent system** has one solution.

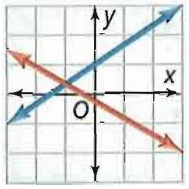
A **dependent system** has infinitely many solutions.

The graphs for an inconsistent system are parallel lines. So, there are no solutions. For a dependent system, the two equations represent the same line.

take note

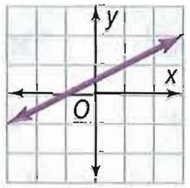
Concept Summary Graphical Solutions of Linear Systems

Intersecting Lines



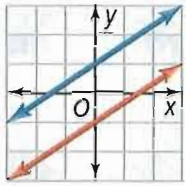
one solution
Consistent
Independent

Coinciding Lines



infinitely many solutions
Consistent
Dependent

Parallel Lines



no solution
Inconsistent

Problem 4 Classifying a System Without Graphing

Plan

What should you compare to classify the system? Compare the slopes and y-intercepts of each line.

Without graphing, is the system *independent*, *dependent*, or *inconsistent*?

$$\begin{cases} 4y - 2x = 6 \\ 8y = 4x - 12 \end{cases}$$

Rewrite each equation in slope-intercept form. Compare slopes and y-intercepts.

$$4y - 2x = 6 \qquad\qquad 8y = 4x - 12$$

$$y = \frac{1}{2}x + \frac{3}{2} \qquad\qquad y = \frac{1}{2}x - \frac{3}{2}$$

$$m = \frac{1}{2}; \text{ } y\text{-intercept is } \frac{3}{2} \qquad\qquad m = \frac{1}{2}; \text{ } y\text{-intercept is } -\frac{3}{2}$$

The slopes are equal and the y-intercepts are different. The lines are different but parallel. The system is inconsistent.

Got It? **4.** Without graphing, is each system *independent*, *dependent*, or *inconsistent*?

a. $\begin{cases} -3x + y = 4 \\ x - \frac{1}{3}y = 1 \end{cases}$ **b.** $\begin{cases} 2x + 3y = 1 \\ 4x + y = -3 \end{cases}$ **c.** $\begin{cases} y = 2x - 3 \\ 6x - 3y = 9 \end{cases}$

Lesson Check

Do you know HOW?

Solve each system of equations by graphing. Check your solution.

1. $\begin{cases} y = x - 1 \\ y = -x + 3 \end{cases}$

2. $\begin{cases} 2x + y = 4 \\ x - y = 2 \end{cases}$

3. You bought a total of 6 pens and pencils for \$4. If each pen costs \$1 and each pencil costs \$.50, how many pens and pencils did you buy?

Do you UNDERSTAND?

4. **Vocabulary** Is it possible for a system of equations to be both independent and inconsistent? Explain.

5. **Open-Ended** Write a system of linear equations that has no solution.

6. **Reasoning** In a system of linear equations, the slope of one line is the negative reciprocal of the slope of the other line. Is this system *independent*, *dependent*, or *inconsistent*? Explain.

Practice and Problem-Solving Exercises

A Practice

Solve each system by graphing or using a table. Check your answers. **See Problem 1.**

7. $\begin{cases} y = x - 2 \\ y = -2x + 7 \end{cases}$

8. $\begin{cases} y = -x + 3 \\ y = \frac{3}{2}x - 2 \end{cases}$

9. $\begin{cases} 2x + 4y = 12 \\ x + y = 2 \end{cases}$

10. $\begin{cases} x = -3 \\ y = 5 \end{cases}$

11. $\begin{cases} 2x - 2y = 4 \\ y - x = 6 \end{cases}$

12. $\begin{cases} 3x + y = 5 \\ x - y = 7 \end{cases}$

Write and solve a system of equations for each situation. Check your answers. **See Problem 2.**

13. A store sells small notebooks for \$8 and large notebooks for \$10. If you buy 6 notebooks and spend \$56, how many of each size notebook did you buy?

14. A shop has one-pound bags of peanuts for \$2 and three-pound bags of peanuts for \$5.50. If you buy 5 bags and spend \$17, how many of each size bag did you buy?

Graphing Calculator **Find linear models for each set of data. In what year will the two quantities be equal?** **See Problem 3.**

15. **U.S. Life Expectancy at Birth (1970–2000)**

Year	1970	1975	1980	1985	1990	1995	2000
Men (years)	67.1	68.8	70.0	71.1	71.8	72.5	74.3
Women (years)	74.7	76.6	77.4	78.2	78.8	78.9	79.7

Source: U.S. Census Bureau

16. **Annual U.S. Consumption of Vegetables**

Year	1980	1985	1990	1995	1998	1999	2000
Broccoli (lb/person)	1.5	2.6	3.4	4.3	5.1	6.5	6.1
Cucumbers (lb/person)	3.9	4.4	4.7	5.6	6.5	6.8	6.4

Source: U.S. Census Bureau

Without graphing, classify each system as *independent*, *dependent*, or *inconsistent*.

See Problem 4.

17. $\begin{cases} 7x - y = 6 \\ -7x + y = -6 \end{cases}$

18. $\begin{cases} -3x + y = 4 \\ x - \frac{1}{3}y = 1 \end{cases}$

19. $\begin{cases} 4x + 8y = 12 \\ x + 2y = -3 \end{cases}$

20. $\begin{cases} y = 2x - 1 \\ y = -2x + 5 \end{cases}$

21. $\begin{cases} x = 6 \\ y = -2 \end{cases}$

22. $\begin{cases} 2y = 5x + 6 \\ -10x + 4y = 8 \end{cases}$

23. $\begin{cases} x - 3y = 2 \\ 4x - 12y = 8 \end{cases}$

24. $\begin{cases} y - x = 0 \\ y = -x \end{cases}$

25. $\begin{cases} 2y - x = 4 \\ \frac{1}{2}x + y = 2 \end{cases}$

26. $\begin{cases} x + 4y = 12 \\ 2x - 8y = 4 \end{cases}$

27. $\begin{cases} 4x + 8y = -6 \\ 6x + 12y = -9 \end{cases}$

28. $\begin{cases} 4y - 2x = 6 \\ 8y = 4x - 12 \end{cases}$

B Apply

Graph and solve each system.

29. $\begin{cases} 3 = 4y + x \\ 4y = -x + 3 \end{cases}$

30. $\begin{cases} y = \frac{1}{2}x + \frac{1}{2} \\ y = \frac{1}{4}x + \frac{3}{2} \end{cases}$

31. $\begin{cases} 3x + 6y - 12 = 0 \\ x + 2y = 8 \end{cases}$

32. $\begin{cases} y = -3x + 3 \\ y = 2x - 7 \end{cases}$

33. $\begin{cases} 3x = -5y + 4 \\ 250 + 150x = 300y \end{cases}$

34. $\begin{cases} x + 3y = 6 \\ 6y + 2x = 12 \end{cases}$

35. $\begin{cases} -x + 3y = 6 \\ 2x - y = 8 \end{cases}$

36. $\begin{cases} y = -\frac{1}{2}x + 8 \\ y = 2x - 6 \end{cases}$

37. $\begin{cases} y = -2x + 6 \\ x - 3y = -6 \end{cases}$

Without graphing, classify each system as *independent*, *dependent*, or *inconsistent*.

38. $\begin{cases} 3x - 2y = 8 \\ 4y = 6x - 5 \end{cases}$

39. $\begin{cases} 2x + 8y = 6 \\ x = -4y + 3 \end{cases}$

40. $\begin{cases} 3m = -5n + 4 \\ n - \frac{6}{5} = -\frac{3}{5}m \end{cases}$

41. Think About a Plan You and a friend are both reading a book. You read 2 pages each minute and have already read 55 pages. Your friend reads 3 pages each minute and has already read 35 pages. Graph and solve a system of equations to find when the two of you will have read the same number of pages. Since the number of pages you have read depends on how long you have been reading, let x represent the number of minutes it takes to read y pages.

- How can you describe the relationship between x and y for you?
- How can you describe the relationship between x and y for your friend?
- How can a graph help you solve this problem?

42. Sports You can choose between two tennis courts at two university campuses to learn how to play tennis. One campus charges \$25 per hour. The other campus charges \$20 per hour plus a one-time registration fee of \$10.

a. Write a system of equations to represent the cost c for h hours of court use at each campus.

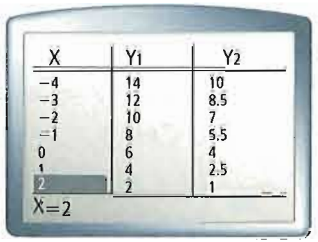
b. Graphing Calculator Find the number of hours for which the costs are the same.

c. Reasoning If you want to practice for a total of 10 hours, which university campus should you choose? Explain.

43. **Error Analysis** Your friend used a graphing calculator to solve a system of linear equations, shown below. After using the **TABLE** feature, your friend says that the system has no solution. Explain what your friend did wrong. What is the solution of the system?

$2x + y = 6$ $\qquad$ $3x + 2y = 8$

$y = 6 - 2x$ $\qquad$ $y = \frac{8 - 3x}{2}$



```
X    | Y1  | Y2
-4   | 14  | 10
-3   | 12  | 8.5
-2   | 10  | 7
-1   | 8   | 5.5
0    | 6   | 4
1    | 4   | 2.5
2    | 2   | 1
X=2
```

44. **Reasoning** Is it possible for an inconsistent linear system to contain two lines with the same *y*-intercept? Explain.

45. **Writing** Summarize the possible relationships for the *y*-intercepts, slopes, and number of solutions in a system of two linear equations in two variables.

Reasoning **Determine whether each statement is *always, sometimes* or *never* true for the following system.**

$$\begin{cases} y = x + 3 \\ y = mx + b \end{cases}$$

46. If $m = 1$, the system has no solution.

47. If $b = 3$, the system has exactly one solution.

48. If $m \neq 1$, the system has no solution.

49. If $m \neq 1$ and $b = 2$, the system has infinitely many solutions.

Challenge

Open-Ended **Write a second equation for each system so that the system will have the indicated number of solutions.**

50. infinite number of solutions

$$\begin{cases} \frac{x}{4} + \frac{y}{3} = 1 \\ \underline{\quad ? \quad} \end{cases}$$

51. no solutions

$$\begin{cases} 5x + 2y = 10 \\ \underline{\quad ? \quad} \end{cases}$$

52. Write a system of linear equations with the solution set $\{(x, y) \mid y = 5x + 2\}$.

53. **Reasoning** What relationship exists between the equations in a dependent system?

54. **Economics** Research shows that in a certain market only 2000 widgets can be sold at \$8 each, but if the price is reduced to \$3, then 10,000 can be sold.

a. Let *p* represent price and *n* represent the number of widgets. Identify the independent and dependent variables.

b. Write a linear equation that relates price and the quantity demanded. This type of equation is called a *demand* equation.

c. A shop can make 2000 widgets for \$5 each and 20,000 widgets for \$2 each. Use this information to write a linear equation that relates price and the quantity supplied. This type of equation is called a *supply* equation.

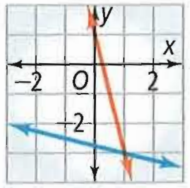
d. Find the equilibrium point where supply is equal to demand. Explain the meaning of the coordinates of this point within the context of the exercise.

Sunshine State Standards Practice

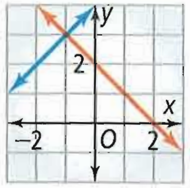
MA.912.A.3.14

55. Which graph shows the solution of the following system? $\begin{cases} 4x + y = 1 \\ x + 4y = -11 \end{cases}$

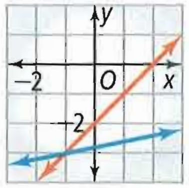
(A)



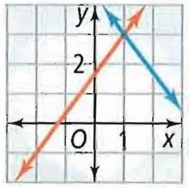
(B)



(C)

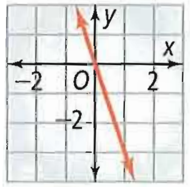


(D)



MA.912.A.3.10

56. Which is the equation of a line that is perpendicular to the line in the graph?



(F) $y = -3x + 2$

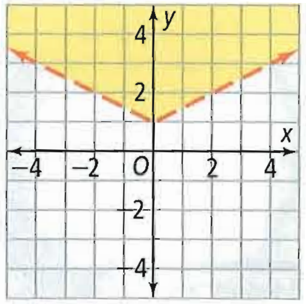
(G) $y = \frac{1}{3}x + 5$

(H) $y = -\frac{1}{3}x - 4$

(I) $y = 3x - 1$

MA.912.A.2.5

57. Which inequality represents the graph at the right?



(A) $y \geq \frac{1}{2}|x| + 1$

(B) $y \leq \frac{1}{2}|x| + 1$

(C) $y > \frac{1}{2}|x| + 1$

(D) $y < \frac{1}{2}|x| + 1$

MA.912.A.3.15

58. Extended Response Amy ordered prints of a total of 6 photographs in two different sizes, 5 × 7 and 4 × 6, from an online site. She paid \$7.50 for her order. The cost of a 5 × 7 print is \$1.75 and the cost of a 4 × 6 print is \$.25. Explain how to solve a system of equations using tables to find the number of 4 × 6 prints Amy ordered.

Mixed Review

Graph each inequality on a coordinate plane.

See Lesson 2-8.

59. $3x - 4y \geq 16$

60. $-5x > 8y + 4$

61. $x < -4$

Solve each inequality. Check your solution.

See Lesson 1-5.

62. $3n < -4(2 + n)$

63. $\frac{x}{3} + 5 \geq \frac{1}{6}$

64. $4x - 2 > \frac{1}{2}$

Find the slope of the line through each pair of points.

See Lesson 2-3.

65. $(-2, -4)$ and $(1, 2)$

66. $(0, 0)$ and $(5, -3)$

67. $(1, 3)$ and $(4, 9)$

Get Ready! **To prepare for Lesson 3-2, do Exercises 68 and 69.**

See Lesson 1-3.

68. What is the value of $a + b - 2c$ for $a = 3$, $b = 1$, and $c = -3$?

69. Substitute −3 for x in each of the following equations. What is the value of y?

a. $y = 2x + 3$

b. $y = -x + 5$

c. $y = 3x - 1$

3-2 Solving Systems Algebraically

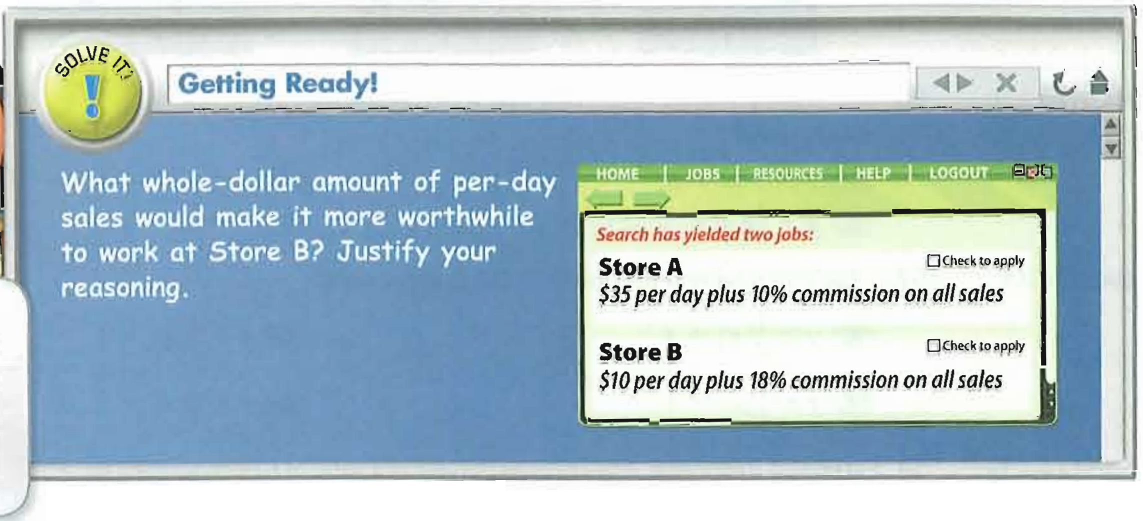
Sunshine State Standards

MA.912.A.3.14 Solve systems of linear equations in two variables using substitution and elimination.

MA.912.A.3.15 Solve real-world problems involving systems of linear equations in two variables.

Objective To solve linear systems algebraically

You can draw graphs of income from each store, but the dollar amounts might be difficult to read.



Solve It! Getting Ready!

What whole-dollar amount of per-day sales would make it more worthwhile to work at Store B? Justify your reasoning.

Search has yielded two jobs:

Store A ☐ Check to apply

$35 per day plus 10% commission on all sales

Store B ☐ Check to apply

$10 per day plus 18% commission on all sales

Lesson Vocabulary
- equivalent systems

When you try to solve a system of equations by graphing, the coordinates of the point of intersection may not be obvious.

Essential Understanding You can solve a system of equations by writing equivalent systems until the value of one variable is clear. Then substitute to find the value(s) of the other variable(s).

You can use the substitution method to solve a system of equations when it is easy to isolate one of the variables. After isolating the variable, substitute for that variable in the other equation. Then solve for the other variable.

Think

Which variable should you solve for first?
In the second equation, the coefficient of y is 1. It is the easiest variable to isolate.

Problem 1 Solving by Substitution

What is the solution of the system of equations? $\begin{cases} 3x + 4y = 12 \\ 2x + y = 10 \end{cases}$

Step 1

Solve one equation for one of the variables.

$$\begin{aligned} 2x + y &= 10 \\ y &= -2x + 10 \end{aligned}$$

Step 2

Substitute the expression for y in the other equation. Solve for x.

$$\begin{aligned} 3x + 4y &= 12 \\ 3x + 4(-2x + 10) &= 12 \\ 3x - 8x + 40 &= 12 \\ x &= 5.6 \end{aligned}$$

Step 3

Substitute the value for x into one of the original equations. Solve for y.

$$\begin{aligned} 2x + y &= 10 \\ 2(5.6) + y &= 10 \\ 11.2 + y &= 10 \\ y &= -1.2 \end{aligned}$$

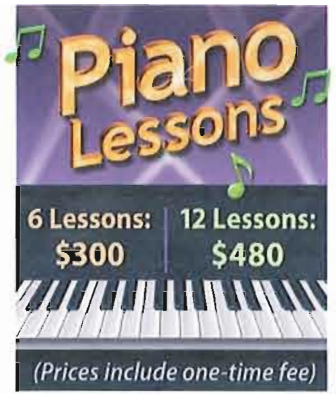
The solution is (5.6, −1.2).

Got It? **1.** What is the solution of the system of equations? $\begin{cases} x + 3y = 5 \\ -2x - 4y = -5 \end{cases}$

Problem 2 Using Substitution to Solve a Problem

Dynamic Activity
Special Types of Solutions to Linear Systems

Music A music store offers piano lessons at a discount for customers buying new pianos. The costs for lessons and a one-time fee for materials (including music books, CDs, software, etc.) are shown in the advertisement. What is the cost of each lesson and the one-time fee for materials?



Relate 6 · cost of one lesson + one-time fee = $300

12 · cost of one lesson + one-time fee = $480

Define Let c = the cost of one lesson.

Let f = the one-time fee.

Write

$$\begin{cases} 6 \cdot c + f = 300 \\ 12 \cdot c + f = 480 \end{cases}$$

$6c + f = 300$
$f = 300 - 6c$

Choose one equation. Solve for f in terms of c.

$12c + (300 - 6c) = 480$
$c = 30$

Substitute the expression for f into the other equation, $12c + f = 480$. Solve for c.

$6(30) + f = 300$
$f = 120$

Substitute the value of c into one of the equations. Solve for f.

Think

Which equation should you use to find *f*?
Use the equation with numbers that are easier to work with.

Check Substitute $c = 30$ and $f = 120$ in the original equations.

$$\begin{aligned} 6c + f &= 300 \\ 6(30) + 120 &\stackrel{?}{=} 300 \\ 180 + 120 &\stackrel{?}{=} 300 \\ 300 &= 300 \checkmark \end{aligned}$$

$$\begin{aligned} 12c + f &= 480 \\ 12(30) + 120 &\stackrel{?}{=} 480 \\ 360 + 120 &\stackrel{?}{=} 480 \\ 480 &= 480 \checkmark \end{aligned}$$

The cost of each lesson is $30. The one-time fee for materials is $120.

Got It? **2.** An online music company offers 15 downloads for $19.75 and 40 downloads for $43.50. Each price includes the same one-time registration fee. What is the cost of each download and the registration fee?

You can use the Addition Property of Equality to solve a system of equations. If you add a pair of additive inverses or subtract identical terms, you can eliminate a variable.

Problem 3 Solving by Elimination

What is the solution of the system of equations? $\begin{cases} 4x + 2y = 9 \\ -4x + 3y = 16 \end{cases}$

Think

How can you use the Addition Property of Equality?
Since $-4x + 3y$ is equal to 16, you can add the same value to each side of $4x + 2y = 9$.

$$\begin{array}{rl} 4x + 2y = 9 & \\ \underline{-4x + 3y = 16} & \text{One equation has } 4x \text{ and the other has } -4x\text{. Add to eliminate the variable } x. \\ 5y = 25 & \\ y = 5 & \text{Solve for } y. \\ 4x + 2y = 9 & \text{Choose one of the original equations.} \\ 4x + 2(5) = 9 & \text{Substitute for } y. \\ 4x = -1 & \text{Solve for } x. \\ x = -\frac{1}{4} & \end{array}$$

The solution is $\left(-\frac{1}{4}, 5\right)$.

Got It? **3.** What is the solution of the system of equations? $\begin{cases} -2x + 8y = -8 \\ 5x - 8y = 20 \end{cases}$

When you multiply each side of one or both equations in a system by the same nonzero number, the new system and the original system have the same solutions. The two systems are called **equivalent systems**. You can use this method to make additive inverses.

Problem 4 Solving an Equivalent System

What is the solution of the system of equations? $\begin{cases} ① \; 2x + 7y = 4 \\ ② \; 3x + 5y = -5 \end{cases}$

Think

By multiplying ① by 3 and ② by −2, the *x*-terms become opposites, and you can eliminate them. Add ③ and ④. Solve for *y*.

Now that you know the value of *y*, use either equation to find *x*.

Write

$$\begin{array}{llll} ① & 2x + 7y = 4 & ③ & 6x + 21y = 12 \\ ② & 3x + 5y = -5 & ④ & \underline{-6x - 10y = 10} \\ & & & 11y = 22 \\ & & & y = 2 \end{array}$$

$$\begin{array}{rl} ① & 2x + 7(2) = 4 \\ & 2x + 14 = 4 \\ & 2x = -10 \\ & x = -5 \end{array}$$

The solution is (−5, 2).

Got It? **4.** **a.** What is the solution of this system of equations? $\begin{cases} 3x + 7y = 15 \\ 5x + 2y = -4 \end{cases}$

b. **Reasoning** In Problem 4, you found that $y = 2$. Substitute this value into equation ② instead of equation ①. Do you still get the same value for x? Explain why.

Solving a system algebraically does not always provide a unique solution. Sometimes you get infinitely many solutions. Sometimes you get no solutions.

Problem 5 Solving Systems Without Unique Solutions

What are the solutions of the following systems? Explain.

Think

How are the two equations in this system related?
Multiplying both sides of the first equation by −1 results in the second equation.

A $$\begin{array}{r} \begin{cases} -3x + y = -5 \\ 3x - y = 5 \end{cases} \\ \hline 0 = 0 \end{array}$$

Elimination gives an equation that is always true. The two equations in the system represent the same line. This is a dependent system with infinitely many solutions.

B $$\begin{array}{r} \begin{cases} 4x - 6y = 6 \\ -4x + 6y = 10 \end{cases} \\ \hline 0 = 16 \end{array}$$

Elimination gives an equation that is always false. The two equations in the system represent parallel lines. This is an inconsistent system. It has no solutions.

Got It? **5.** What are the solutions of the following systems? Explain.

a. $\begin{cases} -x + y = -2 \\ 2x - 2y = 0 \end{cases}$

b. $\begin{cases} 4x + y = 6 \\ 12x + 3y = 18 \end{cases}$

Lesson Check

Do you know HOW?

Solve each system by substitution.

1. $\begin{cases} 3x + 5y = 13 \\ 2x + y = 4 \end{cases}$

2. $\begin{cases} 2x - 3y = 6 \\ x + y = -12 \end{cases}$

Solve each system by elimination.

3. $\begin{cases} 2x + 3y = 7 \\ -2x + 5y = 1 \end{cases}$

4. $\begin{cases} x + 2y = -1 \\ x - y = 8 \end{cases}$

5. $\begin{cases} x - y = -4 \\ 3x + 2y = 7 \end{cases}$

6. $\begin{cases} 3x + 4y = 10 \\ 2x + 3y = 7 \end{cases}$

Do you UNDERSTAND?

7. **Vocabulary** Give an example of two equivalent systems.

8. **Compare and Contrast** Explain how the substitution method of solving a system of equations differs from the elimination method.

9. **Writing** A café sells a regular cup of coffee for $1 and a large cup for $1.50. Melissa and her friends buy 5 cups of coffee and spend a total of $6. Explain how to write and solve a system of equations to find the number of large cups of coffee they bought.

Practice and Problem-Solving Exercises

A Practice

Solve each system by substitution. Check your answers.

See Problem 1.

10. $\begin{cases} 4x + 2y = 7 \\ y = 5x \end{cases}$

11. $\begin{cases} 3c + 2d = 2 \\ d = 4 \end{cases}$

12. $\begin{cases} x + 12y = 68 \\ x = 8y - 12 \end{cases}$

13. $\begin{cases} 4p + 2q = 8 \\ q = 2p + 1 \end{cases}$

14. $\begin{cases} x + 3y = 7 \\ 2x - 4y = 24 \end{cases}$

15. $\begin{cases} x + 6y = 2 \\ 5x + 4y = 36 \end{cases}$

16. $\begin{cases} t = 2r + 3 \\ 5r - 4t = 6 \end{cases}$

17. $\begin{cases} y = 2x - 1 \\ 3x - y = -1 \end{cases}$

18. $\begin{cases} r + s = -12 \\ 4r - 6s = 12 \end{cases}$

19. **Money** A student has some \$1 bills and \$5 bills in his wallet. He has a total of 15 bills that are worth \$47. How many of each type of bill does he have?

See Problem 2.

20. A student took 60 minutes to answer a combination of 20 multiple-choice and extended-response questions. She took 2 minutes to answer each multiple-choice question and 6 minutes to answer each extended-response question.

a. Write a system of equations to model the relationship between the number of multiple choice questions m and the number of extended-response questions r.

b. How many of each type of question was on the test?

21. **Transportation** A youth group with 26 members is going skiing. Each of the five chaperones will drive a van or sedan. The vans can seat seven people, and the sedans can seat five people. Assuming there are no empty seats, how many of each type of vehicle could transport all 31 people to the ski area in one trip?

Solve each system by elimination.

See Problem 3.

22. $\begin{cases} x + y = 12 \\ x - y = 2 \end{cases}$

23. $\begin{cases} x + 2y = 10 \\ x + y = 6 \end{cases}$

24. $\begin{cases} 3a + 4b = 9 \\ -3a - 2b = -3 \end{cases}$

25. $\begin{cases} 4x + 2y = 4 \\ 6x + 2y = 8 \end{cases}$

26. $\begin{cases} 2w + 5y = -24 \\ 3w - 5y = 14 \end{cases}$

27. $\begin{cases} 3u + 3v = 15 \\ -2u + 3v = -5 \end{cases}$

28. $\begin{cases} 3x + 2y = 6 \\ 3x + 3 = y \end{cases}$

29. $\begin{cases} 5x - y = 4 \\ 2x - y = 1 \end{cases}$

30. $\begin{cases} 2r + s = 3 \\ 4r - s = 9 \end{cases}$

Solve each system by elimination.

See Problems 4 and 5.

31. $\begin{cases} 4x - 6y = -26 \\ -2x + 3y = 13 \end{cases}$

32. $\begin{cases} 9a - 3d = 3 \\ -3a + d = -1 \end{cases}$

33. $\begin{cases} 2a + 3b = 12 \\ 5a - b = 13 \end{cases}$

34. $\begin{cases} 2x - 3y = 6 \\ 6x - 9y = 9 \end{cases}$

35. $\begin{cases} 20x + 5y = 120 \\ 10x + 7.5y = 80 \end{cases}$

36. $\begin{cases} 6x - 2y = 11 \\ -9x + 3y = 16 \end{cases}$

37. $\begin{cases} 2x - 3y = -1 \\ 3x + 4y = 8 \end{cases}$

38. $\begin{cases} 5x - 2y = -19 \\ 2x + 3y = 0 \end{cases}$

39. $\begin{cases} r + 3s = 7 \\ 2r - s = 7 \end{cases}$

40. $\begin{cases} y = 4 - x \\ 3x + y = 6 \end{cases}$

41. $\begin{cases} 3x + 2y = 10 \\ 6x + 4y = 15 \end{cases}$

42. $\begin{cases} 3m + 4n = -13 \\ 5m + 6n = -19 \end{cases}$

B Apply

43. Think About a Plan Suppose you have a part-time job delivering packages. Your employer pays you a flat rate of $9.50 per hour. You discover that a competitor pays employees $2 per hour plus $3 per delivery. How many deliveries would the competitor's employees have to make in four hours to earn the same pay you earn in a four-hour shift?

- How can you write a system of equations to model this situation?
- Which method should you use to solve the system?
- How can you interpret the solution in the context of the problem?

Solve each system.

44. $\begin{cases} 5x + y = 0 \\ 5x + 2y = 30 \end{cases}$

45. $\begin{cases} 2m = -4n - 4 \\ 3m + 5n = -3 \end{cases}$

46. $\begin{cases} 7x + 2y = -8 \\ 8y = 4x \end{cases}$

47. $\begin{cases} 2m + 4n = 10 \\ 3m + 5n = 11 \end{cases}$

48. $\begin{cases} -6 = 3x - 6y \\ 4x = 4 + 5y \end{cases}$

49. $\begin{cases} \frac{x}{3} + \frac{4y}{3} = 300 \\ 3x - 4y = 300 \end{cases}$

50. $\begin{cases} 0.02a - 1.5b = 4 \\ 0.5b - 0.02a = 1.8 \end{cases}$

51. $\begin{cases} 4y = 2x \\ 2x + y = \frac{x}{2} + 1 \end{cases}$

52. $\begin{cases} \frac{1}{2}x + \frac{2}{3}y = 1 \\ \frac{3}{4}x - \frac{1}{3}y = 2 \end{cases}$

53. Error Analysis Identify and correct the error shown in finding the solution of $\begin{cases} 3x - 4y = 14 \\ x + y = -7 \end{cases}$ using substitution.

~~x + y = -7~~
~~y = -7 - x~~

~~3x - 4y = 14~~
~~3x - 4(-7 - x) = 14~~
~~3x - 28 - 4x = 14~~
~~-x - 28 = 14~~
~~x = -42~~

~~y = -7 - (-42)~~
~~y = 35~~

54. Break-Even Point Jenny's Bakery sells carrot muffins at $2 each. The electricity to run the oven is $120 per day and the cost of making one carrot muffin is $1.40. How many muffins need to be sold each day to break even?

55. Open-Ended Write a system of equations in which both equations must be multiplied by a number other than 1 or −1 before using elimination. Solve the system.

56. Chemistry A scientist wants to make 6 milliliters of a 30% sulfuric acid solution. The solution is to be made from a combination of a 20% sulfuric acid solution and a 50% sulfuric acid solution. How many milliliters of each solution must be combined to make the 30% solution?

57. Writing Explain how you decide whether to use substitution or elimination to solve a system.

58. The equation $3x - 4y = 2$ and which equation below form a system with no solutions?

Ⓐ $2y = 1.5x - 2$

Ⓑ $2y = 1.5x - 1$

Ⓒ $3x + 4y = 2$

Ⓓ $4y - 3x = -2$

For each system, choose the method of solving that seems easier to use. Explain why you made each choice. Solve each system.

59. $\begin{cases} 3x - y = 5 \\ y = 4x + 2 \end{cases}$

60. $\begin{cases} 2x - 3y = 4 \\ 2x - 5y = -6 \end{cases}$

61. $\begin{cases} 6x - 3y = 3 \\ 5x - 5y = 10 \end{cases}$

Challenge

62. Entertainment In the final round of a singing competition, the audience voted for one of the two finalists, Luke or Sean. Luke received 25% more votes than Sean received. Altogether, the two finalists received 5175 votes. How many votes did Luke receive?

63. Weather The equation $F = \frac{9}{5}C + 32$ relates temperatures on the Celsius and Fahrenheit scales. Does any temperature have the same number reading on both scales? If so, what is the number?

Find the value of *a* that makes each system a dependent system.

64. $\begin{cases} y = 3x + a \\ 3x - y = 2 \end{cases}$

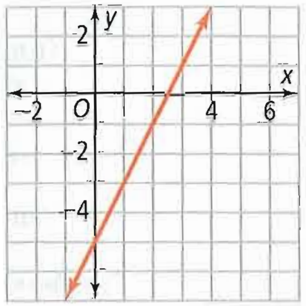
65. $\begin{cases} 3y = 2x \\ 6y - a - 4x = 0 \end{cases}$

66. $\begin{cases} y = \frac{x}{2} + 4 \\ 2y - x = a \end{cases}$

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.3.9 **67.** What is the slope of the line at the right?



MA.912.A.3.14 **68.** What is the x-value of the solution of $\begin{cases} x + y = 7 \\ 3x - 2y = 11 \end{cases}$?

MA.912.A.3.1 **69.** Solve $9(x + 7) - 6(x - 3) = 99$. What is the value of x?

MA.912.A.3.15 **70.** Georgia has only dimes and quarters in her bag. She has a total of 18 coins that are worth \$3. How many more dimes than quarters does she have?

MA.912.A.2.10 **71.** The graph of $g(x)$ is a horizontal translation of $f(x) = 2|x + 1| + 3$, 5 units to the right. What is the x-value of the vertex of $g(x)$?

Mixed Review

Solve each system of equations by graphing. See Lesson 3-1.

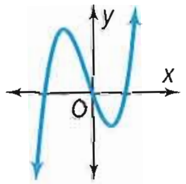
72. $\begin{cases} y = 3x + 4 \\ 2y = 6x - 2 \end{cases}$

73. $\begin{cases} -3y = 9x + 1 \\ 6y = -18x - 2 \end{cases}$

74. $\begin{cases} 4x - y = -5 \\ -8x + 2y = 15 \end{cases}$

Use the vertical line test to determine whether each graph represents a function. See Lesson 2-1.

75.



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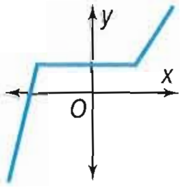

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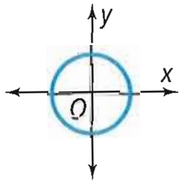


Get Ready! **To prepare for Lesson 3-3, do Exercises 78–80.**

Solve each inequality. Graph the solution. See Lesson 1-5.

78. $-3(2x + 1) > 3$

79. $4x > -2$

80. $8y + 4 < 12$

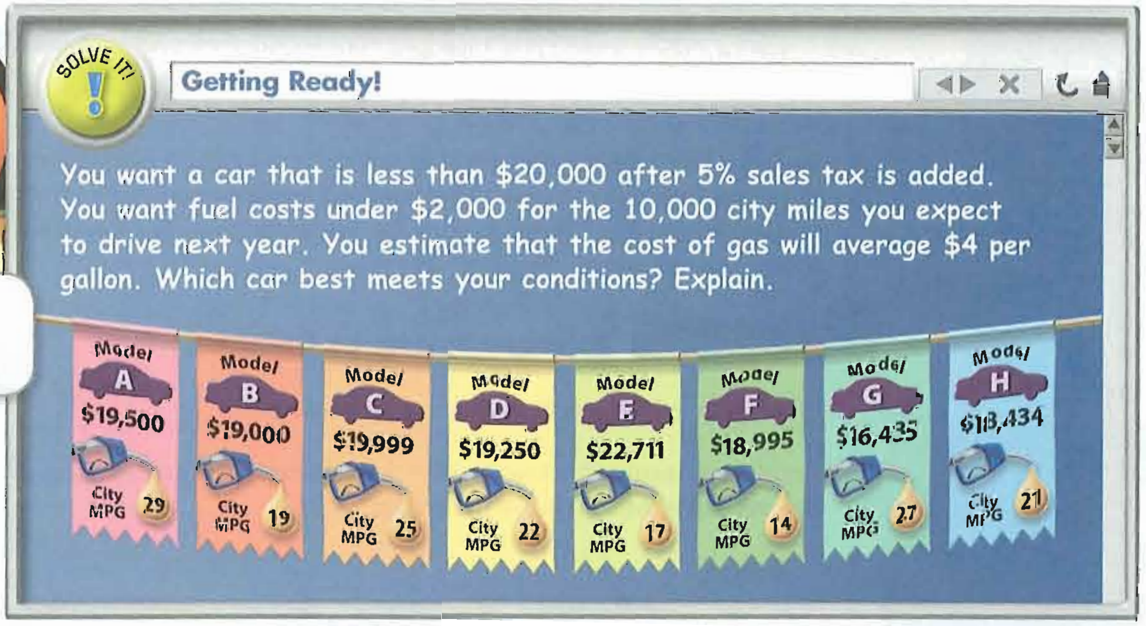
3-3 Systems of Inequalities

Sunshine State Standards

MA.912.A.3.14 Solve systems of linear inequalities in two variables using graphical methods.

MA.912.A.3.15 Solve real-world problems involving systems of linear inequalities in two variables.

Objective To solve systems of linear inequalities



You want a car that is less than $20,000 after 5% sales tax is added. You want fuel costs under $2,000 for the 10,000 city miles you expect to drive next year. You estimate that the cost of gas will average $4 per gallon. Which car best meets your conditions? Explain.

Dynamic Activity
Systems of Linear Inequalities

An inequality and a system of inequalities can each have many solutions. A solution of a system of inequalities is a solution for each inequality in the system.

Essential Understanding You can solve a system of inequalities in more than one way. Graphing the solution is usually the most appropriate method. The solution is the set of all points that are solutions of each inequality in the system.

Problem 1 Solving a System by Using a Table

Plan

Which inequality should you use to build a table?
The first inequality has an infinite number of whole number solutions. The second one has a finite number of solutions. Use the second inequality.

Assume that g and m are whole numbers. What is the solution of the system of inequalities? $\begin{cases} g + m \geq 6 \\ 5g + 2m \leq 20 \end{cases}$

Make a table of values for g and m that satisfy the second inequality. The values for g and m must be whole numbers.

If $g = 0$, then $5(0) + 2m \leq 20$, and $m \leq 10$.
If $m = 0$, then $5g + 2(0) \leq 20$, and $g \leq 4$.

g	m
0	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
1	0, 1, 2, 3, 4, 5, 6, 7
2	0, 1, 2, 3, 4, 5
3	0, 1, 2
4	0

In the table, highlight each pair of values that satisfies the first inequality. The highlighted pairs are the solutions of both inequalities.

g	m
0	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
1	0, 1, 2, 3, 4, 5, 6, 7
2	0, 1, 2, 3, 4, 5
3	0, 1, 2
4	0

Got It? **1.** Assume that x and y are whole numbers. What is the solution of the system of inequalities?

$$\begin{cases} x + y > 4 \\ 3x + 7y \leq 21 \end{cases}$$

You can solve a system of linear inequalities by graphing. Recall that when the variables of a linear inequality represent real numbers, a graphed solution consists of a half-plane and possibly its boundary line. Thus, for two inequalities, the solution is the overlap of the two half-planes.

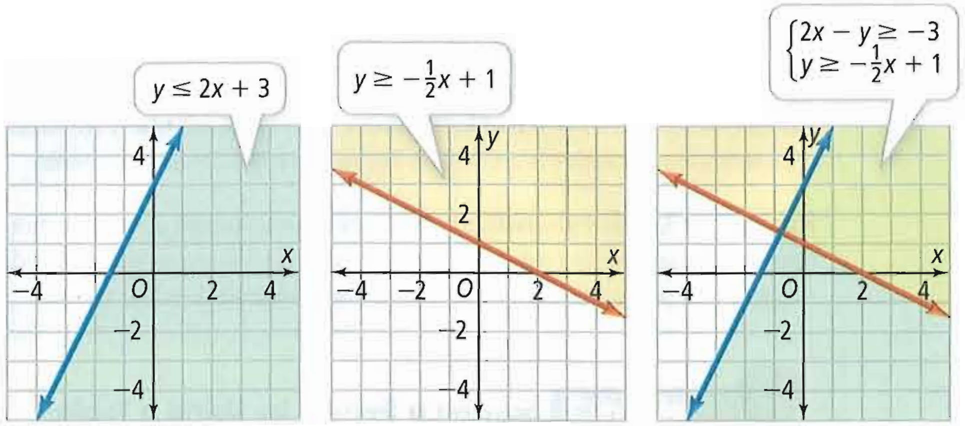
Problem 2 Solving a System by Graphing

Plan

How can you be sure to shade the correct half-planes?
If the inequality is in slope-intercept form, shade above the boundary if $y >$ or $y \geq$ and shade below the boundary if $y <$ or $y \leq$. If not, use a test point.

What is the solution of the system of inequalities? $\begin{cases} 2x - y \geq -3 \\ y \geq -\frac{1}{2}x + 1 \end{cases}$

Graph each inequality. Rewrite $2x - y \geq -3$ in slope-intercept form as $y \leq 2x + 3$. The overlap is the solution of the system.



Check Pick a point in the overlap region, such as (0, 2), and check it in both inequalities of the system.

$$2x - y \geq -3 \qquad y \geq \frac{1}{2}x + 1$$

$$2(0) - 2 \geq -3 \qquad 2 \geq \frac{1}{2}(0) + 1$$

$$-2 \geq -3 \checkmark \qquad 2 \geq 1 \checkmark$$

Got It? **2.** What is the solution of the system of inequalities? $\begin{cases} x + 2y \leq 4 \\ y \geq -x - 1 \end{cases}$

Sometimes, you can model a real situation with a system of linear inequalities. Solutions to real-world problems are often whole numbers, so only certain points in the region of overlap will solve the problem.

Problem 3 Using a System of Inequalities

Fundraising **Your city's cultural center is sponsoring a concert to raise at least \$30,000 for the city's Youth Services. Tickets are \$20 for balcony seats and \$30 for orchestra seats. If the center has 500 orchestra seats, how many of each type of seat must they sell?**

Know	Need	Plan
Must raise at least \$30,000. There are at most 500 orchestra seats.	The possible sales of balcony and orchestra seats	- Model the problem with a system of inequalities. - Graph the inequalities on your calculator.

Relate $20 \cdot \boxed{\text{balcony seats}} + 30 \cdot \boxed{\text{orchestra seats}} \geq 30{,}000$

$\boxed{\text{orchestra seats}} \leq 500$

Define Let $\boxed{x}$ = the number of balcony seats sold.

Let $\boxed{y}$ = the number of orchestra seats sold.

Write $20 \cdot \boxed{x} + 30 \cdot \boxed{y} \geq 30{,}000$

$\boxed{y} \leq 500$

Rewrite $20x + 30y \geq 30{,}000$ in slope intercept form as $y \geq -\frac{2}{3}x + 1000$.

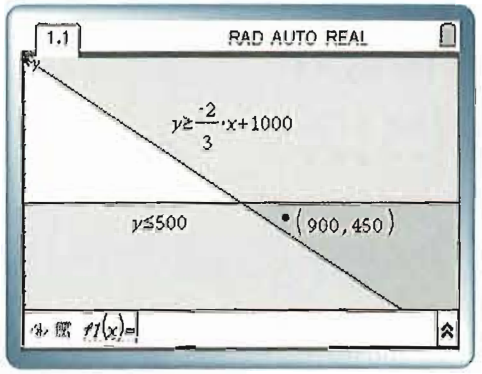
The system of inequalities is $\begin{cases} y \geq -\frac{2}{3}x + 1000 \\ y \leq 500 \end{cases}$.

Think

What do points in the overlap represent?
The points represent combinations of balcony and orchestra seats that have a total value of at least \$30,000.

Use your graphing calculator to graph the inequalities.

The solution is the overlap.



Test a point. If the cultural center sells 900 balcony and 450 orchestra tickets, will the Youth Services meet its goal?

$$20(900) + 30(450) \overset{?}{\geq} 30{,}000 \qquad 450 \overset{?}{\leq} 500$$
$$18{,}000 + 13{,}250 \overset{?}{\geq} 30{,}000 \qquad 450 \leq 500 \ ✓$$
$$31{,}250 \geq 30{,}000 \ ✓$$

Because the number of seats must be a whole number, only the points in the overlap that represent whole numbers are solutions.

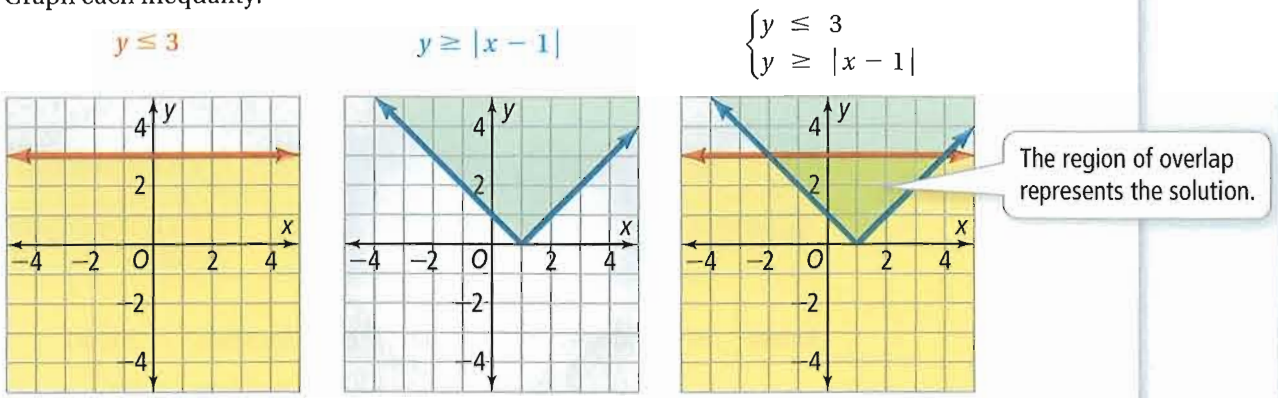
Got It? **3.** A pizza parlor charges \$1 for each vegetable topping and \$2 for each meat topping. You want at least five toppings on your pizza. You have \$10 to spend on toppings. How many of each type of topping can you get on your pizza?

A system of inequalities can include nonlinear inequalities. You can also solve these systems graphically.

Problem 4 Solving a Linear/Absolute-Value System

What is the solution of the system of inequalities? $\begin{cases} y \leq 3 \\ y \geq |x - 1| \end{cases}$

Graph each inequality.


$y \leq 3$

$y \geq |x - 1|$

$\begin{cases} y \leq 3 \\ y \geq |x - 1| \end{cases}$

The region of overlap represents the solution.

Think

Why is the shape of the overlap different from that of a system of linear inequalities? The overlap is not formed by 2 half-planes so it is more varied in shape.

Got It? **4.** What is the solution of the system of inequalities? $\begin{cases} y < -\frac{1}{3}x + 1 \\ y > 2|x - 1| \end{cases}$

Lesson Check

Do you know HOW?

Solve each system of inequalities by graphing.

1. $\begin{cases} x + y \geq 2 \\ 2x + y \leq 5 \end{cases}$

2. $\begin{cases} y > x \\ y < x + 1 \end{cases}$

3. $\begin{cases} y \geq -3x - 1 \\ y < x + 2 \end{cases}$

4. You spend no more than 3 hours each day watching TV and playing football. You play football for at least 1 hour each day. What are the possible numbers of hours you can spend on each activity in one day?

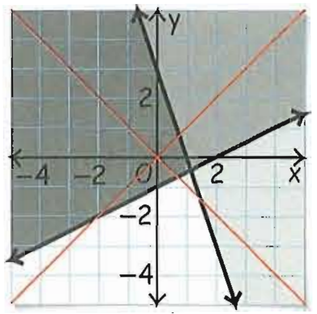
Do you UNDERSTAND?

5. Reasoning Is the solution of a system of linear inequalities the union or intersection of the solutions of the two inequalities? Justify your answer.

6. Compare and Contrast Explain how the graphical solution of a system of inequalities is different from the graphical solution of a system of equations.

7. Error Analysis Describe and correct the error made in solving this system of inequalities.

$\begin{cases} y < \frac{1}{2}x - 1 \\ y > -3x + 3 \end{cases}$



Practice and Problem-Solving Exercises

A Practice

Find all whole number solutions of each system using a table.

See Problem 1.

8. $\begin{cases} y + 3x \leq 8 \\ y - 3 > 2x \end{cases}$

9. $\begin{cases} x + y < 8 \\ 3x \leq y + 6 \end{cases}$

10. $\begin{cases} y \geq x + 2 \\ 3y < -6x + 6 \end{cases}$

Solve each system of inequalities by graphing.

See Problem 2.

11. $\begin{cases} y \leq 2x + 2 \\ y < -x + 1 \end{cases}$

12. $\begin{cases} y > -2 \\ x < 1 \end{cases}$

13. $\begin{cases} y \leq 3 \\ y \leq \frac{1}{2}x + 1 \end{cases}$

14. $\begin{cases} y \leq 3x + 1 \\ -6x + 2y > 5 \end{cases}$

15. $\begin{cases} x + 2y \leq 10 \\ x + y \leq 3 \end{cases}$

16. $\begin{cases} -x - y \leq 2 \\ y - 2x > 1 \end{cases}$

17. $\begin{cases} y > -2x \\ 2x - y \geq 2 \end{cases}$

18. $\begin{cases} c \geq d - 3 \\ c < \frac{1}{2}d + 3 \end{cases}$

19. $\begin{cases} 2x + y < 1 \\ y > -2x + 3 \end{cases}$

See Problem 3.

20. You want to decorate a party hall with a total of at least 40 red and yellow balloons, with a minimum of 25 yellow balloons. Write and graph a system of inequalities to model the situation.

21. A gardener wants to plant at least 50 tulips and rose plants in a garden, but no more than 20 rose plants. Write and graph a system of inequalities to model the situation.

Solve each system of inequalities by graphing.

See Problem 4.

22. $\begin{cases} y > 4 \\ y < |x - 1| \end{cases}$

23. $\begin{cases} y < -\frac{1}{3}x + 1 \\ y > |2x - 1| \end{cases}$

24. $\begin{cases} y > x - 2 \\ y \geq |x + 2| \end{cases}$

25. $\begin{cases} y \leq -\frac{4}{3}x \\ y \geq -|x| \end{cases}$

26. $\begin{cases} 3y < -x - 1 \\ y \leq |x + 1| \end{cases}$

27. $\begin{cases} y > -2 \\ y \leq -|x - 3| \end{cases}$

28. $\begin{cases} -2y < 4x + 2 \\ y > |2x + 1| \end{cases}$

29. $\begin{cases} -x \geq 4 - y \\ y \geq |3x - 6| \end{cases}$

30. $\begin{cases} y \leq x - 4 \\ y > |x - 6| \end{cases}$

B Apply

31. **Think About a Plan** The food pyramid suggests that you eat 4–6 servings of fruits and vegetables a day for a healthy diet. It also says that the number of servings of vegetables should be greater than the number of servings of fruits. Find the number of servings of fruits and vegetables that could make a healthy diet. Use whole numbers only.

- How can you write two inequalities that model the information in the problem?
- How can you use a graph to find combinations of fruits and vegetable servings that may help in having a healthy diet?

32. **College Admissions** An entrance exam has two sections, a verbal section and a mathematics section. You can score a maximum of 1600 points. For admission, the school of your choice requires a math score of at least 600. Write a system of inequalities to model scores that meet the school's requirements. Then solve the system by graphing.

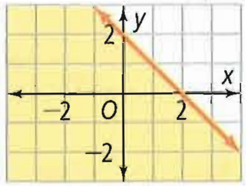
33. **Open-Ended** Write and graph a system of inequalities for which the solution is bounded by a dashed vertical line and a solid horizontal line.

34. **Writing** Explain how you determine where to shade when solving a system of inequalities.

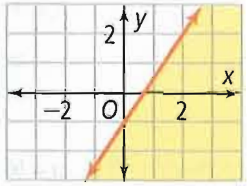
35. Given a system of two linear inequalities, explain how you can pick test points in the plane to determine where to shade the solution set.

In Exercises 36–45, identify the inequalities A, B, and C for which the given ordered pair is a solution.

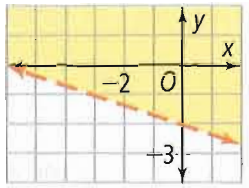
A. $x + y \leq 2$



B. $y \leq \frac{3}{2}x - 1$



C. $y > -\frac{1}{3}x - 2$



36. (0, 0)
37. (−2, −5)
38. (−2, 0)
39. (0, −2)
40. (−15, 15)
41. (3, 2)
42. (2, 0)
43. (−6, 0)
44. (4, −1)
45. (−8, −11)

Solve each system of inequalities by graphing.

46. $\begin{cases} x + y < 8 \\ x \geq 0 \\ y \geq 0 \end{cases}$

47. $\begin{cases} 2y - 4x \leq 0 \\ x \geq 0 \\ y \geq 0 \end{cases}$

48. $\begin{cases} y \geq -2x + 4 \\ x > -3 \\ y \geq 1 \end{cases}$

49. $\begin{cases} y \leq \frac{2}{3}x + 2 \\ y \geq |x| + 2 \end{cases}$

50. $\begin{cases} y < x - 1 \\ y > -|x - 2| + 1 \end{cases}$

51. $\begin{cases} 2x + y \leq 3 \\ y > |x + 3| - 2 \end{cases}$

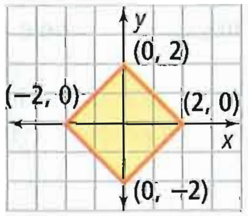
52. $\begin{cases} y < |x - 1| + 2 \\ x \geq 0 \end{cases}$

53. $\begin{cases} y > |x - 1| + 1 \\ y \leq -|x - 3| + 4 \end{cases}$

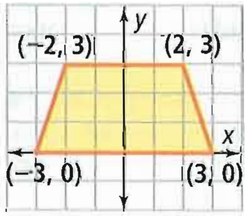
54. $\begin{cases} y \leq |x| - 2 \\ y \leq |x| + 2 \end{cases}$

Challenge **Geometry** **Write a system of inequalities to describe each shaded figure.**

55.



56.



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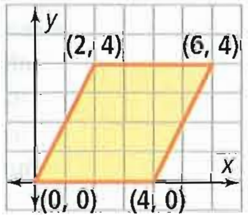
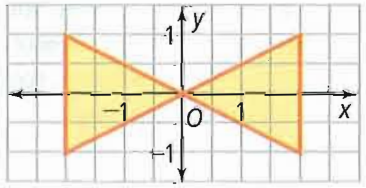
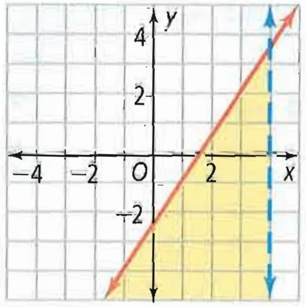


58. a. Graph the "bowtie" inequality, $|y| \leq |x|$.
b. Write a system of inequalities to describe the graph shown at the right.



Sunshine State Standards Practice

MA.912.A.3.14 **59.** Which system of inequalities is shown in the graph?



Ⓐ $\begin{cases} x \geq 4 \\ 3x - 2y > 5 \end{cases}$

Ⓑ $\begin{cases} x > 4 \\ 3x - 2y \leq 5 \end{cases}$

Ⓒ $\begin{cases} x < 4 \\ 3x - 2y \geq 5 \end{cases}$

Ⓓ $\begin{cases} x \leq 4 \\ 3x - 2y < 5 \end{cases}$

MA.912.A.3.10 **60.** What is the equation of the line that passes through the point (4, −3) and has slope $\frac{1}{2}$?

Ⓕ $y = \frac{1}{2}x + 3$

Ⓖ $y = \frac{1}{2}x - 1$

Ⓗ $y = \frac{1}{2}x - 5$

Ⓘ $y = x - \frac{1}{2}$

MA.912.A.2.10 **61.** Which equation is a vertical translation of $y = -5x$?

Ⓐ $y = -\frac{5}{2}x$

Ⓑ $y = -5x + 2$

Ⓒ $y = -10x$

Ⓓ $y = 5x - 2$

MA.912.A.3.15 **62. Short Response** The cost of renting a pool at an aquatic center is either \$30 per hour or \$20 per hour with a \$40 non-refundable deposit. For how many hours is the cost of renting a pool the same for both plans?

Mixed Review

Solve each system by elimination or substitution.

See Lesson 3-2.

63. $\begin{cases} y = 3x + 1 \\ 2x - y = 8 \end{cases}$

64. $\begin{cases} 3x + y = 4 \\ 2x - 4y = 7 \end{cases}$

65. $\begin{cases} -x + 5y = 3 \\ 2x - 10y = 4 \end{cases}$

66. $\begin{cases} 2x + 4y = -8 \\ -5x + 4y = 6 \end{cases}$

67. $\begin{cases} y - 3 = x \\ 4x + y = -2 \end{cases}$

68. $\begin{cases} 2 = 4y - 3x \\ 5x = 2y - 3 \end{cases}$

Get Ready! **To prepare for Lesson 3-4, do Exercises 69–72.**

Write an ordered pair that is a solution of each system of inequalities.

See Lesson 3-3.

69. $\begin{cases} x + y > 2 \\ 3x + 2y \leq 6 \end{cases}$

70. $\begin{cases} 2y > 4 \\ 3x + 4y \leq 14 \end{cases}$

71. $\begin{cases} x \geq 2 \\ 5x + 2y \leq 9 \end{cases}$

72. $\begin{cases} x + 3y < 6 \\ y < x \end{cases}$

3 Mid-Chapter Quiz

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

Solve each system by graphing.

1. $\begin{cases} 3x - y = 8 \\ 10 + 2y = 4x \end{cases}$

2. $\begin{cases} y + 5 = 2x \\ 3y + 6x = -3 \end{cases}$

3. $\begin{cases} 14x - 2y = 6 \\ 6y - 9x = 15 \end{cases}$

Without graphing, classify each system as *independent, dependent,* or *inconsistent.*

4. $\begin{cases} 3y + 2x = 12 \\ 36 - 9y = -6x \end{cases}$

5. $\begin{cases} -2y = 20 - 2x \\ 3y - 6x = -30 \end{cases}$

6. $\begin{cases} 15x = 10y - 20 \\ 18 + 9x = 6y \end{cases}$

Solve each system by substitution.

7. $\begin{cases} 5m - n = 7 \\ 3 + 3n = 6m \end{cases}$

8. $\begin{cases} 4y - 6 = 2x \\ y - 3x = 9 \end{cases}$

9. $\begin{cases} 3u + 8 = 4v \\ 24v = 6 - 3u \end{cases}$

Solve each system by elimination.

10. $\begin{cases} 5c - 4t = 8 \\ 14 + 4t = 3c \end{cases}$

11. $\begin{cases} 8y + 10 = 6x \\ 8y - 4x = -12 \end{cases}$

12. $\begin{cases} 11 - 2c = 3d \\ 2c - 7d = -9 \end{cases}$

Graph the solutions to each of the following systems.

13. $\begin{cases} y < 2 + 3x \\ y \geq |x - 3| \end{cases}$

14. $\begin{cases} 2x + y > 7 \\ x < 4 \\ y \leq 5 \end{cases}$

Do you UNDERSTAND?

15. Which equation below combines with the equation $-4x + 6y = 3$ to form a system with an infinite number of solutions?

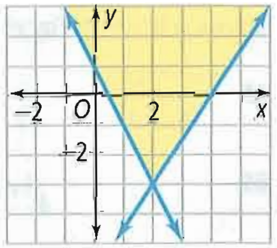
Ⓐ $0.5 + x = 1.5y$

Ⓑ $0.75 + 2x = 1.5y$

Ⓒ $0.5 + 2x = 1.5y$

Ⓓ $0.75 + x = 1.5y$

16. Write a system of inequalities to describe the shaded region.



17. An ordinary refrigerator costs \$503 and has an estimated annual operating cost of \$92. An energy-efficient model costs \$615 with an estimated annual operating cost of \$64. Write a system of equations to represent this situation.

a. What is the solution of the system?

b. What does the solution mean?

c. Which model would you choose for your family? Why?

18. **Writing** Explain how to classify a linear system as independent, dependent, or inconsistent without graphing.

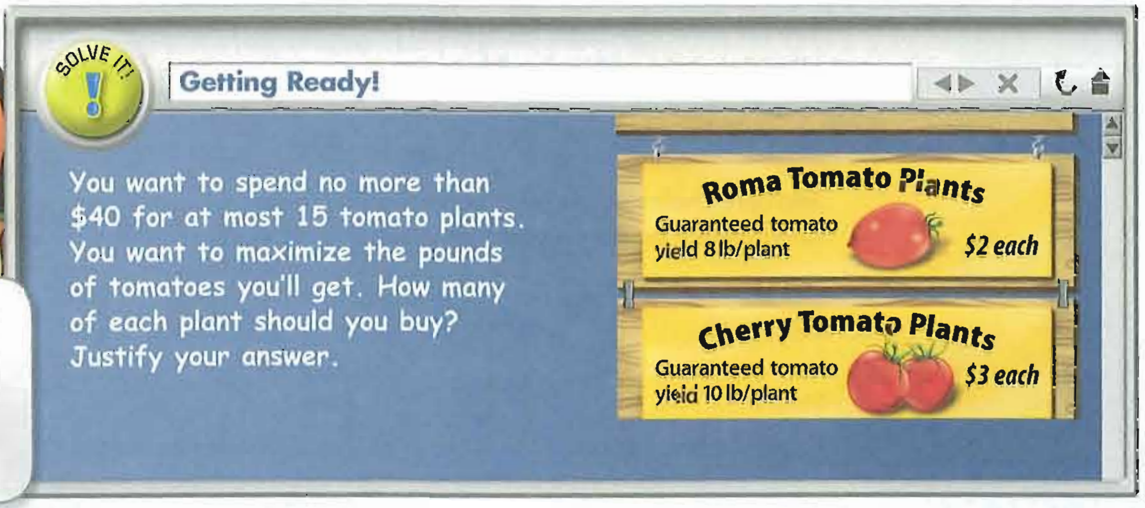
3-4 Linear Programming

Sunshine State Standards

MA.912.A.3.14 Solve systems of linear inequalities in two variables using graphical methods.

MA.912.A.3.15 Solve real-world problems involving systems of linear inequalities in two variables.

Objective To solve problems using linear programming



You want to spend no more than $40 for at most 15 tomato plants. You want to maximize the pounds of tomatoes you'll get. How many of each plant should you buy? Justify your answer.

Lesson Vocabulary

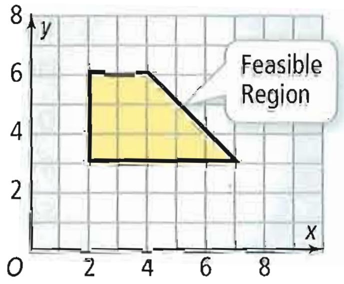
- constraint
- linear programming
- feasible region
- objective function

In the Solve It, you maximized your tomato production given some limits, or **constraints**. **Linear programming** is a method for finding a minimum or maximum value of some quantity, given a set of constraints.

Essential Understanding Some real-world problems involve multiple linear relationships. Linear programming accounts for all of these linear relationships and gives the solution to the problem.

The constraints in a linear programming situation form a system of inequalities, like the one at the right. The graph of the system is the **feasible region**. It contains all the points that satisfy all the constraints.

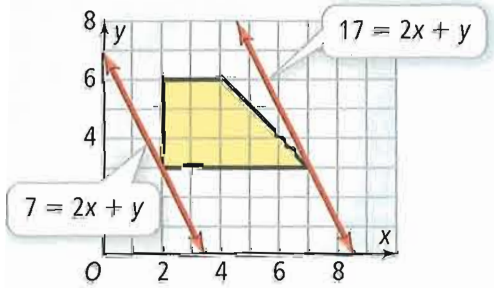
$$\begin{cases} x \geq 2 \\ y \geq 3 \\ y \leq 6 \\ x + y \leq 10 \end{cases}$$



The quantity you are trying to maximize or minimize is modeled with an **objective function**. Often this quantity is cost or profit. Suppose the objective function is $C = 2x + y$.

Graphs of the objective function for various values of C are parallel lines. Lines closer to the origin represent smaller values of C.

The graphs of the equations $7 = 2x + y$ and $17 = 2x + y$ intersect the feasible region at (2, 3) and (7, 3). These vertices of the feasible represent the least and the greatest values for the objective function.



Dynamic Activity
Linear Programming

take note

Key Concept Vertex Principle of Linear Programming

If there is a maximum or a minimum value of the linear objective function, it occurs at one or more vertices of the feasible region.

You can solve a problem using linear programming by testing in the objective function all of the vertices of the feasible region.

Problem 1 Testing Vertices

Multiple Choice What point in the feasible region maximizes P for the objective function $P = 2x + y$?

Constraints $\begin{cases} x + 2y \leq 5 \\ x - y \leq 2 \\ x \geq 0 \\ y \geq 0 \end{cases}$

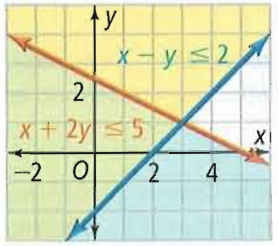
Ⓐ (2, 0)   Ⓑ (0, 0)   Ⓒ (3, 1)   Ⓓ (0, 2.5)

Think

What quadrant will the feasible region be in?
The constraints $x \geq 0$ and $y \geq 0$ indicate the first quadrant.

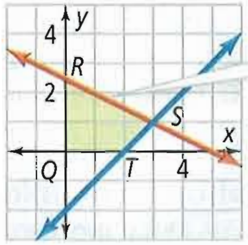
Step 1

Graph the inequalities.



Step 2

Form the feasible region.



The intersections of the boundaries are the vertices of the feasible region.

Step 3

Find the coordinates of each vertex.

$Q(0, 0)$

$R(0, 2.5)$

$S(3, 1)$

$T(2, 0)$

Step 4

Evaluate P at each vertex.

$P = 2(0) + 0 = 0$

$P = 2(0) + 2.5 = 2.5$

$P = 2(3) + 1 = 7$ Maximum Value

$P = 2(2) + 0 = 4$

P has a maximum value of 7 when $x = 3$ and $y = 1$. The correct choice is C.

Got It? **1. a.** Use the constraints in Problem 1 with the objective function $P = x + 3y$. What values of x and y maximize P?

b. **Reasoning** Can an objective function $P = ax + by + c$ have (the same) maximum value at all four vertex points Q, R, S, and T? At points R and S only? Explain using examples.

Problem 2 Using Linear Programming to Maximize Profit

Business **You are screen-printing T-shirts and sweatshirts to sell at the Polk County Blues Festival and are working with the following constraints.**

- You have at most 20 hours to make shirts.
- You want to spend no more than $600 on supplies.
- You want to have at least 50 items to sell.

How many T-shirts and how many sweatshirts should you make to maximize your profit? How much is the maximum profit?

Organize the information in a table.

	T-Shirts, x	Sweatshirts, y	Total
Minutes	$10x$	$30y$	1200
Number	x	y	50
Cost	$4x$	$20y$	600
Profit	$6x$	$20y$	$6x + 20y$

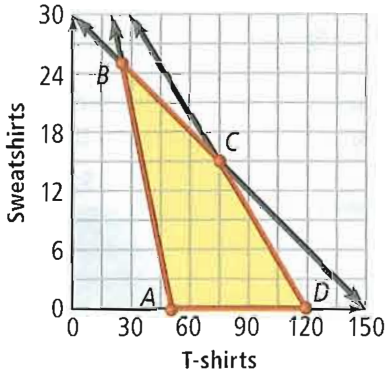
Write the constraints and the objective function.

$$\text{Constraints: } \begin{cases} 10x + 30y \leq 1200 \\ x + y \geq 50 \\ 4x + 20y \leq 600 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Objective Function: $P = 6x + 20y$

Step 1

Graph the constraints to form the feasible region.



Step 2

Find the coordinates of each vertex.

$A(50, 0)$

$B(25, 25)$

$C(75, 15)$

$D(120, 0)$

Step 3

Evaluate P.

$P = 6(50) + 20(0) = 300$

$P = 6(25) + 20(25) = 650$

$P = 6(75) + 20(15) = 750$

$P = 6(120) + 20(0) = 720$

Think

How do you find the coordinates of the vertices if they are hard to read off the graph?

Solve the system of equations related to the lines that intersect to form the vertex.

You can maximize your profit by selling 75 T-shirts and 15 sweatshirts. The maximum profit is $750.

Got It? **2.** If it took you 20 minutes to make a sweatshirt, how many of each type of shirt should you make to maximize your profit?

Lesson Check

Do you know HOW?

Graph each system of inequalities.

1. $\begin{cases} x + y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$

2. $\begin{cases} 2x - y \leq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$

3. $\begin{cases} x + 2y \leq 10 \\ x \geq 1 \\ y \geq 2 \end{cases}$

4. $\begin{cases} 2x + 3y \leq 18 \\ 0 \leq x \leq 5 \\ 0 \leq y \leq 4 \end{cases}$

Graph each system of constraints. Then name the vertices of the feasible region.

5. $\begin{cases} x \leq 5 \\ y \leq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$

6. $\begin{cases} x + y \leq 8 \\ y \geq 5 \\ x \geq 0 \end{cases}$

Do you UNDERSTAND?

7. **Vocabulary** Explain why the inequalities of a linear programming problem are called constraints. *Hint*: Use the definition of *constraint* as part of your answer.

8. **Compare and Contrast** What are some similarities between solving a linear programming problem and solving a system of linear inequalities? What are some differences?

9. **Open-Ended** Write a system of constraints whose graphs determine a trapezoid. Write an objective function and evaluate it at each vertex.

Practice and Problem-Solving Exercises

A Practice

Graph each system of constraints. Name all vertices. Then find the values of x and y that maximize or minimize the objective function.

See Problem 1.

10. $\begin{cases} x + y \leq 8 \\ 2x + y \leq 10 \\ x \geq 0 \\ y \geq 0 \end{cases}$

Maximum for
$N = 100x + 40y$

11. $\begin{cases} x + 2y \geq 8 \\ x \geq 2 \\ y \geq 0 \end{cases}$

Minimum for
$C = x + 3y$

12. $\begin{cases} 2 \leq x \leq 6 \\ 1 \leq y \leq 5 \\ x + y \leq 8 \end{cases}$

Maximum for
$P = 3x + 2y$

13. **Air Quality** A city wants to plant maple and spruce trees to absorb carbon dioxide. It has $2100 to spend on planting spruce and maple trees. The city has 45,000 ft^2 available for planting.

See Problem 2.

a. Use the data from the table. Write the constraints for the situation.

b. Write the objective function.

c. Graph the feasible region and find the vertices.

d. How many of each tree should the city plant to maximize carbon dioxide absorption?

Spruce and Maple Tree Data

	Spruce	Maple
Planting Cost	$30	$40
Area Required	600 ft^2	900 ft^2
Carbon Dioxide Absorption	650 lb/yr	300 lb/yr

SOURCE: Auburn University and Anderson Associates

B Apply

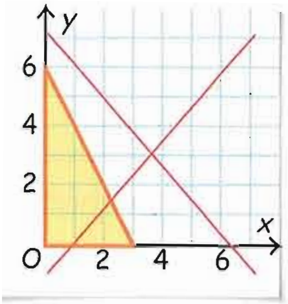
14. Think About a Plan A biologist is developing two new strains of bacteria. Each sample of Type I bacteria produces four new viable bacteria, and each sample of Type II produces three new viable bacteria. Altogether, at least 240 new viable bacteria must be produced. At least 30, but not more than 60, of the original samples must be Type I. Not more than 70 of the original samples can be Type II. A sample of Type I costs \$5 and a sample of Type II costs \$7. How many samples of Type II bacteria should the biologist use to minimize the cost?

- What are the unknowns?
- What constraints do you get from each condition in the problem?
- Are there any implicit constraints?

15. Error Analysis Your friend is trying to find the maximum value of $P = -x + 3y$ subject to the following constraints.

$$\begin{cases} y \leq -2x + 6 \\ y \leq x + 3 \\ x \geq 0, y \geq 0 \end{cases}$$

What error did your friend make? What is the correct solution?



16. Cooking Baking a tray of corn muffins takes 4 cups of milk and 3 cups of wheat flour. Baking a tray of bran muffins takes 2 cups of milk and 3 cups of wheat flour. A baker has 16 cups of milk and 15 cups of wheat flour. He makes \$3 profit per tray of corn muffins and \$2 profit per tray of bran muffins. How many trays of each type of muffin should the baker make to maximize his profit?

Graph each system of constraints. Name all vertices. Then find the values of x and y that maximize or minimize the objective function. Find the maximum or minimum value.

17. $\begin{cases} 3x + y \leq 7 \\ x + 2y \leq 9 \\ x \geq 0, y \geq 0 \end{cases}$

Maximum for
$P = 2x + y$

18. $\begin{cases} 25 \leq x \leq 75 \\ y \leq 110 \\ 8x + 6y \geq 720 \end{cases}$

Minimum for
$C = 8x + 5y$

19. $\begin{cases} x + y \leq 11 \\ 2y \geq x \\ x \geq 0, y \geq 0 \end{cases}$

Maximum for
$P = 3x + 2y$

20. $\begin{cases} 2x + y \leq 300 \\ x + y \leq 200 \\ x \geq 0, y \geq 0 \end{cases}$

Maximum for
$P = x + 2y$

21. $\begin{cases} 5x + y \geq 10 \\ x + y \leq 6 \\ x + 4y \geq 12 \\ x \geq 0, y \geq 0 \end{cases}$

Minimum for
$C = 10{,}000x + 20{,}000y$

22. $\begin{cases} 6 \leq x + y \leq 13 \\ x \geq 3 \\ y \geq 1 \end{cases}$

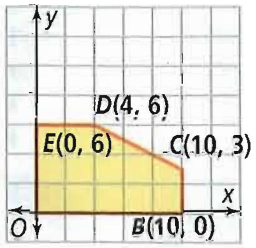
Maximum for
$P = 4x + 3y$

Challenge

23. A vertex of a feasible region does not always have whole-number coordinates. Sometimes you may need to round coordinates to find the solution. Using the objective function and the constraints at the right, find the whole-number values of x and y that minimize C. Then find C for those values of x and y.

$$C = 6x + 9y$$
$$\begin{cases} x + 2y \geq 50 \\ 2x + y \geq 60 \\ x \geq 0, y \geq 0 \end{cases}$$

24. Reasoning Sometimes two corners of a graph both yield the maximum profit. In this case, many other points may also yield the maximum profit. Evaluate the profit formula $P = x + 2y$ for the graph shown. Find four points that yield the maximum profit.

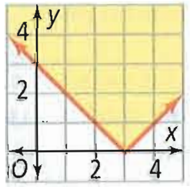


Sunshine State Standards Practice

MA.912.A.3.3 **25.** Solve the equation $\frac{1}{2}(a + b) = c$ for b.

Ⓐ $b = \frac{1}{2}c - a$ Ⓑ $b = 2a - c$ Ⓒ $b = 2c - a$ Ⓓ $b = 2ca$

MA.912.A.2.5 **26.** Which is the graph of $y \leq |x - 3|$?

Ⓕ


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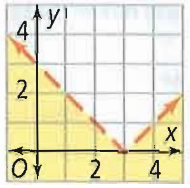

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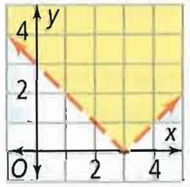

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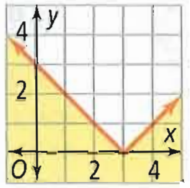


MA.912.A.3.15 **27. Short Response** What are the vertices of the feasible region bounded by the constraints at the right?

$$\begin{cases} x + y \leq 3 \\ 2x + y \leq 4 \\ x \geq 0, y \geq 0 \end{cases}$$

Mixed Review

Solve each system of inequalities by graphing. See Lesson 3-3.

28. $\begin{cases} y < -2x + 8 \\ 3y \geq 4x - 6 \end{cases}$ **29.** $\begin{cases} x - 2y \geq 11 \\ 5x + 4y < 27 \end{cases}$ **30.** $\begin{cases} 2x + 6y > 12 \\ 3x + 9y \leq 27 \end{cases}$

Evaluate each expression for $a = 3$ and $b = -5$. See Lesson 1-3.

31. $2a + b$ **32.** $-4 + 2ab$ **33.** $3(a - b)$ **34.** $b(2b - a)$

Get Ready! To prepare for Lesson 3-5, do Exercises 35–37.

Find the x- and y-intercepts of the graph of each linear equation. See Lesson 2-4.

35. $y = 2x + 6$ **36.** $2x + 9y = 36$ **37.** $y = x - 1$

Concept Byte

For Use With Lesson 3-4

TECHNOLOGY

Linear Programming

Sunshine State Standards
MA.912.A.3.14 Solve systems of linear inequalities in two variables using graphical methods.

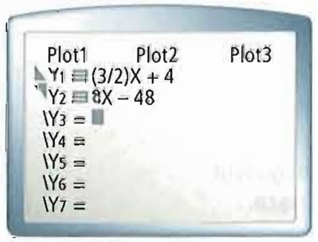
You can solve linear programming problems using your graphing calculator.

Activity

Find the values of x and y that will maximize the objective function $P = 13x + 2y$ for the constraints at the right. What is the value of P at this maximum point?

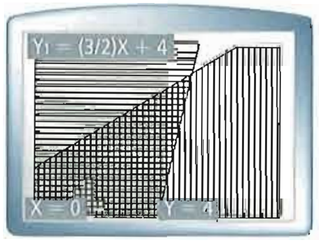
$$\begin{cases} -3x + 2y \leq 8 \\ -8x + y \geq -48 \\ x \geq 0, y \geq 0 \end{cases}$$

Step 1 Rewrite the first two inequalities to isolate y. Enter the inequalities.

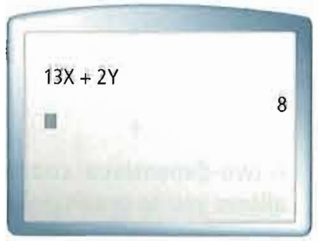


```
Plot1   Plot2   Plot3
\Y1 ≡(3/2)X + 4
\Y2 ≡8X – 48
\Y3 = ▮
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

Step 2 Use the **VALUE** option of **CALC** to find the upper left vertex. Press 0 enter.

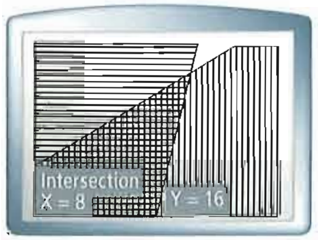


Step 3 Enter the objective function on the home screen. Press enter for the value of P at the vertex.

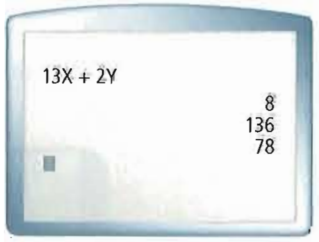


```
13X + 2Y
                8
▮
```

Step 4 Use the **INTERSECT** option of **CALC** to find the upper right vertex. Go to the home screen and press enter for the value of P.



Step 5 Use the **ZERO** option of **CALC** to find the lower right vertex. Go to the home screen and press enter for the value of P. The objective function has a value of 0 when the vertex is at the origin. The maximum value of P is 136.



```
13X + 2Y
                8
              136
               78
▮
```

Exercises

Find the values of x and y that maximize or minimize the objective function.

1. $\begin{cases} 4x + 3y \geq 30 \\ x + 3y \geq 21 \\ x \geq 0, y \geq 0 \end{cases}$

Minimum for $C = 5x + 8y$

2. $\begin{cases} 3x + 5y \geq 35 \\ 2x + y \leq 14 \\ x \geq 0, y \geq 0 \end{cases}$

Maximum for $P = 3x + 2y$

3. $\begin{cases} x + y \geq 8 \\ x + 5y \geq 20 \\ x \geq 0, y \geq 2 \end{cases}$

Minimum for $C = 3x + 4y$

4. $\begin{cases} x + 2y \leq 24 \\ 3x + 2y \leq 34 \\ 3x + y \leq 29 \\ x \geq 0 \end{cases}$

Maximum for $P = 2x + 3y$

Concept Byte

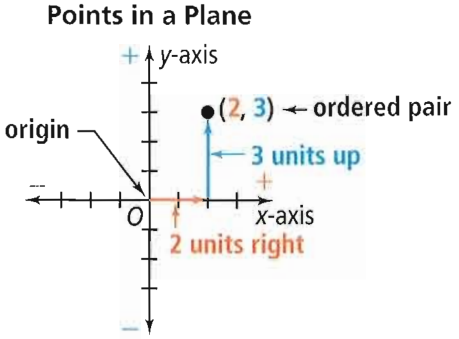
For Use With Lesson 3-5

ACTIVITY

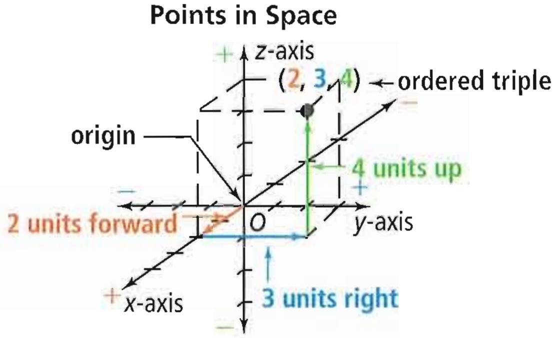
Graphs in Three Dimensions

Sunshine State Standards
Prepares for MA.912.A.3.14 Solve systems of linear equations in three variables.

To describe positions in space, you need a three-dimensional coordinate system. You have learned to graph on an *xy*-coordinate plane using ordered pairs. Adding a third axis, the *z*-axis, to the *xy*-coordinate plane creates **coordinate space**. In coordinate space you graph points using **ordered triples** of the form (*x*, *y*, *z*).


Points in a Plane

A two-dimensional coordinate system allows you to graph points in a plane.


Points in Space

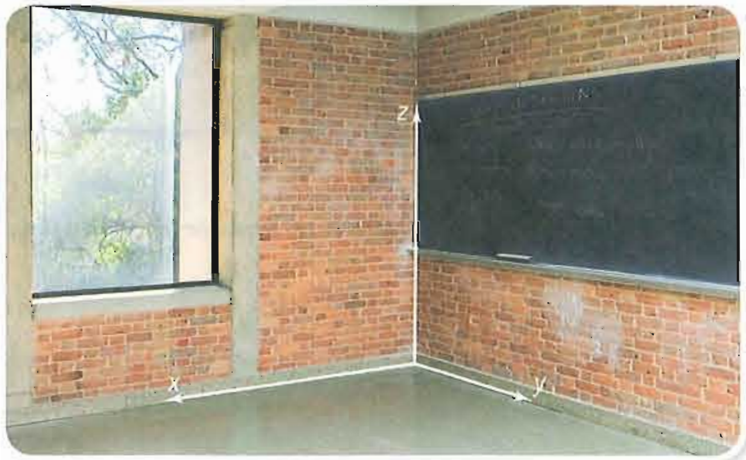
A three-dimensional coordinate system allows you to graph points in space.

In the coordinate plane, point (2, 3) is two units right and three units up from the origin. In coordinate space, point (2, 3, 4) is two units forward, three units right, and four units up.

Activity 1

Define one corner of your classroom as the origin of a three-dimensional coordinate system like the classroom shown. Write the coordinates of each item in your coordinate system.

1. each corner of your classroom
2. each corner of your desk
3. one corner of the blackboard
4. the clock
5. the waste-paper basket
6. Pick 3 items in your classroom and write the coordinates of each.



An equation in two variables represents a line in a plane. An equation in three variables represents a plane in space.

Activity 2

Given the following equation in three variables, draw the plane in a coordinate space. $x + 2y - z = 6$

7. Let $x = 0$. Graph the resulting equation in the *yz* plane.

8. Let $y = 0$. Graph the resulting equation in the *xz* plane.

From geometry you know that two lines determine a plane.

9. Sketch the plane $x + 2y - z = 6$.
(If you need help, find a third line by letting $z = 0$ and then graph the resulting equation in the *xy* plane.)

Activity 3

Two equations in three variables represent two planes in space.

10. Draw the two planes determined by the following equations:
$2x + 3y - z = 12$
$2x - 4y + z = 8$

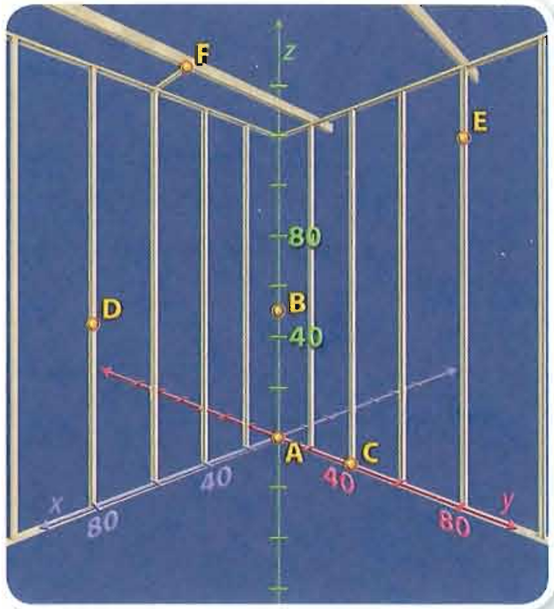
11. Describe the intersection of the two planes above.

Exercises

Find the coordinates of each point in the diagram.

12. *A* **13.** *B* **14.** *C*

15. *D* **16.** *E* **17.** *F*



Sketch the graph of each equation.

18. $x - y - 4z = 8$

19. $x + y + z = 2$

20. $-3x + 5y + 10z = 15$

21. $6x + 6y - 12z = 36$

Graph the following pairs of equations in the same coordinate space and describe their intersection, if any.

22. $-x + 3y + z = 6$
$-3x + 5y - 2z = 60$

23. $-2x - 3y + 5z = 7$
$2x - 3y - 4z = -4$

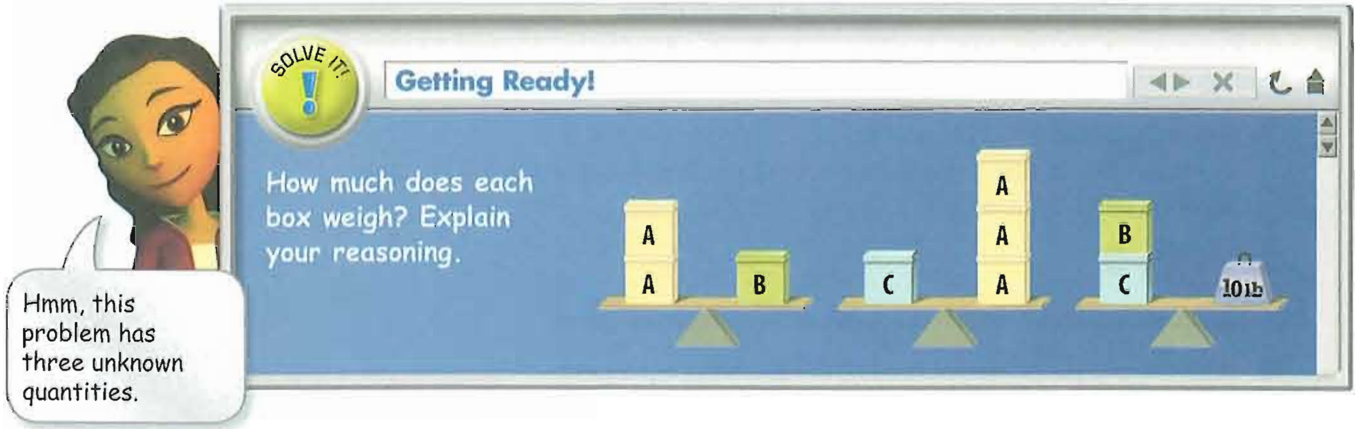
3-5 Systems With Three Variables

Sunshine State Standards

MA.912.A.3.14 Solve systems of linear equations in three variables using substitution and elimination methods.

MA.912.A.3.15 Solve real-world problems involving systems of linear equations in three variables.

Objectives To solve systems in three variables using elimination
To solve systems in three variables using substitution

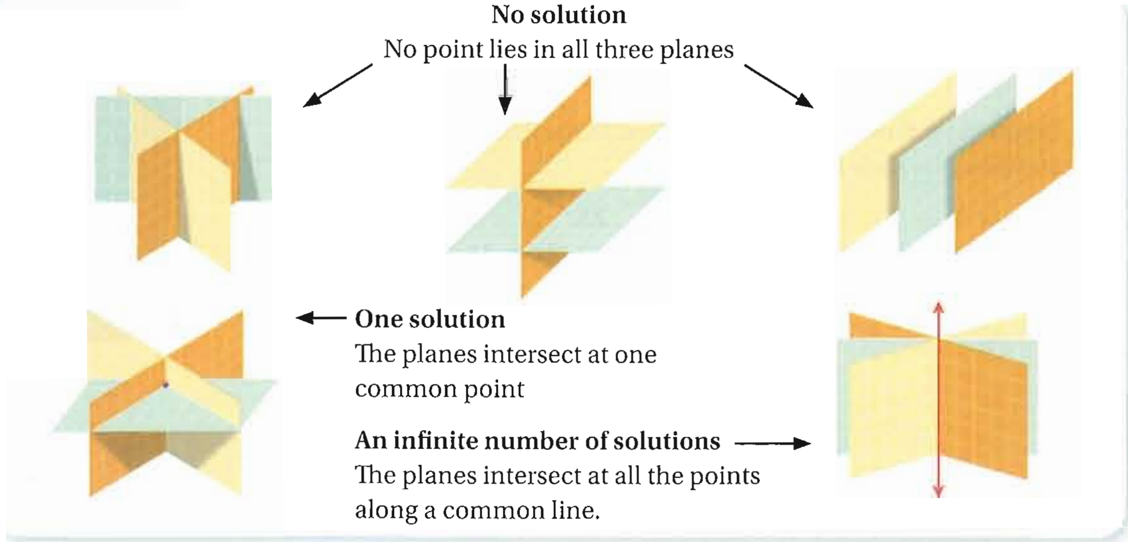


How much does each box weigh? Explain your reasoning.

You can represent three relationships involving three unknowns with a system of equations.

Essential Understanding To solve systems of three equations in three variables, you can use some of the same algebraic methods you used to solve systems of two equations in two variables.

You can represent systems of equations in three variables as graphs in three dimensions. The graph of an equation of the form $Ax + By + Cz = D$, where A, B, and C are not all zero, is a plane. You can show the solutions of a three-variable system graphically as the intersection of planes.



No solution
No point lies in all three planes

One solution
The planes intersect at one common point

An infinite number of solutions
The planes intersect at all the points along a common line.

You can use the elimination and substitution methods to solve a system of three equations in three variables by working with the equations in pairs. You will use one of the equations *twice*. When one point represents the solution of a system of equations in three variables, write it as an ordered triple (x, y, z).

Problem 1 Solving a System Using Elimination

What is the solution of the system? Use elimination. The equations are numbered to make the procedure easy to follow.

$$\begin{array}{r} ① \\ ② \\ ③ \end{array}\begin{cases} 2x - y + z = 4 \\ x + 3y - z = 11 \\ 4x + y - z = 14 \end{cases}$$

Think

Which variable do you eliminate first?
Eliminate the variable for which the process requires the fewest steps.

Step 1 Pair the equations to eliminate z. Then you will have two equations in x and y.

Add.

$$\begin{array}{r} ① \\ ② \\ \\ ④ \end{array}\begin{array}{rcl} 2x - y + z &=& 4 \\ x + 3y - z &=& 11 \\ \hline 3x + 2y &=& 15 \end{array}$$

Subtract.

$$\begin{array}{r} ② \\ ③ \\ \\ ⑤ \end{array}\begin{array}{rcl} x + 3y - z &=& 11 \\ 4x + y - z &=& 14 \\ \hline -3x + 2y &=& -3 \end{array}$$

Step 2 Write the two new equations as a system. Solve for x and y.

Add and solve for y.

$$\begin{array}{r} ④ \\ ⑤ \\ \\ \\ \end{array}\begin{array}{rcl} 3x + 2y &=& 15 \\ -3x + 2y &=& -3 \\ \hline 4y &=& 12 \\ y &=& 3 \end{array}$$

Substitute $y = 3$ and solve for x.

$$\begin{array}{rrcl} ④ & 3x + 2y &=& 15 \\ & 3x + 2(3) &=& 15 \\ & 3x &=& 9 \\ & x &=& 3 \end{array}$$

Think

Does it matter which equation you substitute into to find z?
No, you can substitute into any of the original three equations.

Step 3 Solve for z. Substitute the values of x and y into one of the original equations.

$$\begin{array}{rrcll} ① & 2x - y + z &=& 4 & \text{Use equation ①.} \\ & 2(3) - 3 + z &=& 4 & \text{Substitute.} \\ & 6 - 3 + z &=& 4 & \text{Simplify.} \\ & z &=& 1 & \text{Solve for } z. \end{array}$$

Step 4 Write the solution as an ordered triple. The solution is $(3, 3, 1)$.

Got It? 1. What is the solution of the system? Use elimination. Check your answer in all three original equations.

$$\begin{cases} x - y + z = -1 \\ x + y + 3z = -3 \\ 2x - y + 2z = 0 \end{cases}$$

You can apply the method in Problem 1 to most systems of three equations in three variables. You may need to multiply in one, two, or all three equations by one, two, or three nonzero numbers. Your goal is to obtain a system—equivalent to the original system—with coefficients that allow for the easy elimination of variables.

Problem 2 Solving an Equivalent System

What is the solution of the system? Use elimination.

$$\begin{array}{r} \text{①}\\ \text{②}\\ \text{③} \end{array}\begin{cases} x + y + 2z = 3\\ 2x + y + 3z = 7\\ -x - 2y + z = 10 \end{cases}$$

Think

You are trying to get two equations in x and z. Multiply ① so you can add it to ② and eliminate y. Do the same with ② and ③.

Write

$$\begin{array}{r} \text{①}\\ \text{②} \end{array}\begin{cases} x + y + 2z = 3\\ 2x + y + 3z = 7 \end{cases} \longrightarrow \begin{array}{rr} & -x - y - 2z = -3\\ & \underline{2x + y + 3z = 7}\\ \text{④} & x + z = 4 \end{array}$$

$$\begin{array}{r} \text{②}\\ \text{③} \end{array}\begin{cases} 2x + y + 3z = 7\\ -x - 2y + z = 10 \end{cases} \longrightarrow \begin{array}{rr} & 4x + 2y + 6z = 14\\ & \underline{-x - 2y + z = 10}\\ \text{⑤} & 3x + 7z = 24 \end{array}$$

Multiply ④ so you can add it to ⑤ and eliminate x.

$$\begin{array}{r} \text{④}\\ \text{⑤} \end{array}\begin{cases} x + z = 4\\ 3x + 7z = 24 \end{cases} \longrightarrow \begin{array}{r} -3x - 3z = -12\\ \underline{3x + 7z = 24}\\ 4z = 12\\ z = 3 \end{array}$$

Substitute $z = 3$ into ④. Solve for x.

$$\begin{aligned} x + 3 &= 4\\ x &= 1 \end{aligned}$$

Substitute the values for x and z into ① to find y. Check the answer in the three original equations.

$$\begin{aligned} x + y + 2z &= 3\\ 1 + y + 2(3) &= 3\\ y &= -4 \end{aligned}$$

Check

$$\begin{aligned} 1 + (-4) + 2(3) &= 3 \ \checkmark\\ 2(1) + (-4) + 3(3) &= 7 \ \checkmark\\ -(1) - 2(-4) + 3 &= 10 \ \checkmark \end{aligned}$$

The solution is $(1, -4, 3)$.

Got It? **2. a.** What is the solution of the system? Use elimination.

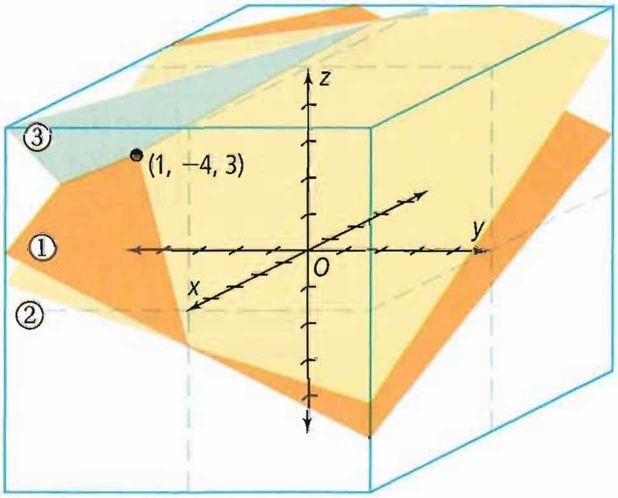
b. **Reasoning** Could you have used elimination in another way? Explain.

$$\begin{cases} x - 2y + 3z = 12\\ 2x - y - 2z = 5\\ 2x + 2y - z = 4 \end{cases}$$

Here is a graphical representation of the solution of Problem 2. The graphs are enclosed in a 10-by-10-by-10 cube with the origin of the coordinate axes at the center.

The graphs of equations ①, ②, and ③, are planes ①, ②, and ③, respectively.

Each pair of planes intersects in a line. The three lines intersect in $(1, -4, 3)$, the solution of the system.



You can also use substitution to solve a system of three equations. Substitution is the best method to use when you can easily solve one of the equations for a single variable.

Problem 3 Solving a System Using Substitution

Multiple Choice What is the x-value in the solution of the system?

$$\begin{array}{l} ① \\ ② \\ ③ \end{array}\begin{cases} 2x + 3y - 2z = -1 \\ x + 5y = 9 \\ 4z - 5x = 4 \end{cases}$$

Ⓐ 1

Ⓑ 4

Ⓒ 6

Ⓓ 10

Think

Which equation should you solve for one of its variables?
Look for an equation that has a variable with coefficient 1.

Step 1 Choose equation ②. Solve for x.

$$\begin{aligned} ② \quad x + 5y &= 9 \\ x &= 9 - 5y \end{aligned}$$

Step 2 Substitute the expression for x into equations ① and ③ and simplify.

$$\begin{aligned} ① \quad 2x + 3y - 2z &= -1 \\ 2(9 - 5y) + 3y - 2z &= -1 \\ 18 - 10y + 3y - 2z &= -1 \\ 18 - 7y - 2z &= -1 \\ ④ \quad -7y - 2z &= -19 \end{aligned}$$

$$\begin{aligned} ③ \quad 4z - 5x &= 4 \\ 4z - 5(9 - 5y) &= 4 \\ 4z - 45 + 25y &= 4 \\ 4z + 25y &= 49 \\ ⑤ \quad 25y + 4z &= 49 \end{aligned}$$

Step 3 Write the two new equations as a system. Solve for y and z.

$$\begin{array}{l} ④ \\ ⑤ \end{array}\begin{cases} -7y - 2z = -19 \\ 25y + 4z = 49 \end{cases}$$

$$\begin{aligned} -14y - 4z &= -38 && \text{Multiply by 2.} \\ \underline{25y + 4z} &\underline{= 49} && \text{Then add.} \\ 11y &= 11 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} ④ \quad -7y - 2z &= -19 \\ -7(1) - 2z &= -19 && \text{Substitute the value of } y \text{ into ④.} \\ -2z &= -12 \\ z &= 6 \end{aligned}$$

Step 4 Use one of the original equations to solve for x.

$$\begin{aligned} ② \quad x + 5y &= 9 \\ x + 5(1) &= 9 && \text{Substitute the value of } y \text{ into ②.} \\ x &= 4 \end{aligned}$$

The solution of the system is (4, 1, 6), and $x = 4$.

The correct answer is B.

Got It? **3. a.** What is the solution of the system? Use substitution.

$$\begin{cases} x - 2y + z = -4 \\ -4x + y - 2z = 1 \\ 2x + 2y - z = 10 \end{cases}$$

b. Reasoning In Problem 3, was it necessary to find the value of z to solve the problem? Explain.

Problem 4 Solving a Real-World Problem

Business You manage a clothing store and budget $6000 to restock 200 shirts. You can buy T-shirts for $12 each, polo shirts for $24 each, and rugby shirts for $36 each. If you want to have twice as many rugby shirts as polo shirts, how many of each type of shirt should you buy?

Think

How many unknowns are there?
There are three unknowns: the number of each type of shirt.

Relate T-shirts + polo shirts + rugby shirts = 200
rugby shirts = 2 · polo shirts
12 · T-shirts + 24 · polo shirts + 36 · rugby shirts = 6000

Define
Let x = the number of T-shirts.
Let y = the number of polo shirts.
Let z = the number of rugby shirts.

Write

$$\begin{cases} ① & x + y + z = 200 \\ ② & z = 2 \cdot y \\ ③ & 12 \cdot x + 24 \cdot y + 36 \cdot z = 6000 \end{cases}$$

Step 1 Since 12 is a common factor of all the terms in equation ③, write a simpler equivalent equation.

$$\begin{cases} ③ & 12x + 24y + 36z = 6000 \\ ④ & x + 2y + 3z = 500 \end{cases} \quad \text{Divide by 12.}$$

Step 2 Substitute $2y$ for z in equations ① and ④. Simplify to find equations ⑤ and ⑥.

$$\begin{aligned} ① \quad x + y + z &= 200 \\ x + y + (2y) &= 200 \\ ⑤ \quad x + 3y &= 200 \end{aligned}$$

$$\begin{aligned} ④ \quad x + 2y + 3z &= 500 \\ x + 2y + 3(2y) &= 500 \\ ⑥ \quad x + 8y &= 500 \end{aligned}$$

Step 3 Write ⑤ and ⑥ as a system. Solve for x and y.

$$\begin{cases} ⑤ & x + 3y = 200 \\ ⑥ & x + 8y = 500 \end{cases}$$

$$\begin{aligned} -x - 3y &= -200 \quad \text{Multiply by } -1. \\ x + 8y &= 500 \quad \text{Then add.} \\ \hline 5y &= 300 \\ y &= 60 \end{aligned}$$

$$\begin{aligned} ⑤ \quad x + 3y &= 200 \\ x + 3(60) &= 200 \quad \text{Substitute the value of } y \text{ into ⑤.} \\ x &= 20 \end{aligned}$$

Step 4 Substitute the value of y in ② and solve for z.

$$\begin{aligned} ② \; z &= 2y \\ z &= 2(60) = 120 \end{aligned}$$

You should buy 20 T-shirts, 60 polo shirts, and 120 rugby shirts.

Got It? **4.** Suppose you want to have the same number of T-shirts as polo shirts. Buying 200 shirts with a budget of $5400, how many of each shirt should you buy?

Lesson Check

Do you know HOW?

Solve each system.

1. $\begin{cases} 2y - 3z = 0 \\ x + 3y = -4 \\ 3x + 4y = 3 \end{cases}$

2. $\begin{cases} 3x + y - 2z = 22 \\ x + 5y + z = 4 \\ x = -3z \end{cases}$

3. $\begin{cases} 2x + 3y - 2z = 1 \\ -x - y + 2z = 5 \\ 3x + 2y - 3z = -6 \end{cases}$

4. $\begin{cases} 2x - y + z = -2 \\ x + 3y - z = 10 \\ x + 2z = -8 \end{cases}$

Do you UNDERSTAND?

5. Reasoning How do you decide whether substitution is the best method to solve a system in three variables?

6. Error Analysis A classmate says that the system consisting of $x = 0$, $y = 0$, and $z = 0$ has no solution. Explain the student's error.

7. Writing How many solutions does this system have? Explain your answer in terms of intersecting planes. (*Hint:* Is the system dependent? inconsistent?)

$$\begin{array}{l} \text{①} \\ \text{②} \\ \text{③} \end{array}\begin{cases} 2x - 3y + z = 5 \\ 2x - 3y + z = -2 \\ -4x + 6y - 2z = 10 \end{cases}$$

8. The graph of a system is shown. How many solutions does this system have? Explain.

Practice and Problem-Solving Exercises

A Practice

Solve each system by elimination. Check your answers.

See Problems 1 and 2.

9. $\begin{cases} x - y + z = -1 \\ x + y + 3z = -3 \\ 2x - y + 2z = 0 \end{cases}$

10. $\begin{cases} x - y - 2z = 4 \\ -x + 2y + z = 1 \\ -x + y - 3z = 11 \end{cases}$

11. $\begin{cases} -2x + y - z = 2 \\ -x - 3y + z = -10 \\ 3x + 6z = -24 \end{cases}$

12. $\begin{cases} a + b + c = -3 \\ 3b - c = 4 \\ 2a - b - 2c = -5 \end{cases}$

13. $\begin{cases} 6q - r + 2s = 8 \\ 2q + 3r - s = -9 \\ 4q + 2r + 5s = 1 \end{cases}$

14. $\begin{cases} x - y + 2z = -7 \\ y + z = 1 \\ x = 2y + 3z \end{cases}$

15. $\begin{cases} 3x + 3y + 6z = 9 \\ 2x + y + 3z = 7 \\ x + 2y - z = -10 \end{cases}$

16. $\begin{cases} 3x - y + z = 3 \\ x + y + 2z = 4 \\ x + 2y + z = 4 \end{cases}$

17. $\begin{cases} x - 2y + 3z = 12 \\ 2x - y - 2z = 5 \\ 2x + 2y - z = 4 \end{cases}$

18. $\begin{cases} x + 2y = 2 \\ 2x + 3y - z = -9 \\ 4x + 2y + 5z = 1 \end{cases}$

19. $\begin{cases} 3x + 2y + 2z = -2 \\ 2x + y - z = -2 \\ x - 3y + z = 0 \end{cases}$

20. $\begin{cases} x + 4y - 5z = -7 \\ 3x + 2y + 3z = 7 \\ 2x + y + 5z = 8 \end{cases}$

Solve each system by substitution. Check your answers. **See Problems 3 and 4.**

21. $\begin{cases} x + 2y + 3z = 6 \\ y + 2z = 0 \\ z = 2 \end{cases}$

22. $\begin{cases} 3a + b + c = 7 \\ a + 3b - c = 13 \\ b = 2a - 1 \end{cases}$

23. $\begin{cases} 5r - 4s - 3t = 3 \\ t = s + r \\ r = 3s + 1 \end{cases}$

24. $\begin{cases} 13 = 3x - y \\ 4y - 3x + 2z = -3 \\ z = 2x - 4y \end{cases}$

25. $\begin{cases} x + 3y - z = -4 \\ 2x - y + 2z = 13 \\ 3x - 2y - z = -9 \end{cases}$

26. $\begin{cases} x - 4y + z = 6 \\ 2x + 5y - z = 7 \\ 2x - y - z = 1 \end{cases}$

27. $\begin{cases} x - y + 2z = 7 \\ 2x + y + z = 8 \\ x - z = 5 \end{cases}$

28. $\begin{cases} x + y + z = 2 \\ x + 2z = 5 \\ 2x + y - z = -1 \end{cases}$

29. $\begin{cases} 5x - y + z = 4 \\ x + 2y - z = 5 \\ 2x + 3y - 3z = 5 \end{cases}$

30. Manufacturing In a factory there are three machines, *A*, *B*, and *C*. When all three machines are working, they produce 287 bolts per hour. When only machines *A* and *C* are working, they produce 197 bolts per hour. When only machines *A* and *B* are working, they produce 202 bolts per hour. How many bolts can each machine produce per hour?

B Apply

31. Think About a Plan In triangle *PQR*, the measure of angle *Q* is three times the measure of angle *P*. The measure of angle *R* is 20°more than the measure of angle *P*. Find the measure of each angle.
- What are the unknowns in this problem?
- What system of equations represents this situation?
- Which method of solving looks easier for this problem?

32. Sports A stadium has 49,000 seats. Seats sell for $25 in Section A, $20 in Section B, and $15 in Section C. The number of seats in Section A equals the total number of seats in Sections B and C. Suppose the stadium takes in $1,052,000 from each sold-out event. How many seats does each section hold?

Solve each system using any method.

33. $\begin{cases} x - 3y + 2z = 11 \\ -x + 4y + 3z = 5 \\ 2x - 2y - 4z = 2 \end{cases}$

34. $\begin{cases} x + 2y + z = 4 \\ 2x - y + 4z = -8 \\ -3x + y - 2z = -1 \end{cases}$

35. $\begin{cases} 4x - y + 2z = -6 \\ -2x + 3y - z = 8 \\ 2y + 3z = -5 \end{cases}$

36. $\begin{cases} 4a + 2b + c = 2 \\ 5a - 3b + 2c = 17 \\ a - 5b = 3 \end{cases}$

37. $\begin{cases} 4x - 2y + 5z = 6 \\ 3x + 3y + 8z = 4 \\ x - 5y - 3z = 5 \end{cases}$

38. $\begin{cases} 2\ell + 2w + h = 72 \\ \ell = 3w \\ h = 2w \end{cases}$

39. $\begin{cases} 6x + y - 4z = -8 \\ \frac{y}{4} - \frac{z}{6} = 0 \\ 2x - z = -2 \end{cases}$

40. $\begin{cases} 4y + 2x = 6 - 3z \\ x + z - 2y = -5 \\ x - 2z = 3y - 7 \end{cases}$

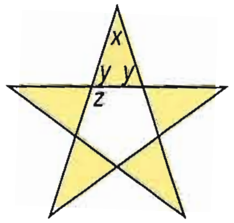
41. $\begin{cases} 4x - y + z = -5 \\ -x + y - z = 5 \\ 2x - z - 1 = y \end{cases}$

42. Finance A worker received a $10,000 bonus and decided to split it among three different accounts. He placed part in a savings account paying 4.5% per year, twice as much in government bonds paying 5%, and the rest in a mutual fund that returned 4%. His income from these investments after one year was $455. How much did the worker place in each account?

Challenge

43. **Open-Ended** Write your own system with three variables. Begin by choosing the solution. Then write three equations that are true for your solution. Use elimination to solve the system.

44. **Geometry** Refer to the regular five-pointed star at the right. Write and solve a system of three equations to find the measure of each labeled angle.



45. **Geometry** In the regular polyhedron described below, all faces are congruent polygons. Use a system of three linear equations to find the numbers of vertices, edges, and faces.

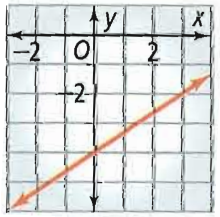
Every face has five edges and every edge is shared by two faces. Every face has five vertices and every vertex is shared by three faces. The sum of the number of vertices and faces is two more than the number of edges.

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.3.14 **46.** What is the value of z in the solution of the system? $\begin{cases} y = -2x + 10 \\ -x + y - 2z = -2 \\ 3x - 2y + 4z = 7 \end{cases}$

MA.912.A.2.10 **47.** What is the x-intercept of the line at the right after it is translated up 3 units?



MA.912.A.2.12 **48.** Suppose y varies directly with x, and $y = 15$ when $x = 10$. What is y when $x = 22$?

MA.912.A.3.15 **49.** A theater has 490 seats. Seats sell for $25 on the floor, $20 in the mezzanine, and $15 in the balcony. The number of seats on the floor equals the total number of seats in the mezzanine and balcony. Suppose the theater takes in $10,520 from each sold-out event. How many seats does the mezzanine section hold?

Mixed Review

50. Maximize the objective function $P = x + 3y$ under the given constraints. At what vertex does this maximum value occur? $\begin{cases} x + y \le 5 \\ x + 2y \le 8 \\ x \ge 0, y \ge 0 \end{cases}$ **See Lesson 3-4.**

Solve each inequality. Graph the solution on a number line. **See Lesson 1-5.**

51. $-4x + 3 \le 9$

52. $-(x + 4) - 3 \ge 11$

53. $2(3x - 1) < x - 7$

Get Ready! **To prepare for Lesson 3-6, do Exercises 54–56.**

Solve each system using elimination. **See Lesson 3-2.**

54. $\begin{cases} x + 4y = 12 \\ 2x - 8y = 4 \end{cases}$

55. $\begin{cases} 4x + 8y = -6 \\ 6x + 12y = -9 \end{cases}$

56. $\begin{cases} 4y - 2x = 6 \\ 8y = 4x - 12 \end{cases}$

3-6 Solving Systems Using Matrices

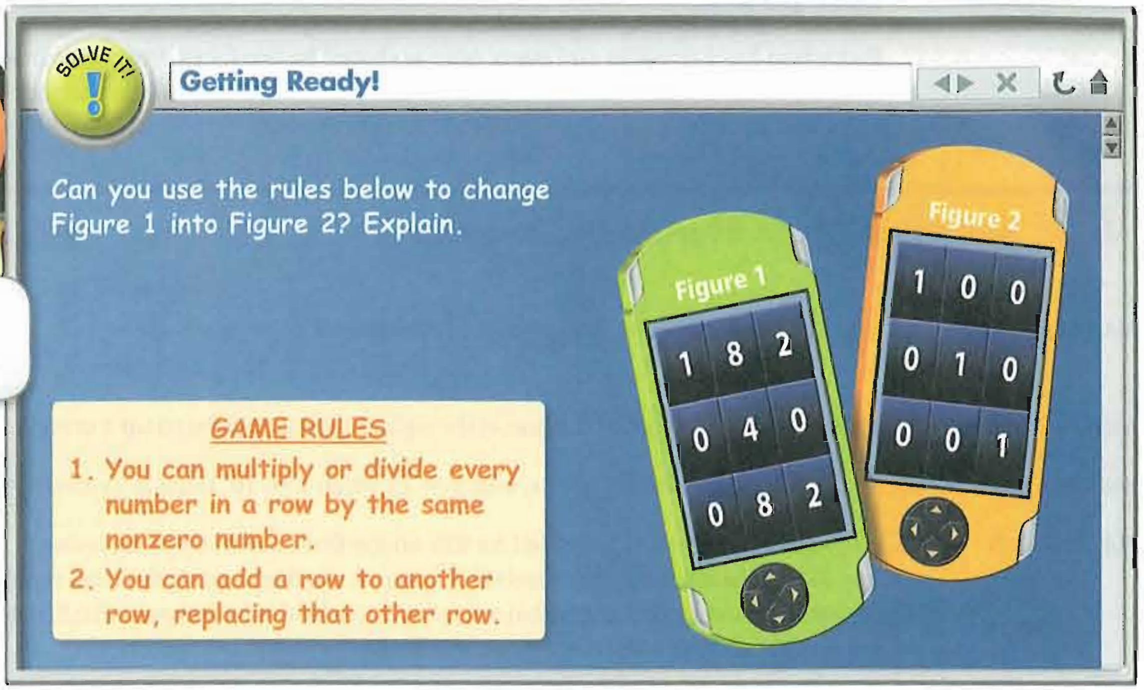
Sunshine State Standards

MA.912.A.3.14 Solve systems of linear equations in two and three variables.

MA.912.A.3.15 Solve real-world problems involving systems of linear equations in two and three variables.

Objectives To represent a system of linear equations with a matrix

To solve a system of linear equations using matrices



Can you use the rules below to change Figure 1 into Figure 2? Explain.

GAME RULES

1. You can multiply or divide every number in a row by the same nonzero number.
2. You can add a row to another row, replacing that other row.

Lesson Vocabulary

- matrix
- matrix element
- row operation

An array of numbers, such as each of those suggested by the tile arrangements in the Solve It, is a matrix.

Essential Understanding You can use a *matrix* to represent and solve a system of equations without writing the variables.

A **matrix** is a rectangular array of numbers. You usually display the array within brackets. The dimensions of a matrix are the numbers of rows and columns in the array.

$$\begin{array}{cc} & \text{3 columns} \\ & \downarrow\ \ \downarrow\ \ \downarrow \\ A = & \begin{bmatrix} 2 & 4 & 1 \\ 6 & 5 & 3 \end{bmatrix} \begin{array}{l} \leftarrow \text{ 2 rows} \\ \leftarrow \end{array} \end{array}$$

Matrix A has 2 rows and 3 columns and is a 2×3 matrix, read "2 by 3." You can write it as A or $A_{2 \times 3}$.

Each number in a matrix is a **matrix element**. You can identify a matrix element by its row and column numbers. In matrix A, a_{12} is the element in row 1 and column 2. a_{12} is the element 4.

Problem 1 Identifying a Matrix Element

GRIDDED RESPONSE

Think

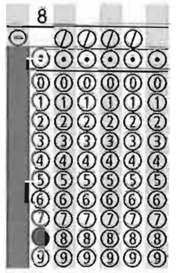
Does the order of the subscript in a_{23} matter?
Yes. a_{23} and a_{32} are different elements.

What is element a_{23} in matrix A?

$$A = \begin{bmatrix} 4 & -9 & 17 & 1 \\ 0 & 5 & 8 & 6 \\ -3 & -2 & 10 & 0 \end{bmatrix}$$

A_{23} is in Row 2 and Column 3.

a_{23} is 8.



Got It? 1. What is element a_{13} in matrix A?

You can represent a system of equations efficiently with a matrix. Each matrix row represents an equation. The last matrix column shows the constants to the right of the equal signs. Each of the other columns shows the coefficients of one of the variables.

System of Equations

$$\begin{aligned} x + 3y &= 7 \\ 3x + y &= -8 \end{aligned}$$

x-coefficients

y-coefficients

constants

Matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 1 & -8 \end{array}\right]$$

The 1's are coefficients of x and y.

Draw a vertical bar to replace the equal signs and separate the coefficients from the constants.

Problem 2 Representing Systems With Matrices

Think

Why is the order of elements important in a matrix?
Different orders of elements could correspond to different systems of equations.

How can you represent the system of equations with a matrix?

A $\begin{cases} 2x + y = 9 \\ x - 6y = -1 \end{cases}$

The matrix $\left[\begin{array}{cc|c} 2 & 1 & 9 \\ 1 & -6 & -1 \end{array}\right]$ represents the system above.

B $\begin{cases} x - 3y + z = 6 \\ x + 3z = 12 \\ y = -5x + 1 \end{cases}$

Step 1 Write each equation in the same variable order. Line up the variables. Leave space where a coefficient is 0.

$$\begin{cases} x - 3y + z = 6 \\ x \qquad\quad + 3z = 12 \\ 5x + y \qquad\quad = 1 \end{cases}$$

Step 2 Write the matrix using the coefficients and constants. Notice the 1's and 0's.

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 6 \\ 1 & 0 & 3 & 12 \\ 5 & 1 & 0 & 1 \end{array}\right]$$

Got It? **2.** How can you represent the system of equations with a matrix?

a. $\begin{cases} -4x - 2y = 7 \\ 3x + y = -5 \end{cases}$

b. $\begin{cases} 4x - y + 2z = 1 \\ y + 5z = 20 \\ 2x = -y + 7 \end{cases}$

Problem 3 Writing a System From a Matrix

What linear system of equations does this matrix represent? $\left[\begin{array}{cc|c} 5 & 2 & 7 \\ 0 & 1 & 9 \end{array}\right]$

Think	Write
Each row shows coefficient-coefficient-constant of one equation.	$5x + 2y = 7$ $0x + 1y = 9$
Simplify. Write the system.	$\begin{cases} 5x + 2y = 7 \\ y = 9 \end{cases}$

Got It? **3.** What linear system does $\left[\begin{array}{cc|c} 2 & 0 & 6 \\ 5 & -2 & 1 \end{array}\right]$ represent?

You can use a matrix that represents a system of equations to solve the system. In this way, you do not have to write the variables. To solve the system using the matrix, use the steps for solving by elimination. Each step is a **row operation**.

Your goal is to use row operations to get a matrix in the form $\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$ or $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$

Notice that the first matrix represents the system $x = a, y = b$, which then will be the solution of a system of two equations in two unknowns. The second matrix represents the system $x = a, y = b$, and $z = c$.

take note

Key Concept Row Operations

Switch any two rows.	$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 5 \end{bmatrix}$ becomes $\begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 3 \end{bmatrix}$
Multiply a row by a constant.	$\begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 3 \end{bmatrix}$ becomes $\begin{bmatrix} 3 & 2 & 5 \\ 2 \cdot 2 & -1 \cdot 2 & 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 5 \\ 4 & -2 & 6 \end{bmatrix}$
Add one row to another.	$\begin{bmatrix} 3 & 2 & 5 \\ 4 & -2 & 6 \end{bmatrix}$ becomes $\begin{bmatrix} 3+4 & 2-2 & 5+6 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 11 \\ 4 & -2 & 6 \end{bmatrix}$
Combine any of these steps.	

Problem 4 Solving a System Using a Matrix

Think

How is solving a system using row operations similar to using elimination?
You use the same steps but the variables don't appear in the matrices.

What is the solution of the system? $\begin{cases} x + 4y = -1 \\ 2x + 5y = 4 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 2 & 5 & 4 \end{array}\right]$$

Write the matrix for the system.

$$\begin{array}{rrrr} -2 & (1 & 4 & -1) \\ + & 2 & 5 & 4 \\ \hline & 0 & -3 & 6 \end{array}$$

Multiply Row 1 by −2. Add to Row 2. Replace Row 2 with the sum. Write the new matrix.

$$\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & -3 & 6 \end{array}\right]$$

$$-\frac{1}{3}\ (0 \quad -3 \quad 6) = 0 \quad 1 \quad -2$$

Multiply Row 2 by $-\frac{1}{3}$. Write the new matrix.

$$\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 1 & -2 \end{array}\right]$$

$$\begin{array}{rrrr} & 1 & 4 & -1 \\ + -4 & (0 & 1 & -2) \\ \hline & 1 & 0 & 7 \end{array}$$

Multiply Row 2 by −4. Add to Row 1. Replace Row 1 with the sum. Write the new matrix.

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

The solution to the system is (7, −2).

Check	$x + 4y = -1$	$2x + 5y = 4$	Use the original equations.
	$7 + 4(-2) \stackrel{?}{=} -1$	$2(7) + 5(-2) \stackrel{?}{=} 4$	Substitute.
	$7 + (-8) \stackrel{?}{=} -1$	$14 + (-10) \stackrel{?}{=} 4$	Multiply.
	$-1 = -1$ ✔	$4 = 4$ ✔	Simplify.

Got It? **4. a.** What is the solution of the system? $\begin{cases} 9x - 2y = 5 \\ 3x + 7y = 17 \end{cases}$

b. **Reasoning** Which method is more similar to solving a system using row operations: *elimination* or *substitution*? Justify your reasoning.

Matrices that represent the solution of a system are in *reduced row echelon form*. Many calculators have a **rref** (reduced row echelon form) function for working with matrices. This function will do all the row operations for you. You can use **rref** to solve a system of equations.

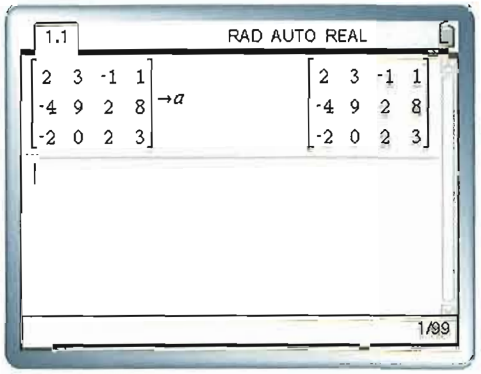
Problem 5 Using a Calculator to Solve a Linear System

Think

How do you enter missing variables into a matrix?
If a variable is not present in an equation, enter its coefficient as 0 in the matrix.

What is the solution of the system of equations? $\begin{cases} 2a + 3b - c = 1 \\ -4a + 9b + 2c = 8 \\ -2a + 2c = 3 \end{cases}$

Step 1 Enter the system into a calculator as a matrix.



1.1 RAD AUTO REAL

$\begin{bmatrix} 2 & 3 & -1 & 1 \\ -4 & 9 & 2 & 8 \\ -2 & 0 & 2 & 3 \end{bmatrix} \rightarrow a \qquad \begin{bmatrix} 2 & 3 & -1 & 1 \\ -4 & 9 & 2 & 8 \\ -2 & 0 & 2 & 3 \end{bmatrix}$

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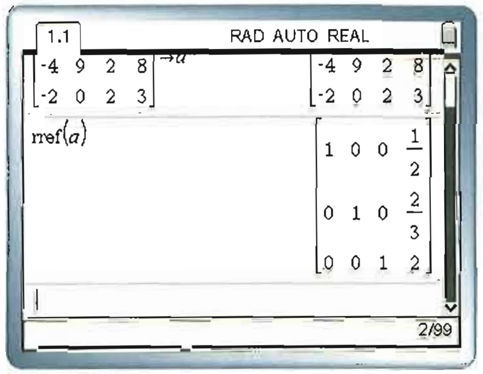
Step 2 Apply the **rref()** function to the matrix. Put the matrix elements in fraction form if some are not integers.



1.1 RAD AUTO REAL

$\begin{matrix} -4 & 9 & 2 & 8 \\ -2 & 0 & 2 & 3 \end{matrix} \rightarrow a \qquad \begin{matrix} -4 & 9 & 2 & 8 \\ -2 & 0 & 2 & 3 \end{matrix}$

rref(a) $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 2 \end{bmatrix}$

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Step 3 List the solution.

The solution of the system is $a = \frac{1}{2}, b = \frac{2}{3}, c = 2$.

Check

$$\begin{aligned} 2a + 3b - c &= 1 \\ 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{2}{3}\right) - 2 &\overset{?}{=} 1 \\ 1 + 2 - 2 &\overset{?}{=} 1 \\ 1 &= 1 \checkmark \end{aligned}$$

$$\begin{aligned} -4a + 9b + 2c &= 8 \\ -4\left(\tfrac{1}{2}\right) + 9\left(\tfrac{2}{3}\right) + 2(2) &\overset{?}{=} 8 \\ -2 + 6 + 4 &\overset{?}{=} 8 \\ 8 &= 8 \checkmark \end{aligned}$$

$$\begin{aligned} -2a + 2c &= 3 \\ -2\left(\tfrac{1}{2}\right) + 2(2) &\overset{?}{=} 3 \\ -1 + 4 &\overset{?}{=} 3 \\ 3 &= 3 \checkmark \end{aligned}$$

Got It? **5.** What is the solution of the system of equations?

$$\begin{cases} a + 4b + 6c = 21 \\ 2a - 2b + c = 4 \\ \quad - 8b + c = -1 \end{cases}$$

Lesson Check

Do you know HOW?

State the dimensions of each matrix.

1. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

2. $\left[\begin{array}{ccc|c} 6 & 9 & 0 & 3 \\ 4 & 6 & 2 & 7 \end{array}\right]$

Write a matrix to represent each system.

3. $\begin{cases} 3a + 5b = 0 \\ a + b = 2 \end{cases}$

4. $\begin{cases} x + 3y - z = 2 \\ x + 2z = 8 \\ 2y - z = 1 \end{cases}$

Do you UNDERSTAND?

5. How many elements are in a 4 × 4 matrix?

6. **Writing** Using Matrix A in Problem 1, describe the difference in identifying element a_{21} and element a_{12}.

7. **Open-Ended** Write a situation that can be modeled by the matrix. $\left[\begin{array}{cc|c} 4 & 2 & 8 \\ 0 & 1 & 2 \end{array}\right]$

Practice and Problem-Solving Exercises

A Practice

Identify the indicated element. $A = \begin{bmatrix} 3 & 12 & 6 \\ 1 & 0 & 9 \\ 8 & 7 & 4 \end{bmatrix}$

See Problem 1.

8. a_{32}

9. a_{21}

10. a_{13}

11. a_{31}

Write a matrix to represent each system.

See Problem 2.

12. $\begin{cases} x + 2y = 11 \\ 2x + 3y = 18 \end{cases}$

13. $\begin{cases} 3x + 2y = 16 \\ y = 5 \end{cases}$

14. $\begin{cases} 2a - 3b = 6 \\ a + b = 2 \end{cases}$

15. $\begin{cases} r - s + t = 150 \\ 2r + t = 425 \\ s + 3t = 0 \end{cases}$

16. $\begin{cases} y = 3x - 7 \\ x = 2 \end{cases}$

17. $\begin{cases} x - y + z = 0 \\ x - 2y - z = 5 \\ 2x - y + 2z = 8 \end{cases}$

Write the system of equations represented by each matrix.

See Problem 3.

18. $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -6 \end{array}\right]$

19. $\left[\begin{array}{cc|c} 5 & 1 & -3 \\ -2 & 2 & 4 \end{array}\right]$

20. $\left[\begin{array}{cc|c} -1 & 2 & -6 \\ 1 & 1 & 7 \end{array}\right]$

21. $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & -2 \end{array}\right]$

22. $\left[\begin{array}{ccc|c} 0 & 1 & 2 & 4 \\ -2 & 3 & 6 & 9 \\ 1 & 0 & 1 & 3 \end{array}\right]$

23. $\left[\begin{array}{ccc|c} 5 & 2 & 1 & 5 \\ 4 & 1 & 2 & 8 \\ 1 & 3 & -6 & 2 \end{array}\right]$

Solve the system of equations using a matrix.

See Problems 4 and 5.

24. $\begin{cases} x + 3y = 5 \\ x + 4y = 6 \end{cases}$

25. $\begin{cases} p - 3q = -1 \\ -5p + 16q = 5 \end{cases}$

26. $\begin{cases} 300x - y = 130 \\ 200x + y = 120 \end{cases}$

27. $\begin{cases} x + 3y = 22 \\ 2x - y = 2 \end{cases}$

28. $\begin{cases} x + 3y = 6 \\ 2x + 4y = 12 \end{cases}$

29. $\begin{cases} x + y = 5 \\ -2x + 4y = 8 \end{cases}$

B Apply

30. **Business** A manufacturer sells pencils and erasers in packages. The price of a package of five erasers and two pencils is \$.23. The price of a package of seven erasers and five pencils is \$.41. Write a system of equations to represent this situation. Then write a matrix to represent the system.

31. **Think About a Plan** Last year your town invested a total of \$25,000 into two separate funds. The return on one fund was 4% and the return on the other was 6%. If the town earned a total of \$1300 in interest, how much money was invested in each fund?
    - What variables will you use? What will they represent?
    - What equations can you write to model this situation?
    - How can you use a matrix to solve this system?

Graphing Calculator **Solve each system.**

32. $\begin{cases} x + y + z = 2 \\ 2y - 2z = 2 \\ x - 3z = 1 \end{cases}$

33. $\begin{cases} x - y + z = 3 \\ x + 3z = 6 \\ y - 2z = -1 \end{cases}$

34. $\begin{cases} x + y + z = -1 \\ 3x + 4y - z = 8 \\ 6x + 8y - 2z = 16 \end{cases}$

35. $\begin{cases} x - y + 3z = 9 \\ x + 2z = 3 \\ 2x + 2y + z = 10 \end{cases}$

36. $\begin{cases} 2x + 3y + z = 13 \\ 5x - 2y - 4z = 7 \\ 4x + 5y + 3z = 25 \end{cases}$

37. $\begin{cases} -2w + x + y = 0 \\ -w + 2x - y + z = 1 \\ -2w + 3x + 3y + 2z = 6 \\ w + x + 2y + z = 5 \end{cases}$

38. **Snacks** Suppose you want to fill nine 1-lb tins with a snack mix. You have \$15 and plan to buy almonds for \$2.45 per lb, hazelnuts for \$1.85 per lb, and raisins for \$.80 per lb. You want the mix to contain an equal amount of almonds and hazelnuts and twice as much of the nuts as the raisins by weight.
    - a. **Writing** Explain how each equation to the right relates to the problem. What does each variable represent?
    - b. Solve the system.
    - c. How many of each ingredient should you buy?

$$\begin{cases} x + y + z = 9 \\ 2.45x + 1.85y + 0.8z = 15 \\ x + y = 2z \end{cases}$$

39. **Geometry** The coordinates (x, y) of a point in a plane are the solution of the system $\begin{cases} 2x + 3y = 13 \\ 5x + 7y = 31 \end{cases}$. Find the coordinates of the point.

40. **Error Analysis** A classmate writes the matrix at the right to represent a system and says that the solution is $x = 2, y = 0$. Explain your classmate's error and describe how to correct it.

$$\xcancel{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

41. **Paint** A hardware store mixes paints in a ratio of two parts red to six parts yellow to make two gallons of pumpkin orange. A ratio of five parts red to three parts yellow makes two gallons of pepper red. A gallon of pumpkin orange sells for \$25, and a gallon of pepper red sells for \$28. Find the cost of 1 quart of red paint and the cost of 1 quart of yellow paint.

Challenge **Open-Ended** Complete each system for the given number of solutions.

42. infinitely many

$$\begin{cases} x + y = 7 \\ 2x + 2y = \blacksquare \end{cases}$$

43. one solution

$$\begin{cases} x + y + z = 7 \\ y + z = \blacksquare \\ z = \blacksquare \end{cases}$$

44. no solution

$$\begin{cases} x + y + z = 7 \\ y + z = \blacksquare \\ y + z = \blacksquare \end{cases}$$

Solve the system of equations using a matrix. (Hint: Start by substituting $m = \frac{1}{x}$ and $n = \frac{1}{y}$.)

45. $\begin{cases} \frac{4}{x} + \frac{1}{y} = 1 \\ \frac{8}{x} + \frac{4}{y} = 3 \end{cases}$

46. $\begin{cases} \frac{4}{x} - \frac{2}{y} = 1 \\ \frac{10}{x} + \frac{20}{y} = 0 \end{cases}$

47. $\begin{cases} \frac{7}{x} + \frac{3}{y} = 5 \\ \frac{2}{x} + \frac{1}{y} = -1 \end{cases}$

Sunshine State Standards Practice

MA.912.A.3.10

48. Which equation represents a line with a slope of $\frac{1}{2}$ and a y-intercept of $\frac{3}{4}$?

Ⓐ $y = \frac{1}{2}x - \frac{3}{4}$ Ⓑ $y = \frac{3}{4}x - \frac{1}{2}$ Ⓒ $y = \frac{1}{2}x + \frac{3}{4}$ Ⓓ $y = \frac{3}{4}x + \frac{1}{2}$

MA.912.A.2.5

49. Which graph best represents the solution of the inequality $y \leq 2|x - 1| - 4$?

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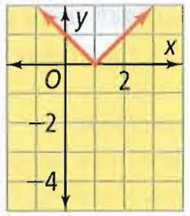

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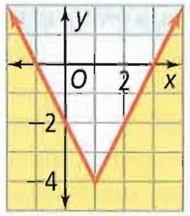

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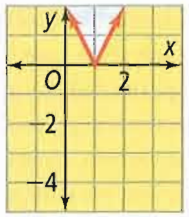

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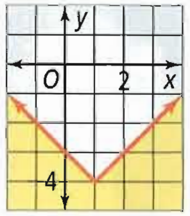


MA.912.A.3.14

50. Short Response At what point do the graphs of the equations $y = 7x - 3$ and $-6x + y = 2$ intersect?

Mixed Review

Solve each inequality. Graph the solution. See Lesson 1-5.

51. $12 \geq 2(4x + 1) + 22$ **52.** $2x - (3x + 5) \leq 30$ **53.** $4x + 5 - 3x \leq 2x + 1$

Solve each equation. Check your answers. See Lesson 1-6.

54. $|2y - 3| = 12$ **55.** $|4x| = 40$ **56.** $|2y - 4| = 16$

Get Ready! **To prepare for Lesson 4-1, do Exercises 57 and 58.**

Write an equation for each transformation of $y = x$. See Lesson 2-6.

57. vertical stretch by a factor of 2.

58. vertical compression by a factor of $\frac{1}{3}$

3

Pull It All Together

MA.912.A.3.14 Solve systems of linear equations and inequalities in two and three variables using graphical, substitution, and elimination methods.

BIG idea Function

The solution of a system of two linear equations corresponds in general to the intersection of the graphs of the corresponding functions.

Task 1

You are given a linear system of two equations in two unknowns. Before solving, describe how you can mentally check whether the system is independent and consistent. In which order would you do your check? Why?

BIG idea Equivalence

You can solve a system of equations by representing the system in some form that is equivalent to the original form but easier to solve. There are different ways to do this.

Task 2

During a back-to-school shopping trip, a group of friends spent \$245.86 on 14 shirts and pants. Each shirt cost \$11.99. Each pair of pants cost \$24.99. How many shirts and pairs of pants did the group buy?

a. Write a system of equations to model the information in the problem.

b. Study the system. Explain, without solving, which method you think would be most efficient for solving the system: *substitution, elimination, graphing*, or *making a table*. Explain why the other methods would be less efficient.

c. How could you simplify the numbers used in this system to simplify the system? Does this new system change your answers to part (b)? Explain.

BIG idea Solving Equations and Inequalities

You can represent a system of equations with a matrix. Transforming the matrix to reduced row echelon form gives you an equivalent system for which the solution is obvious.

Task 3

Solve this system using a matrix. $\begin{cases} 4x + 10y = 3 \\ 7x - 2y = 2 \end{cases}$

Make three columns on your paper. In the first column, show each step, changing one matrix row at a time. In the second column, write the two equations that correspond to each matrix in the first column. In the third column, describe how you could transform each set of equations to the next.

3 Chapter Review

Connecting BIG ideas and Answering the Essential Questions

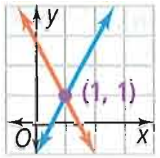
1 Function
Find a point of intersection (x, y) of the graphs of functions f and g and you have found a solution of the system $y = f(x)$, $y = g(x)$.

2 Equivalence
If the equations of two systems are equivalent, then a solution of the system that is easier to solve is also a solution of the more difficult system.

3 Solving Equations and Inequalities
The matrix row operations of adding rows and multiplying a row by a constant are equivalent to addition and multiplication properties of equality.

Solving Systems Using Tables and Graphs (Lesson 3-1)

$$\begin{cases} y = -2x + 3 \\ y = 2x - 1 \end{cases}$$



The solution is (1, 1).

Solving Systems Algebraically (Lesson 3-2)

$$\begin{cases} -y = -x + 2 \\ 3y = 2x - 2 \end{cases} \rightarrow \begin{array}{rl} -2y = & -2x + 4 \\ \underline{3y =} & \underline{2x - 2} \\ y = & 2 \end{array}$$

$$3(2) = 2x - 2 \quad \rightarrow \quad x = 4$$

The solution is $x = 4$, $y = 2$

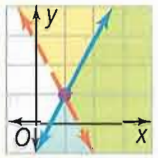
Solving Systems Using Matrices (Lesson 3-6)

$$\left[\begin{array}{rr|r} -2 & 3 & 1 \\ 2 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \quad x = 1, y = 1$$

Systems of Inequalities and Linear Programming (Lessons 3-3 and 3-4)

$$\begin{cases} y > -2x + 3 \\ y \leq 2x - 1 \end{cases}$$



Systems With Three Variables (Lesson 3-5)

$$\begin{cases} -2x + y + z = -3 \\ 2x - y + z = -1 \\ -2x, - y - z = -1 \end{cases}$$

$x = 1, y = 1, z = -2$

Chapter Vocabulary

- consistent system (p. 137)
- constraint (p. 157)
- dependent system (p. 137)
- equivalent systems (p. 144)
- feasible region (p. 157)
- inconsistent system (p. 137)
- independent system (p. 137)
- linear programming (p. 157)
- linear system (p. 134)
- matrix (p. 174)
- matrix element (p. 174)
- objective function (p. 157)
- row operation (p. 176)
- solution of a system (p. 134)
- system of equations (p. 134)

Fill in the blank.

1. A consistent system with exactly one solution is a(n) ___________.

2. ___________ is a method for finding a minimum or maximum value, given a system of limits called ___________.

3-1 Solving Systems Using Tables and Graphs

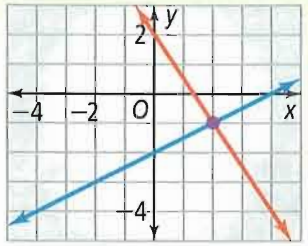
Quick Review

A **system of equations** has two or more equations. Points of intersection are solutions. A **linear system** has linear equations. A **consistent system** can be **dependent**, with infinitely many solutions, or **independent**, with one solution. An **inconsistent system** has no solution.

Example

Solve the system $\begin{cases} 3x + 2y = 4 \\ 2x - 4y = 8 \end{cases}$

Graph the equations.



The only solution, where the lines intersect, is (2, −1).

Exercises

Without graphing, classify each system of equations as *independent, dependent,* or *inconsistent.* Solve independent systems by graphing.

3. $\begin{cases} 6x - 2y = 2 \\ 2 + 6x = y \end{cases}$

4. $\begin{cases} 5 - y = 2x \\ 6x - 15 = -3y \end{cases}$

5. $\begin{cases} 6y + 2x = 8 \\ 12y + 4x = 4 \end{cases}$

6. $\begin{cases} 1.5 + 3x = 0.5y \\ 6 - 2y = -12x \end{cases}$

7. $\begin{cases} 2 - 0.25x = 0.5y \\ -1.5y = 1.5x - 3 \end{cases}$

8. $\begin{cases} 1 + y = x \\ x + y = 1 \end{cases}$

9. For $7.52, you purchased 8 pens and highlighters from a local bookstore. Each highlighter cost $1.09 and each pen cost $.69. How many pens did you buy?

3-2 Solving Systems Algebraically

Quick Review

To solve an independent system by substitution, solve one equation for a variable. Then substitute that expression into the other equation and solve for the remaining variable. To solve by elimination, add two equations with additive inverses as coefficients to eliminate one variable and solve for the other. In both cases you solve for one of the variables and use substitution to solve for the remaining variable.

Example

Solve $\begin{cases} 10 - y = 4x \\ x = 4 + 0.5y \end{cases}$ by substitution.

Substitute for x: $10 - y = 4(4 + 0.5y) = 16 + 2y$.

Solve for y: $y = -2$.

Substitute into the first equation:

$10 - (-2) = 4x$.

Solve for x: $x = 3$. The solution is (3, −2).

Exercises

Solve each system by substitution.

10. $\begin{cases} x - 2y = 3 \\ 3x + y = -5 \end{cases}$

11. $\begin{cases} 14x - 35 = 7y \\ -25 - 6x = 5y \end{cases}$

Solve each system by elimination.

12. $\begin{cases} 11 - 5y = 2x \\ 5y + 3 = -9x \end{cases}$

13. $\begin{cases} 2x + 3y = 4 \\ 4x + 6y = 9 \end{cases}$

14. Roast beef has 25 g of protein and 11 g of calcium per serving. A serving of mashed potatoes has 2 g of protein and 25 g of calcium. How many servings of each are needed to supply exactly 29 g of protein and 61 g of calcium?

3-3 Systems of Inequalities

Quick Review

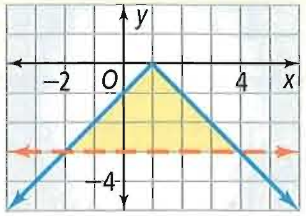
To solve a system of inequalities by graphing, first graph the boundaries for each inequality. Then shade the region(s) of the plane containing solutions valid for both inequalities.

Example

Solve the system of inequalities by graphing.

$$\begin{cases} y > -3 \\ y \leq -|x - 1| \end{cases}$$

Graph both inequalities and shade the region valid for both inequalities.



Exercises

Solve each system of inequalities by graphing.

15. $\begin{cases} y < 4x \\ 3x + y \geq 5 \end{cases}$

16. $\begin{cases} y < |2x - 4| \\ x + 5y \geq -1 \end{cases}$

17. $\begin{cases} y \leq |x + 2| - 3 \\ y \geq 1 + \frac{1}{4}x \end{cases}$

18. $\begin{cases} 2x + 3y > 6 \\ x \leq -1 \\ y \geq 4 \end{cases}$

19. For a community breakfast there should be at least three times as much regular coffee as decaffeinated coffee. A total of ten gallons is sufficient for the breakfast. Write and graph a system of inequalities to model the problem.

3-4 Linear Programming

Quick Review

Linear programming is used to find a minimum or maximum of an **objective function**, given **constraints** as linear inequalities. The maximum or minimum occurs at a vertex of the **feasible region**, which contains the solutions to the system of constraints.

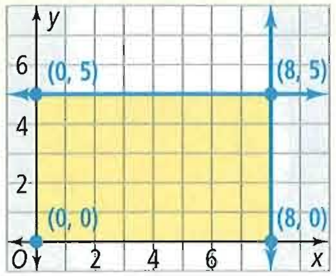
Example

Graph the system of constraints and name the vertices.

Objective function: $P = 2x + y$

$$\begin{cases} x \leq 8 \\ y \leq 5 \\ x \geq 0, y \geq 0 \end{cases}$$

Graph the inequalities and shade the area satisfying all inequalities.



The vertices of the feasible region are (0, 0), (0, 5), (8, 5), and (8, 0).

Evaluate the objective function at each vertex:

$2(0) + 0 = 0$ $\qquad$ $2(0) + 5 = 5$

$2(8) + 5 = 21$ $\qquad$ $2(8) + 0 = 16$

The maximum value occurs at (8, 5).

Exercises

Graph the system of constraints. Name the vertices. Then find the values of *x* and *y* that maximize or minimize the objective function.

20. $\begin{cases} x \geq 2 \\ y \geq 0 \\ 3x + 2y \geq 12 \end{cases}$

Minimum for $C = x + 5y$

21. $\begin{cases} 3x + 2y \leq 12 \\ x + y \leq 5 \\ x \geq 0, y \geq 0 \end{cases}$

Maximum for $P = 3x + 5y$

22. A lunch stand makes $.75 profit on each chef's salad and $1.20 profit on each Caesar salad. On a typical weekday, it sells between 40 and 60 chef's salads and between 35 and 50 Caesar salads. The total number sold has never exceeded 100 salads. How many of each type should be prepared in order to maximize profit?

3-5 Systems With Three Variables

Quick Review

To solve a system of three equations, either pair two equations and eliminate the same variable from both equations, using one equation twice, or choose an equation, solve for one variable, and substitute the expression for that variable into the other two equations. Then, solve the remaining system.

Example

Solve by elimination.
$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{cases} x + y + z = 10 \\ 2x - y + z = 9 \\ -3x + 2y + 2z = 5 \end{cases}$$

Add equations ① and ② to eliminate *y*. ④ $3x + 2z = 19$

Add 2 times ② to ③ to eliminate *y*. ⑤ $x + 4z = 23$

Add -3 times ⑤ to ④ to eliminate *x*. $z = 5$

Substitute $z = 5$ into ⑤. $x = 3$

Substitute $z = 5$ and $x = 3$ into ① or ②. $y = 2$

The solution to the system is (3, 2, 5).

Exercises

Solve each system by elimination.

23. $\begin{cases} x + y - 2z = 8 \\ 5x - 3y + z = -6 \\ -2x - y + 4z = -13 \end{cases}$

24. $\begin{cases} -x + y + 2z = -5 \\ 5x + 4y - 4z = 4 \\ x - 3y - 2z = 3 \end{cases}$

Solve each system by substitution.

25. $\begin{cases} 3x + y - 2z = 22 \\ x + 5y + z = 4 \\ x = -3z \end{cases}$

26. $\begin{cases} x + 2y + z = 14 \\ y = z + 1 \\ x = -3z + 6 \end{cases}$

3-6 Solving Systems Using Matrices

Quick Review

A **matrix** can represent a system of equations where each row stands for a different equation. The columns contain the coefficients of the variables and the constants.

Example

Solve using a matrix. $\begin{cases} 6x + 3y = -15 \\ 2x + 4y = 10 \end{cases}$

Enter coefficients as matrix elements $\left[\begin{array}{cc|c} 6 & 3 & -15 \\ 2 & 4 & 10 \end{array}\right]$.

Divide the first row by 3 to get $\left[\begin{array}{cc|c} 2 & 1 & -5 \\ 2 & 4 & 10 \end{array}\right]$. Subtract the first row from the second row to get $\left[\begin{array}{cc|c} 2 & 1 & -5 \\ 0 & 3 & 15 \end{array}\right]$. Multiply the second row by $\frac{1}{3}$ to get $\left[\begin{array}{cc|c} 2 & 1 & -5 \\ 0 & 1 & 5 \end{array}\right]$. Subtract the second row from the first row to get $\left[\begin{array}{cc|c} 2 & 0 & -10 \\ 0 & 1 & 5 \end{array}\right]$. Divide the first row by 2 to get $\left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 5 \end{array}\right]$. The solution to the system is $(-5, 5)$.

Exercises

Solve each system using a matrix.

27. $\begin{cases} 4x - 12y = -1 \\ 6x + 4y = 4 \end{cases}$

28. $\begin{cases} 7x + 2y = 5 \\ 13x + 14y = -1 \end{cases}$

29. $\begin{cases} -5x + 3y + 4z = 2 \\ 3x - y - z = 4 \\ x - 6y - 5z = -4 \end{cases}$

30. $\begin{cases} x + y + z = 4 \\ 2x - y + z = 5 \\ x + y - 2z = 13 \end{cases}$

3 Chapter Test

Do you know HOW?

Without graphing, classify each system. Then find the solution to each system using a graph.

1. $\begin{cases} y = 5x - 2 \\ y = x + 4 \end{cases}$

2. $\begin{cases} 3x + 2y = 9 \\ 3x + 2y = 4 \end{cases}$

Solve the system by substitution.

3. $\begin{cases} 0.3x - y = 0 \\ y = 2 + 0.25x \end{cases}$

Solve the system by elimination.

4. $\begin{cases} 4x - 2y = 3 \\ y - 2x = -\frac{3}{2} \end{cases}$

5. $\begin{cases} 3x + 4y = 9 \\ 2x + y = 6 \end{cases}$

Graph the solution of each system.

6. $\begin{cases} 2x + y < 3 \\ x < y + 3 \end{cases}$

7. $\begin{cases} |x + 3| > y \\ y > 2x - 1 \end{cases}$

Graph the system of constraints. Identify all vertices. Then find the values of *x* and *y* that maximize or minimize the objective function.

8. $\begin{cases} x \leq 5 \\ y \leq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$

Maximum for $P = 2x + y$

Solve each system.

9. $\begin{cases} x - y + z = 0 \\ 3x - 2y + 6z = 9 \\ -x + y - 2z = -2 \end{cases}$

10. $\begin{cases} 2x + y + z = 8 \\ x + 2y - z = -5 \\ z = 2x - y \end{cases}$

Do you UNDERSTAND?

Write a matrix that represents the system. Then solve the system. Tell what method you used and why.

11. $\begin{cases} -a + 4b + 2c = -8 \\ 3a + b - 4c = 9 \\ b = -1 \end{cases}$

12. Sales A pizza shop makes \$1.50 on each small pizza and \$2.15 on each large pizza. On a typical Friday, it sells between 70 and 90 small pizzas and between 100 and 140 large pizzas. The shop can make no more than 210 pizzas in a day. How many of each size pizza must be sold in order to maximize profit?

13. Investing Your teacher invested \$5000 in three funds. After a year they had \$5450. The growth fund had a return rate of 12%, the income fund had a return rate of 8%, and the money market fund had a return rate of 5%. Your teacher invested twice as much in the income fund as in the money market fund. How much money was invested in each fund?

14. Writing Describe how to identify situations in which substitution may be the best method for solving a system of equations.

15. Open-Ended Write a system of constraints whose graph is a parallelogram.

3 Sunshine State Standards Practice

TIPS FOR SUCCESS

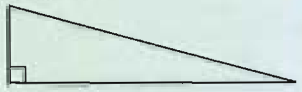
Some problems require the selection of an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) to find a solution.

One angle of a right triangle measures 90°. The measure of the second angle is 5 times the measure of the third. What are the measures of these angles?

- Ⓐ 30° and 60°
- Ⓑ 30° and 150°
- Ⓒ 15° and 75°
- Ⓓ 20° and 100°

TIP 1

Make a drawing.



TIP 2

Write a system:

$$x + y + 90 = 180$$
$$x = 5y$$

Think It Through

A triangle can have only one right angle, so the other two angles must each have a measure less than 90°. Use x and y to represent the unknown angles.

$$\begin{aligned} x &= 5y \\ 5y + y + 90 &= 180 \\ 6y + 90 &= 180 \\ 6y &= 90 \\ y &= 15, x = 75 \end{aligned}$$

The correct answer is C.

Vocabulary Builder

As you solve test items, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. equivalent systems

B. absolute value

C. system of equations

D. linear inequality

I. a number's distance from zero on a number line

II. an inequality in two variables whose graph is a region of the half-plane

III. a set of two or more equations that use the same variables

IV. systems that have the same solution(s)

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

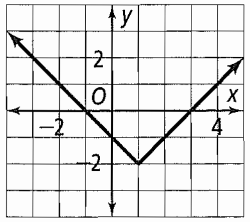
1. Which of the following is true about the given system?

$$\begin{cases} -4y = 12 - 8x \\ y = 2x - 3 \end{cases}$$

The system has

- Ⓐ zero solutions.
- Ⓑ exactly one solution.
- Ⓒ two solutions.
- Ⓓ infinitely many solutions.

2. Which is the graph of $y = -|x - 2| + 1$?

Ⓕ


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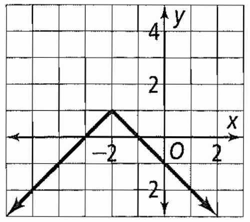

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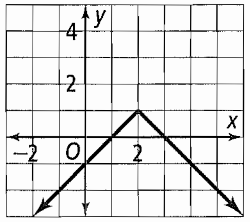

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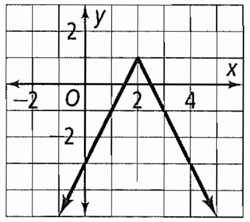
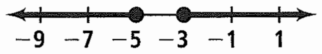
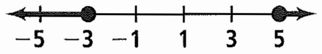


3. Which graph represents the solution of the inequality $|3x + 12| \geq 3$?

Ⓐ


Ⓑ


Ⓒ


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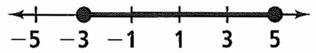


4. Josea wants to solve the system using substitution.

$$\begin{cases} x = -2y + 4 \\ 2x - 3y = 5 \end{cases}$$

Which of the following is the best way for Josea to proceed?

Ⓕ Solve the first equation for y, then substitute into the second equation.

Ⓖ Solve the second equation for y, then substitute into the first equation.

Ⓗ Substitute $-2y + 4$ for x in the second equation.

Ⓘ Substitute $-2y + 4$ for y in the second equation.

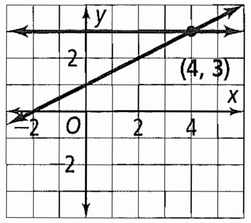
5. A board must be cut so that its length is 40.50 cm. The tolerance is 0.25 cm. Which inequality describes the allowable lengths for the board?

Ⓐ $|x - 0.25| \leq 40.50$ Ⓒ $|x - 40.50| \leq 0.25$

Ⓑ $|x + 0.25| \leq 40.50$ Ⓓ $|x - 0.25| \leq 40.75$

6. Which graph shows the solution to the given system?

$$\begin{cases} \frac{1}{2}x - y = 1 \\ x = 3 \end{cases}$$

Ⓕ


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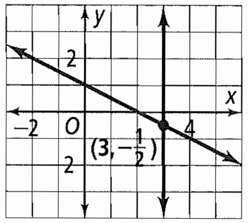

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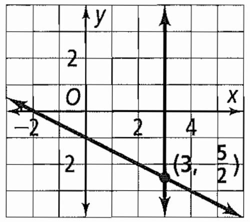

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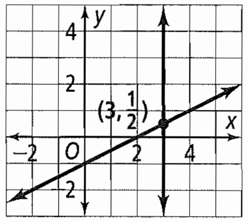


7. Consider the system below.

$$\begin{cases} 4x - 3y - 4z = 17 \\ -2x + 3y - 2z = 1 \\ 4x + 6y + 8z = 2 \end{cases}$$

Which of the following steps would NOT be used to solve the equation by elimination?

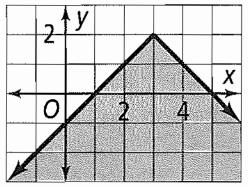
Ⓐ Add the first row to 2 times the second row.

Ⓑ Add the third row to 4 times the second row.

Ⓒ Add the first row to the third row.

Ⓓ Add the first row to the second row.

8. Which of the following inequalities does the graph represent?



Ⓕ $y \leq -|x - 3| + 2$ Ⓗ $y \leq -|x + 3| + 2$

Ⓖ $y \geq -|x - 3| + 2$ Ⓘ $y \geq -|x + 3| + 2$

9. The formula for the area of a trapezoid is $A = \frac{h}{2}(b_1 + b_2)$. Solve this equation for b_2.

(A) $b_2 = \frac{2A}{hb_1}$ (C) $b_2 = \frac{2A}{h - b_1}$

(B) $b_2 = \frac{2A}{h} - b_1$ (D) $b_2 = \frac{2A}{h + b_1}$

10. Which line is parallel to the line $2x - 3y = 9$?

(F) $2x + 3y = 9$ (H) $3x + 2y = 9$

(G) $2x - 3y = 6$ (I) $3x - 2y = 6$

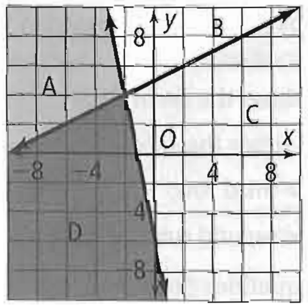
11. What is the equation of the line that passes through $(-2, 4)$ and $(2, 7)$?

(A) $y - 7 = \frac{3}{4}(x + 2)$ (C) $y - 7 = \frac{3}{4}(x - 2)$

(B) $y + 7 = \frac{3}{4}(x - 2)$ (D) $y - 2 = \frac{3}{4}(x - 7)$

12. The graph below shows the boundaries for the system of linear inequalities.

$$\begin{cases} y \leq 0.5x + 5 \\ y \leq -5x - 6 \end{cases}$$



Which of the shaded areas represents a solution to the first inequality but not the second?

(F) Region A (H) Region C

(G) Region B (I) Region D

GRIDDED RESPONSE

13. The nutrition label on a package of crackers shows there are 80 Calories in 16 grams of crackers. How many grams are in a package labeled 100 Calories?

14. A family with 4 adults and 3 children spends $47 for movie tickets at the theater. Another family with 2 adults and 4 children spends $36. What is the price of a child's ticket in dollars?

15. What is the sum of the solutions of $|5 - 3x| = x + 1$?

16. Sofia is buying party favors for her birthday party. The candles cost $1 each, the frames are $2 each, and the mugs are $2.50 each. She has $120 to spend on 75 favors. Also, she wants to buy twice as many candles as mugs. How many frames should she buy?

17. An ice cream shop has regular mix-ins for $.50 each and premium mix-ins for $1 each. You have $2.50 to spend on mix-ins, and you want at least 4 mix-ins. What is the greatest number of premium mix-ins you can get in your ice cream?

18. The equation of line *m* is $y = 3x - 1$. What is the *y*-intercept of a line that goes through the point $(3, -2)$ and is perpendicular to line *m*?

19. Jenna is trying to break her school's record for doing the most push-ups in ten minutes. The current record holder did 350 push-ups in ten minutes. The table shows the number of push-ups Jenna completed in the first 6 minutes.

If Jenna's pattern of push-ups continues, by how many push-ups will Jenna's total exceed the current record? (If Jenna's total falls short of the record, give your answer as a negative number.)

Time (min)	1	2	3	4	5	6
Number of push-ups	50	97	141	182	220	255

CHAPTER 4

Get Ready!

Lesson 1-4

Solving Linear Equations

Solve each equation. Check your answer.

1. $9x - 16 = 8 + 5x$

2. $4(y + 2) + 1 = -5(3 - 2y)$

Lesson 1-6

Solving Absolute Value Inequalities

Solve each inequality. Graph the solution.

3. $|6x - 12| + 6 < 30$

4. $6|4y - 2| \geq 42$

Lesson 2-3

Writing and Graphing Equations in Slope-Intercept Form

Graph the line passing through the given points. Then write its equation in slope-intercept form.

5. $(1, -1)$ and $(3, 17)$

6. $(2, 9)$ and $(6, 11)$

Lesson 2-6

Identifying Translations

Identify each horizontal and vertical translation of the parent function $y = |x|$.

7. $y = |x - 4| + 2$

8. $y = |x + 10| - 3$

Lesson 3-2

Solving Systems of Equations

Solve each system of equations by substitution.

9. $\begin{cases} 2x + 6y = 14 \\ 4x - 8y = 48 \end{cases}$

10. $\begin{cases} x + 2y = -18 \\ 2x - 4y = 12 \end{cases}$

Looking Ahead Vocabulary

11. A *form* is a document with blank spaces to fill in. What types of forms might you use?

12. Something is *imaginary* if it has no factual reality. What are some examples of imaginary items?

13. Many items have a specific *function*, or purpose for use. What is the function of a pencil?

CHAPTER

4

Quadratic Functions and Equations

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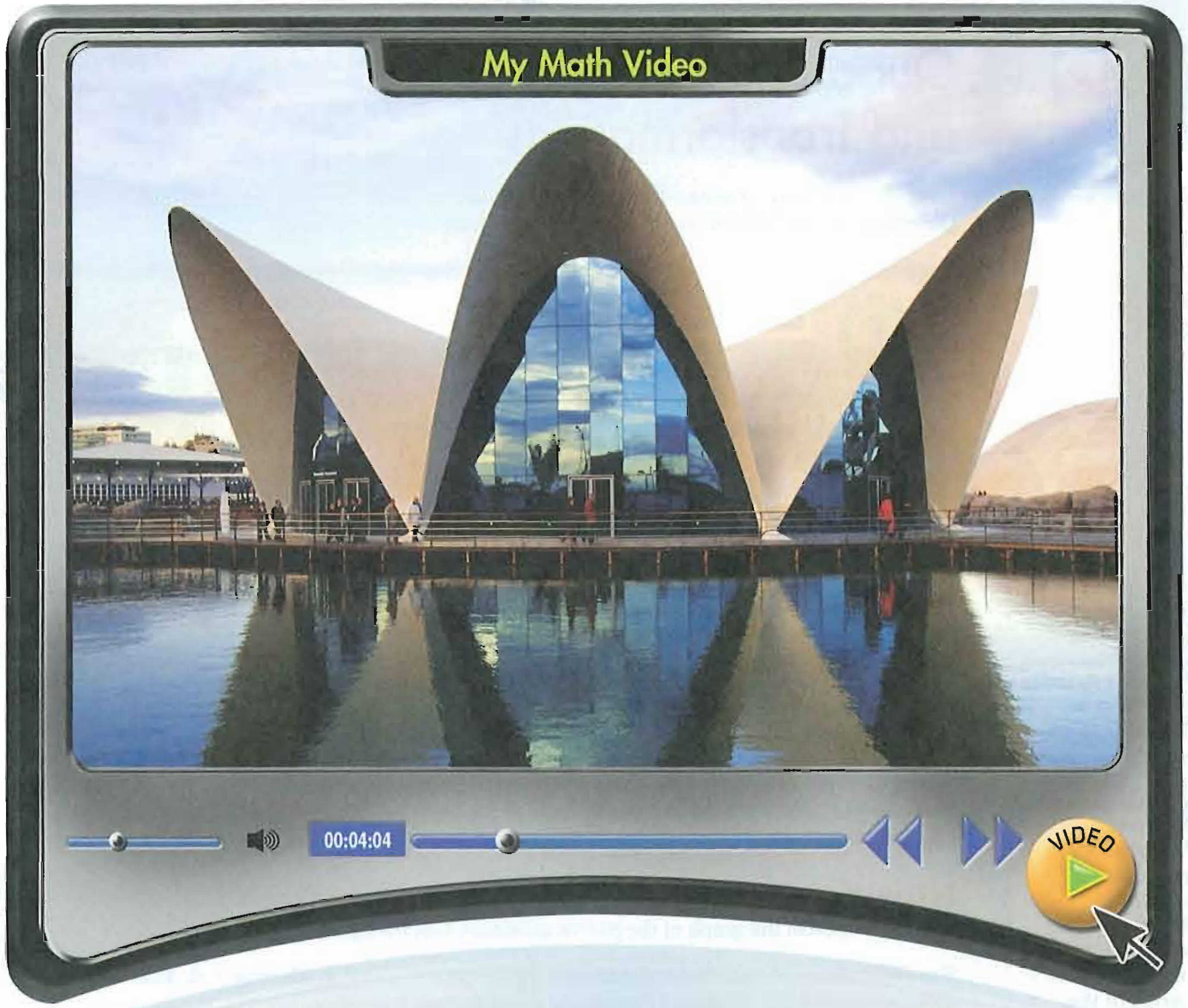
A parabola is the graph of a quadratic function.

Parabolas can be seen all over the place. You can find them in the designs of buildings like the one on the next page.

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
axis of symmetry, *p. 194*	eje de simetría
complex number, *p. 249*	números complejos
discriminant, *p. 242*	discriminante
greatest common factor, *p. 218*	máximo factor común de una expresión
imaginary number, *p. 249*	número imaginario
parabola, *p. 194*	parábola
Quadratic Formula, *p. 240*	fórmula cuadrática
quadratic function, *p. 194*	función cuadrática
standard form, *p. 202*	forma normal
vertex form, *p. 194*	forma del vértice
zero of a function, *p. 226*	cero de una función



BIG ideas

1 **Equivalence**

Essential Question What are the advantages of a quadratic function in vertex form? In standard form?

2 **Function**

Essential Question How is any quadratic function related to the parent quadratic function $y = x^2$?

3 **Solving Equations and Inequalities**

Essential Question How are the real solutions of a quadratic equation related to the graph of the related quadratic function?

Chapter Preview

4-1

Quadratic Functions and Transformations

Sunshine State Standards

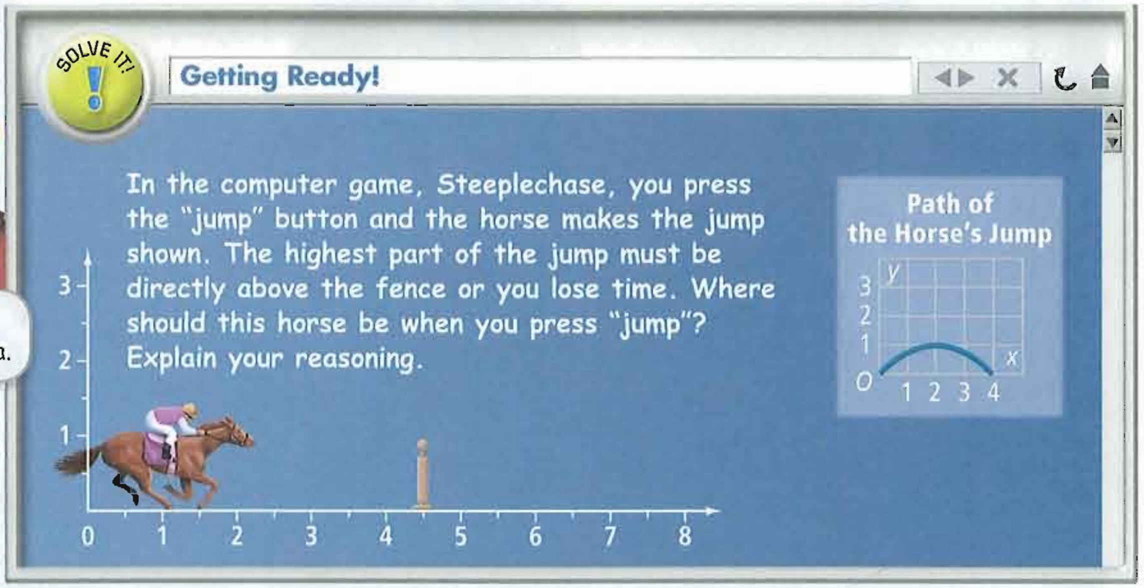
MA.912.A.2.10 Describe and graph transformations of functions.

MA.912.A.2.6 Identify and graph quadratic functions.

MA.912.A.7.6 Identify the axis of symmetry, vertex, domain, and range for a given parabola.

Objective To identify and graph quadratic functions

The path of the jump is a parabola.



In the computer game, Steeplechase, you press the "jump" button and the horse makes the jump shown. The highest part of the jump must be directly above the fence or you lose time. Where should this horse be when you press "jump"? Explain your reasoning.

Dynamic Activity
Quadratics in Vertex Form

Lesson Vocabulary
- parabola
- quadratic function
- vertex form
- axis of symmetry
- vertex of the parabola
- minimum value
- maximum value

In the Solve It, you used the *parabolic* shape of the horse's jump. A **parabola** is the graph of a **quadratic function**, which you can write in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

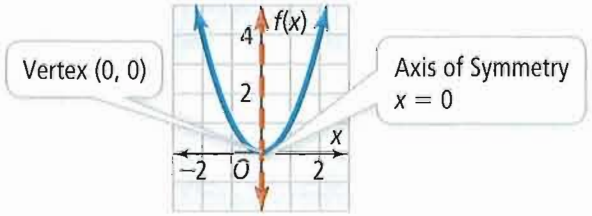
Essential Understanding The graph of any quadratic function is a transformation of the graph of the parent quadratic function, $y = x^2$.

The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The **axis of symmetry** is a line that divides the parabola into two mirror images. The equation of the axis of symmetry is $x = h$. The **vertex of the parabola** is (h, k), the intersection of the parabola and its axis of symmetry.

take note

Key Concept The Parent Quadratic Function

The parent quadratic function is $f(x) = x^2$. Its graph is the parabola shown. The axis of symmetry is $x = 0$. The vertex is $(0, 0)$.



Problem 1 Graphing a Function of the Form $f(x) = ax^2$

Plan

How do you choose points to plot?
Choose the vertex and two points on one side of the axis of symmetry that give integer values of $f(x)$.

What is the graph of $f(x) = \frac{1}{2}x^2$?

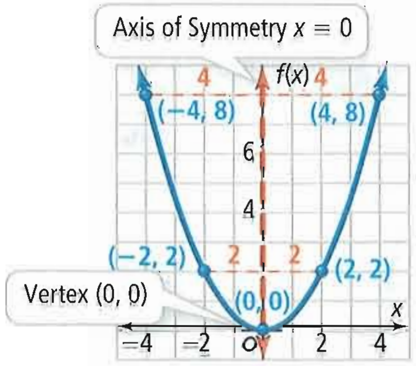
Step 1 Plot the vertex (0, 0). Draw the axis of symmetry, $x = 0$.

Step 2 Find and plot two points on one side of the axis of symmetry.

x	$f(x) = \frac{1}{2}x^2$	$(x, f(x))$
0	$\frac{1}{2}(0)^2 = 0$	(0, 0)
2	$\frac{1}{2}(2)^2 = 2$	(2, 2)
4	$\frac{1}{2}(4)^2 = 8$	(4, 8)

Step 3 Plot the corresponding points on the other side of the axis of symmetry.

Step 4 Sketch the curve.

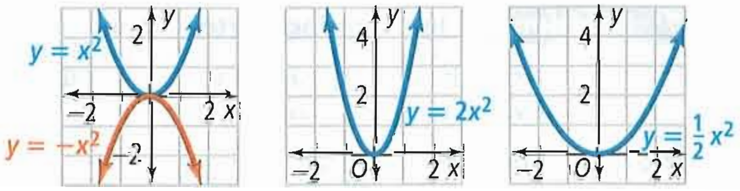


Got It? **1. a.** What is the graph of $f(x) = -\frac{1}{3}x^2$?
b. Reasoning What can you say about the graph of the function $f(x) = ax^2$ if a is a negative number? Explain.

Graphs of $y = ax^2$ and $y = -ax^2$ are reflections of each other across the x-axis. Increasing $|a|$ stretches the graph vertically and narrows it horizontally. Decreasing $|a|$ compresses the graph vertically and widens it horizontally.

take note

Key Concept Reflection, Stretch, and Compression

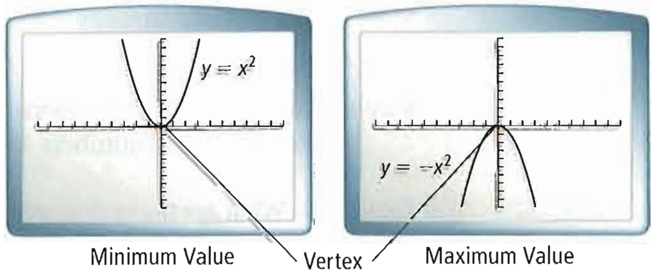


Reflection, a **and** $-a$

Stretch, $a > 1$

Compression, $0 < a < 1$

If $a > 0$, the parabola opens upward. The y-coordinate of the vertex is the **minimum value** of the function. If $a < 0$, the parabola opens downward. The y-coordinate of the vertex is the **maximum value** of the function.



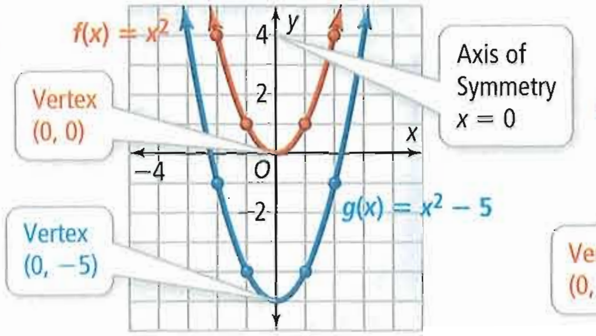
Problem 2 Graphing Translations of $f(x) = x^2$

Think

How does $g(x)$ differ from $f(x)$?
For each value of x, the value of $g(x)$ is 5 less than the value of $f(x)$.

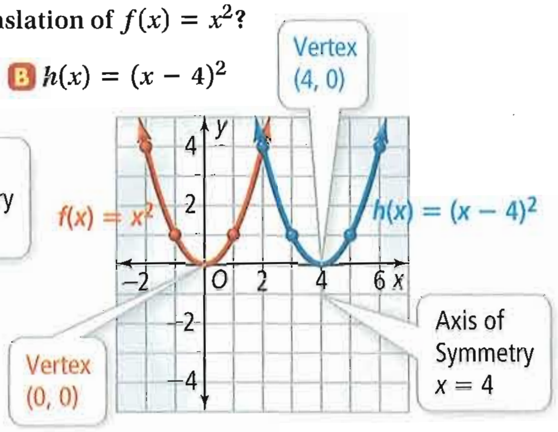
Graph each function. How is each graph a translation of $f(x) = x^2$?

A $g(x) = x^2 - 5$



Translate the graph of f down 5 units to get the graph of $g(x) = x^2 - 5$.

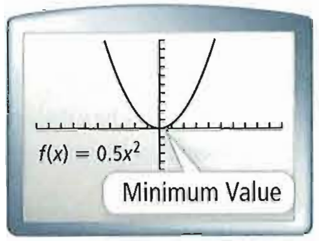
B $h(x) = (x - 4)^2$

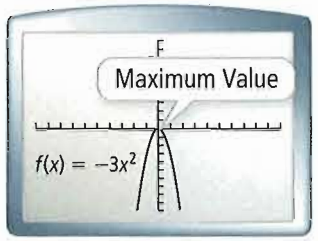


Translate the graph of f to the right 4 units to get the graph of $h(x) = (x - 4)^2$.

Got It? **2.** Graph each function. How is it a translation of $f(x) = x^2$?
a. $g(x) = x^2 + 3$ **b.** $h(x) = (x + 1)^2$

The vertex form, $f(x) = a(x - h)^2 + k$, gives you information about the graph of f without drawing the graph. If $a > 0$, k is the minimum value of the function. If $a < 0$, k is the maximum value.





Problem 3 Interpreting Vertex Form

Plan

How do you use vertex form?
Compare
$y = 3(x - 4)^2 - 2$
to vertex form
$y = a(x - h)^2 + k$
to find values for a, h, and k.

For $y = 3(x - 4)^2 - 2$, what are the vertex, the axis of symmetry, the maximum or minimum value, the domain and the range?

Step 1 Compare: $y = 3(x - 4)^2 - 2$
$y = a(x - h)^2 + k$

Step 2 The vertex is $(h, k) = (4, -2)$.

Step 3 The axis of symmetry is $x = h$, or $x = 4$.

Step 4 Since $a > 0$, the parabola opens upward. $k = -2$ is the minimum value.

Step 5 Domain: All real numbers. There is no restriction on the value of x.
Range: All real numbers ≥ -2, since the minimum value of the function is -2.

Got It? **3.** What are the vertex, axis of symmetry, minimum or maximum, and domain and range of the function $y = -2(x + 1)^2 + 4$?

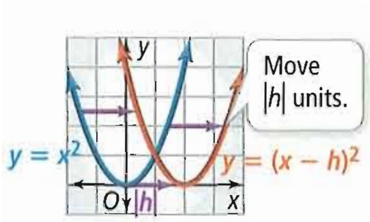
You can use the vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, to transform the graph of the parent function $f(x) = x^2$.

- Stretch or compress the graph of $f(x) = x^2$ vertically by the factor $|a|$.
- If $a < 0$, reflect the graph across the x-axis.
- Shift the graph $|h|$ units horizontally and $|k|$ units vertically.

take note

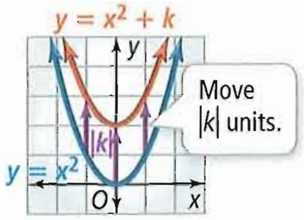
Key Concept Translation of the Parabola

Horizontal



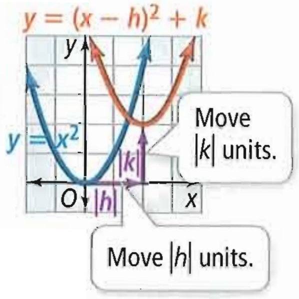
vertex becomes $(h, 0)$

Vertical



vertex becomes $(0, k)$

Horizontal and Vertical



vertex becomes (h, k)

Problem 4 Using Vertex Form

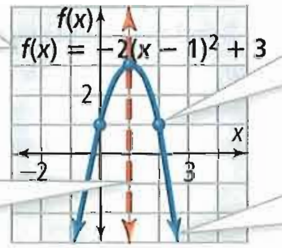
Plan

What do the values of a, h, and k tell you about the graph?
The graph is a stretched reflection of $y = x^2$, shifted 1 unit right and 3 units up.

A **What is the graph of $f(x) = -2(x - 1)^2 + 3$?**

Step 1 Identify the constants $a = -2$, $h = 1$, and $k = 3$. Because $a < 0$, the parabola opens downward.

Step 2 Plot the vertex $(h, k) = (1, 3)$ and draw the axis of symmetry $x = 1$.



Step 3 Plot two points. $f(2) = -2(2 - 1)^2 + 3 = 1$. Plot (2, 1) and the symmetric point (0, 1).

Step 4 Sketch the curve.

B **Multiple Choice** **What steps transform the graph of $y = x^2$ to $y = -2(x + 1)^2 + 3$?**

Ⓐ Reflect across the x-axis, stretch by the factor 2, translate 1 unit to the right and 3 units up.

Ⓑ Stretch by the factor 2, translate 1 unit to the right and 3 units up.

Ⓒ Reflect across the x-axis, translate 1 unit to the left and 3 units up.

Ⓓ Stretch by the factor 2, reflect across the x-axis, translate 1 unit to the left and 3 units up.

The correct choice is D.

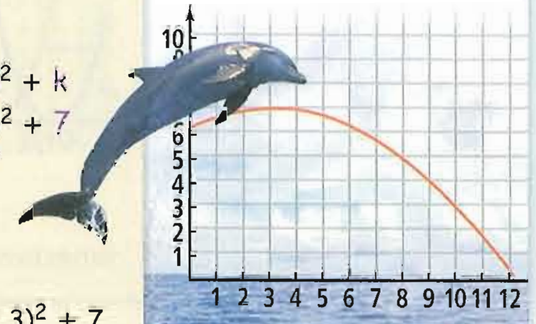
Got It? **4.** What steps transform the graph of $y = x^2$ to $y = 2(x + 2)^2 - 5$?

You can use the vertex form of a quadratic function to model a real-world situation.

Problem 5 Writing a Quadratic Function in Vertex Form

Nature **The picture shows the jump of a dolphin. What quadratic function models the path of the dolphin's jump?**

Think	Write
What is the vertex?	The vertex is (3, 7). $h = 3$, $k = 7$
Choose another point, (9, 4), from the path. Substitute in the vertex form. Solve for *a*.	$f(x) = a(x - h)^2 + k$ $4 = a(9 - 3)^2 + 7$ $4 = 36a + 7$ $-3 = 36a$ $a = -\frac{1}{12}$
Substitute in the vertex form.	$f(x) = -\frac{1}{12}(x - 3)^2 + 7$ models the path of the dolphin's jump.



Got It? **5.** Suppose the path of the jump changes so that the axis of symmetry becomes $x = 2$ and the height stays the same. If the path of the jump also passes through the point (5, 5), what quadratic function would model this path?

Lesson Check

Do you know HOW?

1. Graph the function $f(x) = -3x^2$.

2. Determine whether the function $f(x) = 0.25\,(2x - 15)^2 + 150$ has a maximum or a minimum value.

3. Rewrite $y = -2x^2 + 35$ in vertex form.

Do you UNDERSTAND?

4. **Vocabulary** When does the graph of a quadratic function have a minimum value?

5. **Reasoning** Is $y = 0(x - 4)^2 + 3$ a quadratic function? Explain.

6. **Compare and Contrast** Describe the differences between the graphs of $y = (x + 6)^2$ and $y = (x - 6)^2 + 7$.

Practice and Problem-Solving Exercises

A Practice

Graph each function. See Problem 1.

7. $y = -x^2$ **8.** $f(x) = 5x^2$ **9.** $y = \frac{2}{5}x^2$ **10.** $y = 2x^2$

11. $f(x) = 2\frac{1}{4}x^2$ **12.** $y = -\frac{4}{9}x^2$ **13.** $y = -7x^2$ **14.** $f(x) = 3\frac{2}{5}x^2$

Graph each function. Describe how it was translated from $f(x) = x^2$. See Problem 2.

15. $f(x) = x^2 + 3$ **16.** $f(x) = (x - 2)^2$ **17.** $f(x) = x^2 - 6$ **18.** $f(x) = (x + 3)^2$

19. $f(x) = x^2 - 9$ **20.** $f(x) = (x + 5)^2$ **21.** $f(x) = x^2 + 1.5$ **22.** $f(x) = (x - 2.5)^2$

Identify the vertex, the axis of symmetry, the maximum or minimum value, and the domain and the range of each function. See Problem 3.

23. $y = -1.5(x + 20)^2$ **24.** $f(x) = 0.1(x - 3.2)^2$ **25.** $f(x) = 24(x + 5.5)^2$

26. $y = 0.0035(x + 1)^2 - 1$ **27.** $f(x) = -(x - 4)^2 - 25$ **28.** $y = (x - 125)^2 + 125$

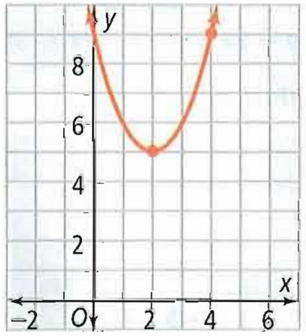
Graph each function. Identify the axis of symmetry. See Problem 4.

29. $f(x) = (x - 1)^2 + 2$ **30.** $y = (x + 3)^2 - 4$ **31.** $f(x) = 2(x - 2)^2 + 5$

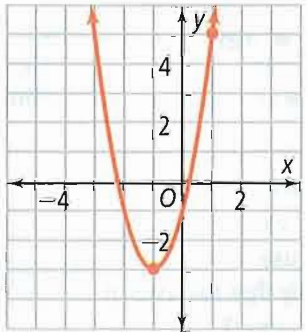
32. $y = -3(x + 7)^2 - 8$ **33.** $y = -(x - 1)^2 + 4$ **34.** $f(x) = -(x - 7)^2 + 10$

Write a quadratic function to model each graph. See Problem 5.

35.



36.



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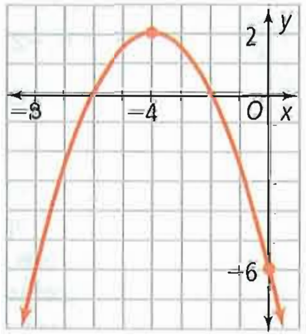


B Apply

38. Think About a Plan A gardener is putting a wire fence along the edge of his garden to keep animals from eating his plants. If he has 20 meters of fence, what is the largest rectangular area he can enclose?

- To find the area of a rectangle, what two quantities do you need? Choose one to be your variable and write the other in terms of this variable.
- How can a graph help you solve this problem?
- What quadratic function represents the area of the garden?

39. Manufacturing The equation for the cost in dollars of producing computer chips is $C = 0.000015x^2 - 0.03x + 35$, where x is the number of chips produced. Find the number of chips that minimizes the cost. What is the cost for that number of chips?

Describe how to transform the parent function $y = x^2$ to the graph of each function below. Graph both functions on the same axes.

40. $y = -2(x - 1)^2$

41. $y = -2(x + 1)^2 + 1$

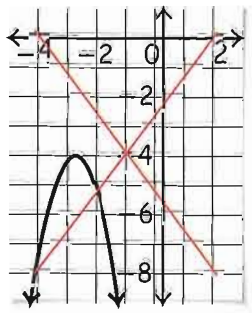
42. $y = 3(x - 2)^2 + 3$

43. $y = -1(x + 4)^2 + 5$

44. $y = -0.25x^2 + 3$

45. $y = 0.2(x - 12)^2 - 3$

46. Error Analysis A classmate graphed $y = -2(x - 3)^2 + 4$ as shown. Explain your classmate's error. Graph the correct parabola.



47. Writing Describe the family of quadratic functions whose members each have (3, 4) as their vertex.

48. Write a quadratic function to represent the areas of all rectangles with a perimeter of 36 ft. Graph the function and describe the rectangle that has the largest area.

Write the equation of each parabola in vertex form.

49. vertex (1, 2), point (2, −5)

50. vertex (−3, 6), point (1, −2)

51. vertex (0, 5), point (1, −2)

52. vertex $\left(\frac{1}{4}, -\frac{3}{2}\right)$, point (1, 3)

In Chapter 2, you graphed absolute value functions as transformations of their parent function $y = |x|$. Similarly, you can graph a quadratic function as a transformation of the parent function $y = x^2$. Graph the following pairs of functions on the same set of axes. Determine how they are similar and how they are different.

53. $y = -|x - 2| + 1; y = -(x - 2)^2 + 1$

54. $y = 3|x + 1| - 2; y = 3(x + 1)^2 - 2$

55. $y = -2|x| + 4; y = -2x^2 + 4$

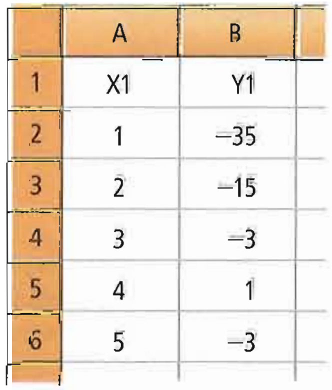
56. $y = |x + 3|; y = (x + 3)^2$

57. Open-Ended Write an equation of a parabola symmetric about $x = -10$.

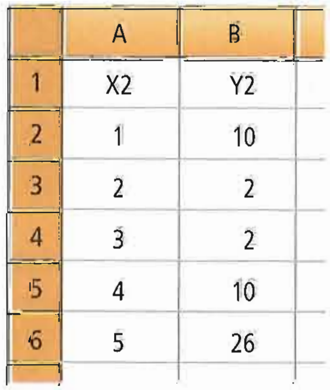
58. a. Technology Determine the axis of symmetry for each parabola defined by the spreadsheet values at the right.

b. How could you use the spreadsheet columns to verify that the axes of symmetry are correct?

c. What functions in vertex form model the data? Check that the axes of symmetry are correct.



	A	B
1	X1	Y1
2	1	−35
3	2	−15
4	3	−3
5	4	1
6	5	−3



	A	B
1	X2	Y2
2	1	10
3	2	2
4	3	2
5	4	10
6	5	26

Challenge **Determine a and k so the given points are on the graph of the function.**

59. (0, 1), (2, 1); $y = a(x - 1)^2 + k$

60. (−3, 2), (0, 11); $y = a(x + 2)^2 + k$

61. (1, 11), (2, −19); $y = a(x + 1)^2 + k$

62. (−2, 6), (3, 1); $y = a(x - 3)^2 + k$

63. a. In the function $y = ax^2 + bx + c$, c represents the y-intercept. Find the value of the y-intercept in the function $y = a(x - h)^2 + k$.

b. Under what conditions does k represent the y-intercept?

Find the quadratic function $y = a(x - h)^2$ whose graph passes through the given points.

64. $(-2, 1)$ and $(2, 1)$

65. $(-5, 2)$ and $(-1, 2)$

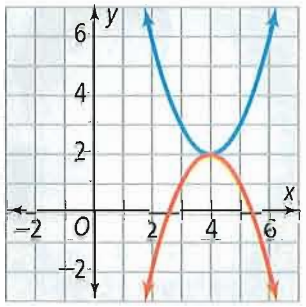
66. $(-1, -4)$ and $(7, -4)$

67. $(2, -1)$ and $(4, 0)$

68. $(-2, 18)$ and $(1, 0)$

69. $(1, -64)$ and $(-3, 0)$

Sunshine State Standards Practice



MA.912.A.2.6

70. One parabola at the right has the equation $y = (x - 4)^2 + 2$. Which equation represents the second parabola?

Ⓐ $y = -(x - 4)^2 + 2$

Ⓑ $y = (-x - 4)^2 + 2$

Ⓒ $y = (x + 4)^2 - 2$

Ⓓ $y = -(x + 4)^2 - 2$

MA.912.A.3.14

71. Which system has the unique solution $(1, 4)$?

Ⓕ $\begin{cases} y = x - 3 \\ x + y = 5 \end{cases}$

Ⓗ $\begin{cases} x + y = 5 \\ y = -x + 3 \end{cases}$

Ⓖ $\begin{cases} y = -x + 5 \\ x - y = -3 \end{cases}$

Ⓘ $\begin{cases} -x + y = 3 \\ 2x - 2y = -6 \end{cases}$

MA.912.A.3.3

72. What is the formula for the surface area of a right circular cylinder, $S = 2\pi rh + 2\pi r^2$, solved for h?

Ⓐ $h = \frac{S}{4\pi r}$

Ⓑ $h = \frac{S}{2\pi r^2}$

Ⓒ $h = \frac{S}{2\pi r} - r$

Ⓓ $h = r - \frac{S}{2\pi r}$

MA.912.A.4.10

73. Short Response An athletic club has 225 feet of fencing to enclose a tennis court. What quadratic function can be used to find the maximum area of the tennis court? Find the maximum area, and the lengths of the sides of the resulting fence.

Mixed Review

Solve each system of equations using a matrix. **See Lesson 3-6.**

74. $\begin{cases} 3x - y = 7 \\ 2x + 2y = 10 \end{cases}$

75. $\begin{cases} 2x + 5y = 10 \\ -3x + y = 36 \end{cases}$

76. $\begin{cases} 3x + y - 2z = -3 \\ x - 3y - z = -2 \\ 2x + 2y + 3z = 11 \end{cases}$

Graph each inequality. **See Lesson 2-8.**

77. $y > 3x + 1$

78. $y < -x + 4$

79. $y \geq \frac{1}{2}x - 2$

Determine whether each relation is a function. **See Lesson 2-1.**

80. $\{(3, 0), (2, -1), (4, 2)\}$

81. $\{(1, 2), (1, 1), (-1, 1)\}$

82. $\{(1, 2), (-1, 2), (-1, 1)\}$

Get Ready! **To prepare for Lesson 4-2, do Exercises 83–85.**

Find the vertex of the graph of each function. **See Lesson 2-7.**

83. $y = -2|x|$

84. $y = |-x - 1|$

85. $y = 5|x - 5|$

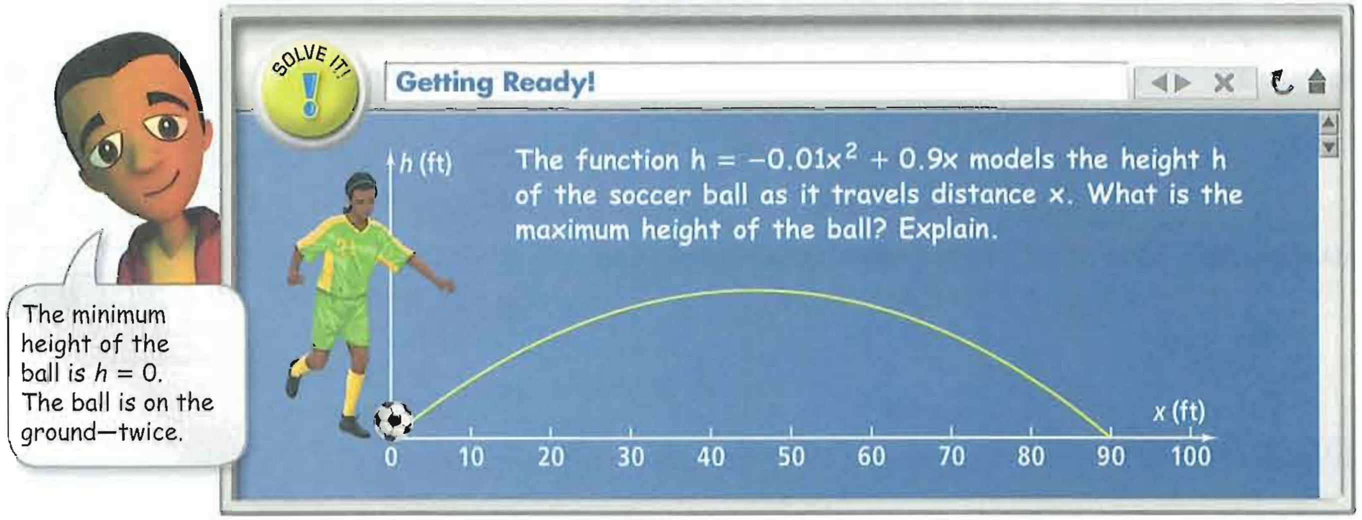
4-2 Standard Form of a Quadratic Function

Sunshine State Standards

MA.912.A.7.6 Identify the axis of symmetry, vertex, domain, range, and intercept(s) for a given parabola.

MA.912.A.2.6 Identify and graph quadratic functions.

Objective To graph quadratic functions written in standard form



The function $h = -0.01x^2 + 0.9x$ models the height h of the soccer ball as it travels distance x. What is the maximum height of the ball? Explain.

The minimum height of the ball is $h = 0$. The ball is on the ground—twice.

Lesson Vocabulary
- standard form

In Lesson 4-1, you worked with quadratic functions written in vertex form. Now you will use quadratic functions in *standard form*. The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$.

Essential Understanding For any quadratic function $f(x) = ax^2 + bx + c$, the values of a, b, and c provide key information about its graph.

You can find information about the graph of a quadratic function (such as the vertex) easily from the vertex form. Such information is "hidden" in standard form. However, standard form is easier to enter into a graphing calculator.

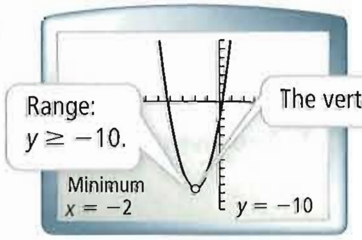
Problem 1 Finding the Features of a Quadratic Function

Graphing Calculator **What are the vertex, the axis of symmetry, the maximum or minimum value, and the range of $y = 2x^2 + 8x - 2$?**

Plan

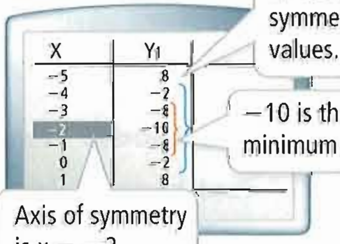
How can you use a calculator to find the features of a quadratic function in standard form?

Graph the function. Then use the **CALC** and **TABLE** features.



Range: $y \geq -10$.

The vertex is $(-2, -10)$.



X	Y1
−5	8
−4	−2
−3	−8
−2	−10
−1	−8
0	−2
1	8

Notice the symmetry of y values.

-10 is the minimum value.

Axis of symmetry is $x = -2$.

Got It? **1.** What are the vertex, axis of symmetry, maximum or minimum value, and range of $y = -3x^2 - 4x + 6$?

You can find information about the quadratic function $f(x) = ax^2 + bx + c$ from the coefficients a and b, and from the constant term c.

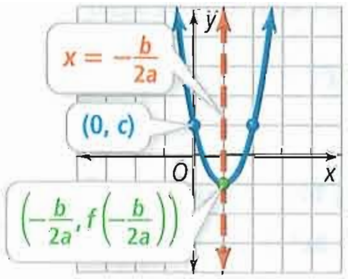
Dynamic Activity
Quadratic Equations in Polynomial Form

take note

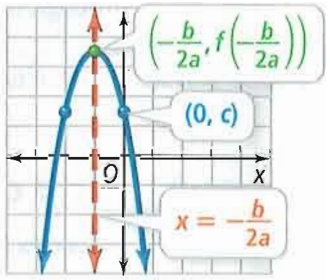
Properties Quadratic Function in Standard Form

- The graph of $f(x) = ax^2 + bx + c, a \neq 0$, is a parabola.
- If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward.
- The axis of symmetry is the line $x = -\frac{b}{2a}$.
- The x-coordinate of the vertex is $-\frac{b}{2a}$. The y-coordinate of the vertex is the y-value of the function for $x = -\frac{b}{2a}$, or $y = f\left(-\frac{b}{2a}\right)$.
- The y-intercept is $(0, c)$.

$y = ax^2 + bx + c, a > 0$



$y = ax^2 + bx + c, a < 0$



Here's Why It Works You can expand the vertex form of a quadratic function to determine properties of the graph of a quadratic function written in standard form.

$$\begin{aligned} f(x) &= a(x - h)^2 + k \\ &= a(x^2 - 2hx + h^2) + k \\ &= ax^2 - 2ahx + ah^2 + k \\ &= ax^2 + (-2ah)x + (ah^2 + k) \end{aligned}$$

Compare to the standard form, $f(x) = ax^2 + bx + c$.

$a = a$ a in standard form is the same as a in vertex form.

$b = -2ah$

$-\frac{b}{2a} = h$ Solve for h.

Since, $h = -\frac{b}{2a}$, the axis of symmetry is $x = -\frac{b}{2a}$ and the vertex is $(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Problem 2 Graphing a Function of the Form $y = ax^2 + bx + c$

What is the graph of $y = x^2 + 2x + 3$?

Think

How can you use the axis of symmetry?
The entire curve on one side of the axis is the mirror image of the curve on the other side.

Step 1 Identify a, b, and c.
$a = 1, b = 2, c = 3$

Step 2 The axis of symmetry is $x = -\frac{b}{2a}$.

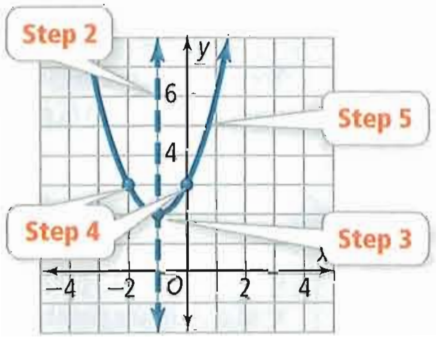
$$x = -\frac{2}{2(1)}$$

Lightly sketch the line $x = -1$.

Step 3 The x-coordinate of the vertex is also $-\frac{b}{2a}$, or -1.

The y-coordinate is
$y = (-1)^2 + 2(-1) + 3 = 2$.

Plot the vertex $(-1, 2)$.



Step 4 Since $c = 3$, the y-intercept is $(0, 3)$. The reflection of $(0, 3)$ across $x = -1$ is $(-2, 3)$. Plot both points.

Step 5 $a > 0$ confirms that the graph opens upward. Draw a smooth curve through the points you found in Steps 3 and 4.

Got It? **2.** What is the graph of $y = -2x^2 + 2x - 5$?

Problem 3 Converting Standard Form to Vertex Form

What is the vertex form of $y = 2x^2 + 10x + 7$?

Plan

How do you find *h*, *k*, and *a*?
Find the vertex. This gives you h and k. The value for a is the same in both forms.

$y = 2x^2 + 10x + 7$	Identify a and b.
$x = -\frac{b}{2a}$	Find the x-coordinate of the vertex.
$= -\frac{10}{2(2)}$	
$= -2.5$	
$y = 2(-2.5)^2 + 10(-2.5) + 7$	Substitute $x = -2.5$ into the equation.
$= -5.5$	

The vertex is $(-2.5, -5.5)$.

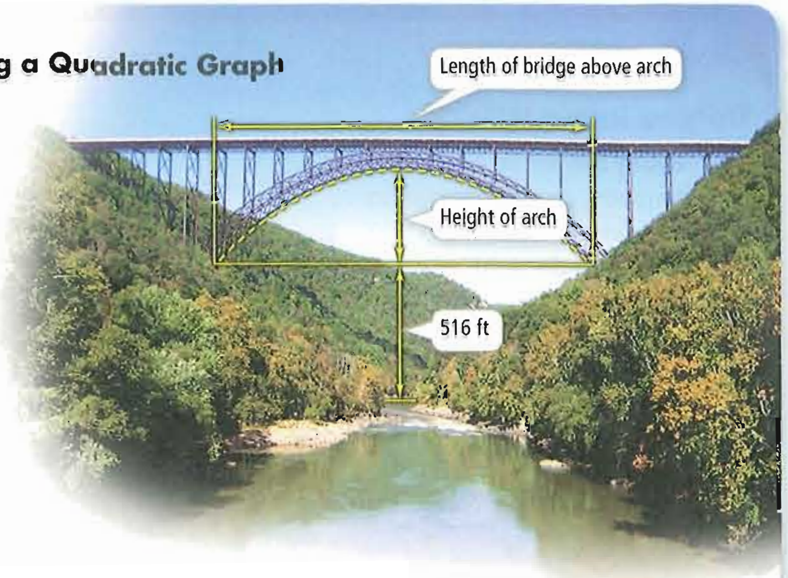
$y = a(x - h)^2 + k$	Write the vertex form.
$y = 2[x - (-2.5)]^2 + (-5.5)$	Substitute $a = 2$, $h = -2.5$, $k = -5.5$.
$y = 2(x + 2.5)^2 - 5.5$	Simplify.

The vertex form is $y = 2(x + 2.5)^2 - 5.5$.

Got It? **3.** What is the vertex form of $y = -x^2 + 4x - 5$?

Problem 4 Interpreting a Quadratic Graph

Bridges The New River Gorge Bridge in West Virginia is the world's largest steel single arch bridge. You can model the arch with the function $y = -0.000498x^2 + 0.847x$, where x and y are in feet. How high above the river is the arch? How long is the section of bridge above the arch?



Know
A function that models the arch and the vertical distance from the base of the supports to the water

Need
The height of the arch above the support base and the length of the bridge above the arch

Plan
Find the vertex. The *y*-coordinate is the height of the arch above the support base. The *x*-coordinate is half the distance between the supports.

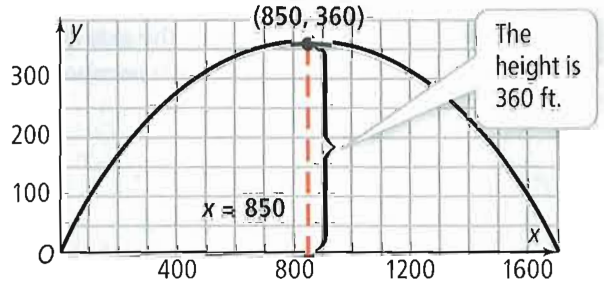
Think
How can you tell that the quadratic function has a maximum value?
Since $a < 0$, the graph of the function opens down. The function has a maximum value.

Step 1 Find the vertex of the arch.

$$x = -\frac{b}{2a} = -\frac{0.847}{2(-0.000498)} \approx 850$$

$$y = -0.000498(850)^2 + 0.847(850) \approx 360$$

The vertex is about (850, 360).



Step 2 Find the height of the arch above its supports.

The *y*-coordinate of the vertex is the height of the arch above its supports. The arch is about 360 ft above its supports.

Step 3 Find the height of the arch above the river.

The arch is about 360 ft + 516 ft = 876 ft above the river.

Step 4 Find the length of the bridge above the arch.

The *x*-coordinate of the vertex is half the length of the bridge above the arch. The length of that part of the bridge is about 850 ft + 850 ft = 1700 ft long.

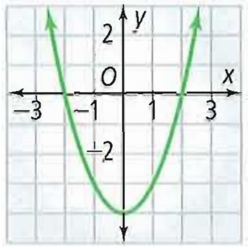
Got It? **4. a.** The Zhaozhou Bridge in China is the oldest known arch bridge, dating to A.D. 605. You can model the support arch with the function $f(x) = -0.001075x^2 + 0.131148x$, where x and y are measured in feet. How high is the arch above its supports?

b. Reasoning Why does the model in part (a) not have a constant term?

Lesson Check

Do you know HOW?

1. Identify the vertex, axis of symmetry, and the maximum or minimum value of the parabola at the right.



Graph each function.

2. $y = x^2 - 2x + 4$

3. $y = -x^2 - 3x + 6$

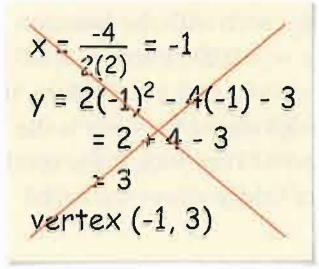
Write each function in vertex form.

4. $y = x^2 - 2x + 9$

5. $y = -x^2 + 3x - 1$

Do you UNDERSTAND?

6. Error Analysis A student graphed the function $y = 2x^2 - 4x - 3$. Find and correct the error.



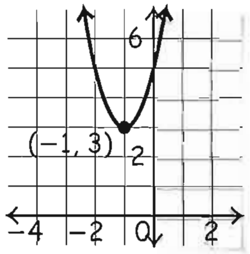
$x = \frac{-4}{2(2)} = -1$

$y = 2(-1)^2 - 4(-1) - 3$

$= 2 + 4 - 3$

$= 3$

vertex (-1, 3)



7. Compare and Contrast Explain the difference between finding the vertex of a function written in vertex form and finding the vertex of a function written in standard form.

Practice and Problem-Solving Exercises

A Practice

Identify the vertex, the axis of symmetry, the maximum or minimum value, and the range of each parabola.

See Problem 1.

8. $y = x^2 + 2x + 1$

9. $y = -x^2 + 2x + 1$

10. $y = x^2 + 4x + 1$

11. $y = -x^2 + 2x + 5$

12. $y = 3x^2 - 4x - 2$

13. $y = -2x^2 - 3x + 4$

14. $y = 2x^2 - 6x + 3$

15. $y = -x^2 - x$

16. $y = 2x^2 + 5$

Graph each function.

See Problem 2.

17. $y = x^2 + 6x + 9$

18. $y = -x^2 - 3x + 6$

19. $y = 2x^2 + 4x$

20. $y = 4x^2 - 12x + 9$

21. $y = -6x^2 - 12x - 1$

22. $y = -\frac{3}{4}x^2 + 6x + 6$

23. $y = 3x^2 - 12x + 10$

24. $y = \frac{1}{2}x^2 + 2x - 8$

25. $y = -4x^2 - 24x - 36$

Write each function in vertex form.

See Problem 3.

26. $y = x^2 - 4x + 6$

27. $y = x^2 + 2x + 5$

28. $y = 4x^2 + 7x$

29. $y = 2x^2 - 5x + 12$

30. $y = -2x^2 + 8x + 3$

31. $y = \frac{9}{4}x^2 + 3x - 1$

32. Economics A model for a company's revenue from selling a software package is $R = -2.5p^2 + 500p$, where p is the price in dollars of the software. What price will maximize revenue? Find the maximum revenue.

See Problem 4.

B Apply

Sketch each parabola using the given information.

33. vertex (3, 6), y-intercept 2

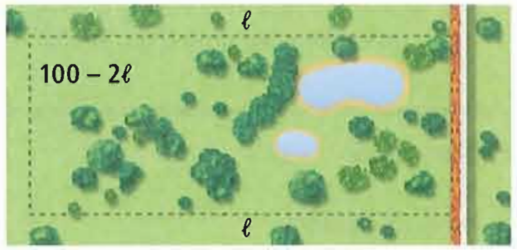
34. vertex (−1, −4), y-intercept 3

35. vertex (0, 5), point (1, −2)

36. vertex (2, 3), point (6, 9)

37. **Think About a Plan** Suppose you work for a packaging company and are designing a box that has a rectangular bottom with a perimeter of 36 cm. The box must be 4 cm high. What dimensions give the maximum volume?
- How can you model the volume of the box with a quadratic function?
- What information can you get from the function to find the maximum volume?

38. **Landscaping** A town is planning a playground. It wants to fence in a rectangular space using an existing wall. What is the greatest area it can fence in using 100 ft of donated fencing?



For each function, the vertex of the function's graph is given. Find the unknown coefficients.

39. $y = x^2 + bx + c$; (3, −4)

40. $y = -3x^2 + bx + c$; (1, 0)

41. $y = ax^2 + 10x + c$; (−5, −27)

42. $y = c - ax^2 - 2x$; (−1, 3)

43. **Physics** The equation for the motion of a projectile fired straight up at an initial velocity of 64 ft/s is $h = 64t - 16t^2$, where h is height in feet and t is time in seconds. Find the time the projectile needs to reach its highest point. How high will it go?

44. A student says that the graph of $y = ax^2 + bx + c$ gets wider as a increases.
- **a.** **Error Analysis** Use examples to show that the student is wrong.
- **b.** **Writing** Summarize the relationship between $|a|$ and the width of the graph of $y = ax^2 + bx + c$.

For each function, find the y-intercept.

45. $y = (x - 1)^2 + 2$

46. $y = -3(x + 2)^2 - 4$

47. $y = -\frac{2}{3}(x - 9)^2$

C Challenge

For each function, the vertex of the function's graph is given. Find a and b.

48. $y = ax^2 + bx - 27$; (2, −3)

49. $y = ax^2 + bx + 5$; (−1, 4)

50. $y = ax^2 + bx + 8$; (2, −4)

51. $y = ax^2 + bx$; (−3, 2)

Sketch each parabola using the given information.

52. axis of symmetry $x = 1$, y-intercept 3, point (−1, 6)

53. axis of symmetry $x = 2$, y-intercept 1, point (3, 2.5)

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.2.12 **54.** The time it takes to chalk a baseball diamond varies directly with the length of the side of the diamond. If it takes 10 minutes to chalk a little league diamond with 60 ft sides, how long will it take to chalk a major league baseball diamond with 90 ft sides?

MA.912.A.7.5 **55.** What is the x-value of the vertex of the quadratic function $y = -5x^2 + \frac{4}{7}$?

MA.912.A.3.15 **56.** Sarah works as a nanny and charges different rates for working during the week and the weekend. One week, she earned $902.50 working 45 hours, of which 5 hours were during the weekend. The following week she earned $1045 working 50 hours, of which 10 hours were during the weekend. What does Sarah charge per hour, in dollars, for working during the week?

Mixed Review

Solve each equation. **See Lesson 1-4.**

57. $0.6(y + 2) - 0.2(2 - y) = 1$

58. $3(a + 4) + 2(a - 1) = a$

For each system, choose the method of solving that seems easier to use. Explain why you made each choice. Solve each system. **See Lessons 3-2 and 3-6.**

59. $\begin{cases} 3x - 5y = 26 \\ -2x - 3y = -11 \end{cases}$

60. $\begin{cases} y = \frac{2}{3}x - 3 \\ -x + 3y = 18 \end{cases}$

61. $\begin{cases} 2m + 3n = 12 \\ -5m + n = -13 \end{cases}$

Get Ready! **To prepare for Lesson 4-3, do Exercises 62–70.**

Identify the vertex, the axis of symmetry, the maximum or minimum value, and the domain and range of each function. **See Lesson 4-2.**

62. $y = 2(x + 2)^2 - 1$

63. $y = -(x - 1)^2 + 3$

64. $y = \frac{1}{2}(x - 3)^2 - 2$

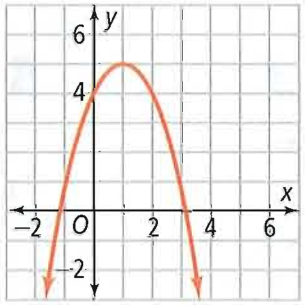
65. $y = -\frac{2}{5}(x + 3)^2 + 5$

66. $y = 3(x + 4)^2$

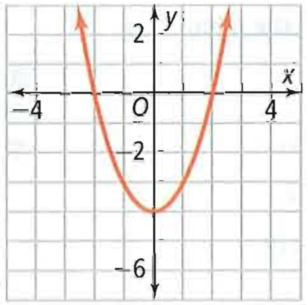
67. $y = -7(x - 4)^2 + 6$

Write an equation in vertex form for each parabola. **See Lesson 4-1.**

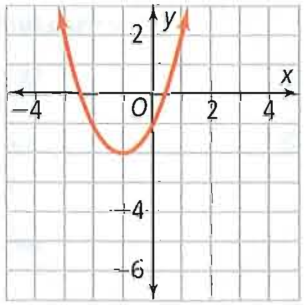
68.



69.



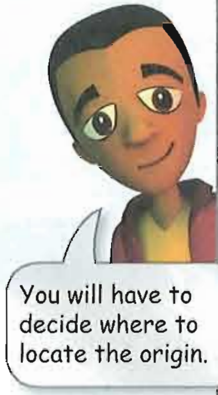
70.

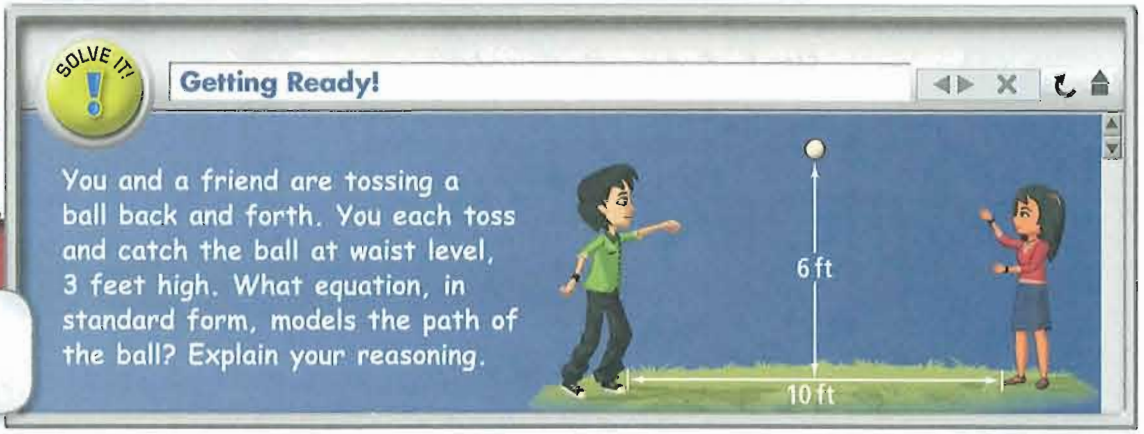


4-3 Modeling With Quadratic Functions

Sunshine State Standard
MA.912.A.3.15 Solve real-world problems involving systems of linear equations in two and three variables.

Objective To model data with quadratic functions





You and a friend are tossing a ball back and forth. You each toss and catch the ball at waist level, 3 feet high. What equation, in standard form, models the path of the ball? Explain your reasoning.

When you know the vertex and a point on a parabola, you can use vertex form to write an equation of the parabola. If you do not know the vertex, you can use standard form and any three points of the parabola to find an equation.

Essential Understanding Three noncollinear points, no two of which are in line vertically, are on the graph of exactly one quadratic function.

Problem 1 Writing an Equation of a Parabola

Plan
How do you use the 3 given points?
Use them to write a system of 3 equations. Solve the system to get *a*, *b*, and *c*.

A parabola contains the points (0, 0), (−1, −2), and (1, 6). What is the equation of this parabola in standard form?

Substitute the (x, y) values into $y = ax^2 + bx + c$ to write a system of equations.

$$\begin{aligned} y &= ax^2 + bx + c & & \\ 0 &= a(0)^2 + b(0) + c & \rightarrow \quad c &= 0 & &\text{Use } (0, 0). \\ -2 &= a(-1)^2 + b(-1) + c & \rightarrow \quad -2 &= a - b + c & &\text{Use } (-1, -2). \\ 6 &= a(1)^2 + b(1) + c & \rightarrow \quad 6 &= a + b + c & &\text{Use } (1, 6). \end{aligned}$$

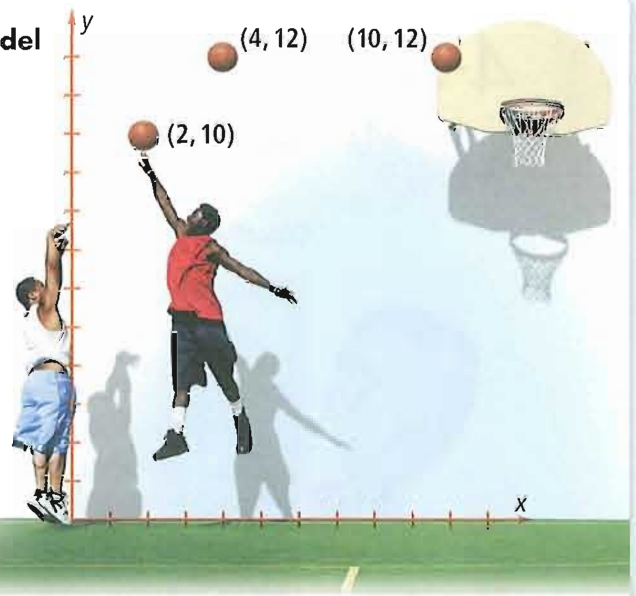
Since $c = 0$, the resulting system has two variables. $\begin{cases} a - b = -2 \\ a + b = 6 \end{cases}$ Use elimination. $a = 2$ and $b = 4$.

Substitute $a = 2$, $b = 4$, and $c = 0$ into standard form: $y = 2x^2 + 4x + 0$.
$y = 2x^2 + 4x$ is the equation of the parabola that contains the given points.

Got It? **1.** What is the equation of a parabola containing the points (0, 0), (1, −2), and (−1, −4)?

Problem 2 Using a Quadratic Model

Basketball **A player throws a basketball toward the hoop. The basketball follows a parabolic path through the points shown. If the center of the hoop is at (12, 10), will the ball pass through the hoop? (You can think of the units as feet.)**



Step 1 Find a quadratic model.

Substitute the x and y values into the standard form of a quadratic function. The result is a system of three linear equations.

$$y = ax^2 + bx + c$$

$$10 = a(2)^2 + b(2) + c \qquad \text{Use } (2, 10).$$

$$12 = a(10)^2 + b(10) + c \qquad \text{Use } (10, 12).$$

$$12 = a(4)^2 + b(4) + c \qquad \text{Use } (4, 12).$$

Use one of the methods from Chapter 3. Solve. $\begin{cases} 4a + 2b + c = 10 \\ 100a + 10b + c = 12 \\ 16a + 4b + c = 12 \end{cases}$

The solution is $a = -0.125$, $b = 1.75$, and $c = 7$.

Substitute the values into the standard form of a quadratic function. An equation of the parabola is $y = -0.125x^2 + 1.75x + 7$.

Think

In terms of the quadratic model, what does it mean for the ball to pass through the hoop?

The point (12, 10) is on the parabola. That is, the point (12, 10) satisfies the quadratic equation.

Step 2 Use the quadratic model to see if the player makes the basket.

$$y = -0.125x^2 + 1.75x + 7$$

$$10 \overset{?}{=} -0.125(12)^2 + 1.75(12) + 7 \qquad \text{Substitute } (x, y) = (12, 10).$$

$$10 = 10 \checkmark$$

The point (12, 10) is on the parabola. The ball will pass through the hoop.

Got It? **2. a.** The parabolic path of a thrown ball can be modeled by the table. The top of a wall is at (5, 6). Will the ball go over the wall? If not, will it hit the wall on the way up, or the way down?

b. **Reasoning** What is a reasonable domain and range for the function that models the path of the ball?

x	y
1	3
2	5
3	6

When more than three data points suggest a quadratic function, you can use the quadratic regression feature of a graphing calculator to find a quadratic model.

Problem 3 Using Quadratic Regression

The table shows a meteorologist's predicted temperatures for an October day in Sacramento, California.

Sacramento, CA

Time	Predicted Temperature (°F)
8 A.M.	52
10 A.M.	64
12 P.M.	72
2 P.M.	78
4 P.M.	81
6 P.M.	76

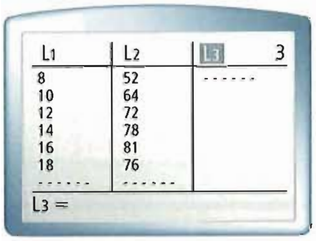
Think

How do you write times using a 24-hour clock?

Add 12 to the number of hours past noon. So, 2 P.M. is 14:00 in the 24-hour clock.

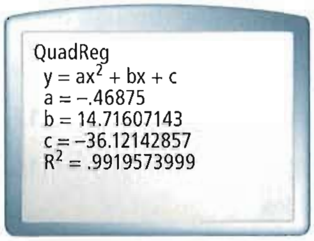
A **What is a quadratic model for this data?**

Step 1 Enter the data. Use the 24-hour clock to represent times after noon.



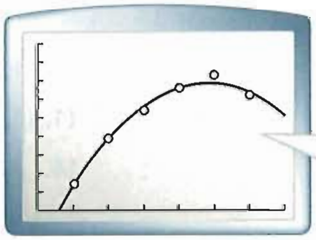
```
L1      L2      L3    3
8       52      ------
10      64
12      72
14      78
16      81
18      76
------  ------
L3 =
```

Step 2 Use **QuadReg.**



```
QuadReg
 y = ax^2 + bx + c
 a = -.46875
 b = 14.71607143
 c = -36.12142857
 R^2 = .9919573999
```

Step 3 Graph the data and the function.

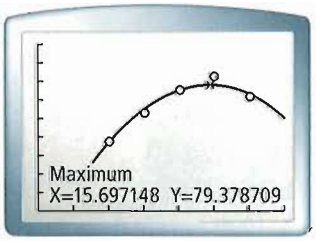


A quadratic model is reasonable.

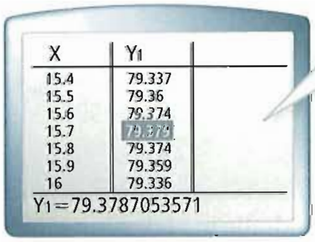
A quadratic model for temperature is $y = -0.469x^2 + 14.716x - 36.121$.

B **Use your model to predict the high temperature for the day. At what time does the high temperature occur?**

Use the **Maximum** feature or tables.



```
Maximum
X=15.697148  Y=79.378709
```



```
X       Y1
15.4    79.337
15.5    79.36
15.6    79.374
15.7    79.379
15.8    79.374
15.9    79.359
16      79.336
Y1=79.3787053571
```

16 represents 4 P.M. The maximum occurs at approximately 15.7, or about 3:42 P.M.

Predict the high temperature for the day to be 79.4°F at about 3:42 P.M.

Got It? **3.** The table shows a meteorologist's predicted temperatures for a summer day in Denver, Colorado. What is a quadratic model for this data? Predict the high temperature for the day. At what time does the high temperature occur?

Denver, CO

Time	Predicted Temperature (°F)
6 A.M.	63
9 A.M.	76
12 P.M.	86
3 P.M.	89
6 P.M.	85
9 P.M.	76

Lesson Check

Do you know HOW?

Find a quadratic function that includes each set of values.

1. $(1, 0), (2, -3), (3, -10)$

2.

x	-2	-1	0	1	2
y	3.5	3.5	7.5	15.5	27.5

3.

x	-2	-1	0	1
y	-41.5	-25.5	-13.5	-5.5

Do you UNDERSTAND?

4. Compare and Contrast How do you know whether to perform a linear regression or a quadratic regression for a given set of data?

5. Reasoning Explain how you can determine if the four points $(2, -8)$, $(4, 3)$, $(7, -1)$ and $(9, 5)$ lie on a single parabola.

6. Error Analysis Your classmate says he can write the equation of a quadratic function that passes through the points $(3, 4)$, $(5, -2)$, and $(3, 0)$. Explain his error.

Practice and Problem-Solving Exercises

A Practice

Find an equation in standard form of the parabola passing through the points.

See Problem 1.

7. $(1, -2), (2, -2), (3, -4)$

8. $(1, -2), (2, -4), (3, -4)$

9. $(-1, 6), (1, 4), (2, 9)$

10. $(1, 1), (-1, -3), (-3, 1)$

11. $(3, -6), (1, -2), (6, 3)$

12. $(-2, 9), (-4, 5), (1, 0)$

13.

x	$f(x)$
-1	-1
1	3
2	8

14.

x	$f(x)$
-1	17
1	17
2	8

15.

x	$f(x)$
-1	-4
1	-2
2	-4

16. Communications The table shows the percent of houses with cable TV in a city.

See Problems 2 and 3.

Cable Television Access

Year	1985	1990	1995	2000	2005
% of Households	46	59	66	68	69

a. Find a quadratic model using 1985 as year 0, 1990 as year 5, and so on.

b. Use the model to estimate the percent of households with cable TV in 1998.

17. Physics A man throws a ball off the top of a building and records the height of the ball at different times, as shown in the table.

a. Find a quadratic model for the data.

b. Use the model to estimate the height of the ball at 2.5 seconds.

c. What is the ball's maximum height?

Height of a Ball

Time(s)	Height (ft)
0	46
1	63
2	48
3	1

B Apply

Determine whether a quadratic model exists for each set of values. If so, write the model.

18. $f(-2) = 16, f(0) = 0, f(1) = 4$

19. $f(0) = 5, f(2) = 3, f(-1) = 0$

20. $f(-1) = -4, f(1) = -2, f(2) = -1$

21. $f(-2) = 7, f(0) = 1, f(2) = 0$

22. a. Geometry Copy and complete the table. It shows the total number of segments whose endpoints are chosen from x points, no three of which are collinear.

Number of points, x	2	3	🟩	🟩
Number of segments, y	1	3	🟩	🟩

b. Write a quadratic model for the data.

c. Predict the number of segments that can be drawn using 10 points.

23. Think About a Plan The table shows the height of a column of water as it drains from its container. Use a quadratic model of this data to estimate the water level at 30 seconds.

- What system of equations can you use to solve this problem?
- How can you determine if your answer is reasonable?

Water Levels

Elapsed Time (s)	Water Level (mm)
0	120
20	83
40	50

24. A parabola contains the points $(-1, 8)$, $(0, 4)$, and $(1, 2)$. Name another point also on the parabola.

25. a. Postal Rates Find a quadratic model for the data. Use 1981 as year 0.

Price of First-Class Stamp

Year	1981	1991	1995	1999	2001	2006	2007	2008
Price (cents)	18	29	32	33	34	39	41	42

Source: United States Postal Service

b. Describe a reasonable domain and range for your model. (*Hint*: This is a discrete, real situation.)

c. Estimation Estimate when first-class postage was 37 cents.

d. Use your model to predict when first-class postage will be 50 cents. Explain why your prediction may not be valid.

26. Road Safety The table below gives the stopping distance for an automobile under certain road conditions.

Speed (mi/h)	20	30	40	50	55
Stopping Distance (ft)	17	38	67	105	127

a. Find a linear model for the data.

b. Find a quadratic model for the data.

c. Writing Compare the models. Which is better? Explain.

27. Open-Ended Write three different quadratic functions, each with a graph that includes $(0, 0)$ and $(5, -1)$.

Challenge

28. **Reasoning** What is the minimum number of data points you need to find a single quadratic model for a data set? Explain.

29. A parabola contains the points (0, −4), (2, 4), and (4, 4). Find the vertex.

30. A model for the height of an arrow shot into the air is $h(t) = -16t^2 + 72t + 5$, where t is time and h is height. Without graphing, answer the following questions.
a. What can you learn by finding the graph's intercept with the h-axis?
b. What can you learn by finding the graph's intercept(s) with the t-axis?

Sunshine State Standards Practice

MA.912.A.7.6 **31.** The graph of a quadratic function has vertex (−3, −2). What is the axis of symmetry?

Ⓐ $x = -3$ Ⓑ $x = 3$ Ⓒ $y = -2$ Ⓓ $y = 2$

MA.912.A.2.6 **32.** Which function is NOT a quadratic function?

Ⓕ $y = (x - 1)(x - 2)$ Ⓗ $y = 3x - x^2$
Ⓖ $y = x^2 + 2x - 3$ Ⓘ $y = -x^2 + x(x - 3)$

MA.912.A.2.8 **33.** Which is the composition $f(g(x))$, if $f(x) = -x - 3$ and $g(x) = 7 + 5x$?

Ⓐ $f(g(x)) = 4x + 4$ Ⓒ $f(g(x)) = -5x - 8$
Ⓑ $f(g(x)) = 4x - 10$ Ⓓ $f(g(x)) = -5x - 10$

MA.912.A.3.15 **34.** **Extended Response** Mark has 42 coins consisting of dimes and quarters. The total value of his coins is $6. How many of each type of coin does he have? Show all your work and explain what method you used to solve the problem.

Mixed Review

Graph each function. **See Lesson 4-2.**

35. $y = x^2 - 6x - 3$ **36.** $y = 2x^2 + 9x - 4$ **37.** $y = 3x^2 - 4x + 1$

Solve each system by elimination. **See Lesson 3-2.**

38. $\begin{cases} x + y = 7 \\ 5x - y = 5 \end{cases}$ **39.** $\begin{cases} 2x - 3y = -14 \\ 3x - y = 7 \end{cases}$ **40.** $\begin{cases} x - 3y = 2 \\ x - 2y = 1 \end{cases}$

For Exercises 41–42, y varies directly with x. **See Lesson 2-2.**

41. If $y = 2$ when $x = 5$, find y when $x = 2$. **42.** If $y = -2$ when $x = 4$, find y when $x = 7$.

Get Ready! **To prepare for Lesson 4-4, do Exercises 43–45.**

Simplify by combining like terms. **See Lesson 1-3.**

43. $x^2 + x + 4x - 1$ **44.** $6x^2 - 4(3)x + 2x - 3$ **45.** $4x^2 - 2(5 - x) - 3x$

Concept Byte

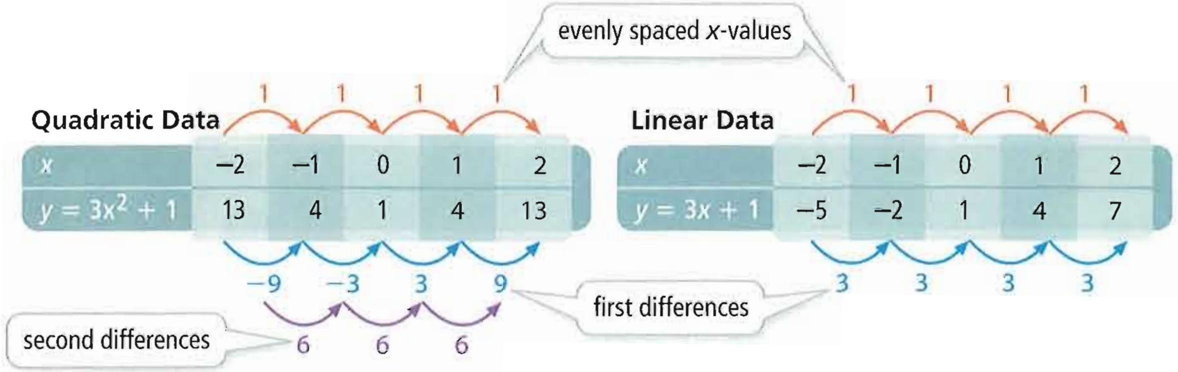
For Use With Lesson 4-3

Identifying Quadratic Data

Sunshine State Standard
Prepares for MA.912.A.2.6 Identify quadratic functions.

You can identify perfect quadratic data when x-values are evenly spaced using the pattern in the differences between y-values.

Example



x	−2	−1	0	1	2
$y = 3x^2 + 1$	13	4	1	4	13

x	−2	−1	0	1	2
$y = 3x + 1$	−5	−2	1	4	7

For linear data, the *first* differences of adjacent y-values are constant. For quadratic data, the *second* differences are constant.

Exercises

Determine if each data set represents perfect quadratic data.

1.

x	y
−2	−5
−1	0
0	5
1	10
2	15

2.

x	y
1	6
2	12
3	22
4	36
5	54
6	76
7	102

3.

x	y
−2	−8
−1	−1
0	0
1	1
2	8
3	27
4	64

4.

x	y
−3	−1
−2.5	−3.75
−2	−6
−1.5	−7.75
−1	−9

5.

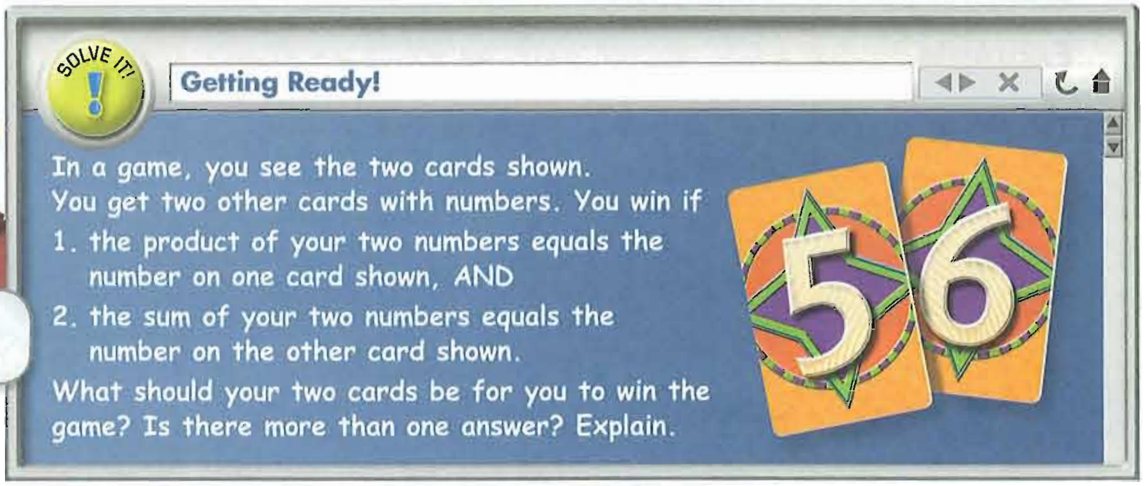
x	y
−3	−13
−1	−8
1	−5
3	−4
5	−5

6. Reasoning Can you use the method above to determine if a data set represents perfect linear or quadratic data if the x-values are *not* evenly spaced? Explain.

4-4 Factoring Quadratic Expressions

Sunshine State Standard
MA.912.A.4.3 Factor polynomial expressions.

Objectives To find common and binomial factors of quadratic expressions
To factor special quadratic expressions



In a game, you see the two cards shown. You get two other cards with numbers. You win if

1. the product of your two numbers equals the number on one card shown, AND
2. the sum of your two numbers equals the number on the other card shown.

What should your two cards be for you to win the game? Is there more than one answer? Explain.

Dynamic Activities
Factoring $x^2 + bx + c$
Factoring $ax^2 + bx + c$

Lesson Vocabulary
- factoring
- greatest common factor (GCF) of an expression
- perfect square trinomial
- difference of two squares

Factors of a given number are numbers that have a product equal to the given number. Factors of a given expression are expressions that have a product equal to the given expression. **Factoring** is rewriting an expression as a product of its factors.

Essential Understanding You can factor many quadratic trinomials ($ax^2 + bx + c$) into products of two binomials.

You can use the Distributive Property or the FOIL method to multiply two binomials. You can use FOIL in reverse to help you factor.

$$(x + 4)(x + 2) = (x + 4)(x) + (x + 4)(2)$$ Use the Distributive Property.

$$= x(x) + 4(x) + x(2) + 4(2)$$

$$= x^2 + 6x + 8$$

$$(x + 4)(x + 2) = x(x) + x(2) + 4(x) + 4(2)$$ F: First; O: Outer; I: Inner; L: Last

$$= x^2 + 6x + 8$$

To factor $x^2 + 6x + 8$, think of FOIL in reverse. Find two binomials for which the first terms have the product x^2, the products of the outer and inner terms have the sum $6x$, and the last terms have the product 8.

$$x^2 + 6x + 8 = (x + 4)(x + 2)$$

When you factor, a table of the different possible factors of the constant term may be helpful.

Problem 1 Factoring $ax^2 + bx + c$ when $a = \pm 1$

Plan

How can you *make a table* to find factors? Use the first row to list sets of factors of the constant. Use the second row to find the sum of each set of factors.

What is the expression in factored form?

A $x^2 + 9x + 20$

Step 1 Find factors of 20 with sum 9.
Since both 20 and 9 are positive, both factors are positive.

Factors of 20	1, 20	2, 10	4, 5
Sum of factors	21	12	9

Step 2 Use the factors you found. Write the expression as the product of two binomials.

$x^2 + 9x + 20 = (x + 4)(x + 5)$ Use the factors 4 and 5.

B $x^2 + 14x - 72$

Step 1 Find factors of −72 with sum 14.

Since $c < 0$, one factor is positive and the other is negative.
Since $b > 0$, the factor with greater absolute value is positive.

Factors of −72	−1, 72	−2, 36	−3, 24	−4, 18	−6, 12	−8, 9
Sum of factors	71	34	21	14	6	1

Step 2 Use the factors you found, −4 and 18. Write
$x^2 + 14x - 72 = (x - 4)(x + 18)$.

Think

Will factoring out −1 change the answer?
No; because the final factored expression will include −1 as a factor.

C $-x^2 + 13x - 12$

Step 1 Rewrite the expression to show a trinomial with leading coefficient 1.

$-(x^2 - 13x + 12)$ Factor out −1.

Step 2 Find factors of 12 with sum −13.

Since $c > 0$, both factors have the same sign.
Since $b < 0$, both factors must be negative.

Factors of 12	−1, −12	−2, −6	−3, −4
Sum of factors	−13	−8	−7

Step 3 Use the factors you found, −1 and −12. Write
$-x^2 + 13x - 12 = -(x^2 - 13x + 12) = -(x - 1)(x - 12)$.

Got It? **1.** What is the expression in factored form?

a. $x^2 + 14x + 40$ **b.** $x^2 - 11x + 30$ **c.** $-x^2 + 14x + 32$

The **greatest common factor (GCF) of an expression** is a common factor of the terms in the expression. It is the common factor with the greatest coefficient and the greatest exponent. You can factor any expression that has a GCF not equal to 1.

Problem 2 Finding Common Factors

What is the expression in factored form?

Plan

Should you factor out a number, a variable, or both?
Both; the two terms have numerical and variable common factors.

A $6n^2 + 9n$

$6n^2 + 9n = 3n(2n) + 3n(3)$	Factor out the GCF, $3n$.
$= 3n(2n + 3)$	Use the Distributive Property.

B $4x^2 + 20x - 56$

$4x^2 + 20x - 56 = 4(x^2) + 4(5x) - 4(14)$	Factor out the GCF, 4.
$= 4(x^2 + 5x - 14)$	Use the Distributive Property.
$= 4(x - 2)(x + 7)$	Factor the trinomial.

Got It? **2.** What is the expression in factored form?

a. $7n^2 - 21$ **b.** $9x^2 + 9x - 18$ **c.** $4x^2 + 8x + 12$

To factor a quadratic trinomial of the form $ax^2 + bx + c$ where $a \neq 1$, and there is no common factor, find factors of ac that have sum b.

Problem 3 Factoring $ax^2 + bx + c$ when $|a| \neq 1$

What is the expression in factored form?

A $2x^2 + 11x + 12$

Step 1 Since there is no common factor, find ac.

$ac = 2(12) = 24$

Step 2 Since both b and ac are positive, find positive factors of 24 that have sum 11.

Think

How should you make your table in this case?
Use the first row to list sets of factors of ac. Use the second row as before, to find the sum of each set of factors.

Factors of 24	1, 24	2, 12	3, 8	4, 6
Sum of factors	25	14	11	10

Step 3 Factor the trinomial as follows.

$2x^2 + 11x + 12$	
$2x^2 + 3x + 8x + 12$	Rewrite bx using $b = 3 + 8$.
$x(2x + 3) + 4(2x + 3)$	Find a common factor for the first two terms and another common factor for the last two terms.
$(x + 4)(2x + 3)$	Rewrite using the Distributive Property.

Think

Do you need positive or negative factors?
You need one negative factor and one positive factor.

B $4x^2 - 4x - 3$

Step 1 Since there is no common factor, find ac.

$ac = 4(-3) = -12$

Step 2 Since $ac = -12 < 0$, find factors of ac with opposite signs. Since $b < 0$, the factor with greater absolute value is negative.

Factors of −12	1, −12	2, −6	3, −4
Sum of factors	−11	−4	−1

Step 3 Factor the trinomial as follows.

$4x^2 - 4x - 3$	
$4x^2 + 2x - 6x - 3$	Rewrite bx using $b = 2 - 6$.
$2x(2x + 1) - 3(2x + 1)$	Find a common factor for the first two terms and another common factor for the last two terms.
$(2x - 3)(2x + 1)$	Rewrite using the Distributive Property.

Check $(2x - 3)(2x + 1) = 4x^2 + 2x - 6x - 3$

$= 4x^2 - 4x - 3$ ✔

Got It? **3.** What is the expression in factored form? Check your answers.

a. $4x^2 + 7x + 3$ **b.** $2x^2 - 7x + 6$

c. **Reasoning** Can you factor the expression $2x^2 + 2x + 2$ into a product of two binomials? Explain.

A **perfect square trinomial** is a trinomial that is the square of a binomial. For example, $x^2 + 10x + 25 = (x + 5)^2$ is a perfect square trinomial.

If $ax^2 + bx + c$ is a perfect square trinomial, then ax^2 and c are squares of the terms of the binomial and thus are both positive. bx is twice the product of the terms of the binomial. b is negative if the binomial terms have opposite signs.

Here is another way to represent the two forms of a perfect square trinomial.

take note

Key Concept **Factoring Perfect Square Trinomials**

$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$

Problem 4 Factoring a Perfect Square Trinomial

What is $4x^2 - 24x + 36$ in factored form?

Think	Write
Write the expression.	$4x^2 - 24x + 36$
You can do this in one step. Is the first term a perfect square? Yes, $4x^2$ is $(2x)^2$. Is the last term a perfect square? Yes, 36 is 6^2. Is the middle term twice the product of 6 and $2x$? Yes, $2 \bullet 6 \bullet 2x = 24x$ There's a minus sign in the middle. You are done.	$(2x - 6)^2$
You should check.	$(2x - 6)^2 = 4x^2 - 24x + 36$ ✔

Got It? **4.** What is $64x^2 - 16x + 1$ in factored form?

The expression $a^2 - b^2$ is the **difference of two squares**. There is a pattern to its factors.

take note

Key Concept Factoring a Difference of Two Squares

$$a^2 - b^2 = (a + b)(a - b)$$

Think

How can a binomial be the product of two binomials?
If the outer and inner products of the binomials are opposites, their sum is zero.

Problem 5 Factoring a Difference of Two Squares

What is $25x^2 - 49$ in factored form?

$25x^2 - 49 = (5x)^2 - 7^2$ Write as the difference of two squares.

$= (5x + 7)(5x - 7)$ Use the pattern for factoring a difference of two squares.

Got It? **5.** What is $16x^2 - 81$ in factored form?

Lesson Check

Do you know HOW?

Factor each expression.

1. $x^2 + 6x + 8$

2. $x^2 - 13x + 12$

3. $x^2 - 81$

4. $25y^2 - 36$

5. $y^2 - 6y + 9$

6. $4x^2 - 4x + 1$

Find the GCF of each expression.

7. $15x^2 - 25x$

8. $4a^3 + 8a^2$

9. $18b^2 - 12b + 24$

10. $21h^3 + 35h^2 - 28h$

Do you UNDERSTAND?

11. **Vocabulary** Is $4b^2 - 26b + 169$ a perfect square trinomial? Explain.

12. **Compare and Contrast** How is factoring a trinomial $ax^2 + bx + c^2$ when $a \neq 1$ different from factoring a trinomial when $a = 1$? How is it similar?

13. **Reasoning** Explain how to rewrite the expression $a^2 - 2ab + b^2 - 25$ as the product of two trinomial factors. (*Hint:* Group the first three terms. What type of expression is this?)

Practice and Problem-Solving Exercises

A Practice

Factor each expression. **See Problem 1.**

14. $x^2 + 3x + 2$

15. $x^2 + 5x + 6$

16. $x^2 + 7x + 10$

17. $x^2 + 10x + 16$

18. $y^2 + 15y + 36$

19. $x^2 + 22x + 40$

20. $x^2 - 3x + 2$

21. $-x^2 + 13x - 12$

22. $-r^2 + 11r - 18$

23. $x^2 - 10x + 24$

24. $d^2 - 12d + 27$

25. $x^2 - 13x + 36$

26. $x^2 - 5x - 14$

27. $-x^2 - x + 20$

28. $-x^2 + 3x + 40$

29. $c^2 + 2c - 63$

30. $x^2 + 10x - 75$

31. $-t^2 + 7t + 44$

Find the GCF of each expression. Then factor the expression. **See Problem 2.**

32. $3a^2 + 9$

33. $25b^2 - 20b$

34. $x^2 - 2x$

35. $5t^2 - 5t - 10$

36. $14y^2 + 7y - 21$

37. $27p^2 - 9p + 18$

Factor each expression. **See Problems 3–5.**

38. $3x^2 + 31x + 36$

39. $2x^2 - 19x + 24$

40. $5r^2 + 23r + 26$

41. $2m^2 - 11m + 15$

42. $5t^2 + 28t + 32$

43. $2x^2 - 27x + 36$

44. $3x^2 + 7x - 20$

45. $5y^2 + 12y - 32$

46. $7x^2 - 8x - 12$

47. $2z^2 + z - 28$

48. $3x^2 + 8x - 16$

49. $28k^2 + 13k - 6$

50. $x^2 + 2x + 1$

51. $t^2 - 14t + 49$

52. $k^2 - 18k + 81$

53. $4z^2 - 20z + 25$

54. $9x^2 + 48x + 64$

55. $81z^2 + 36z + 4$

56. $x^2 - 4$

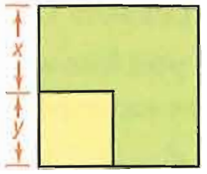
57. $c^2 - 64$

58. $9x^2 - 1$

B Apply

59. Think About a Plan Suppose you cut a small square from a square sheet of cardboard. Find the sides of one rectangle whose area is equal to the area of the remaining part.

- How can you represent the remaining part as a combination of rectangles with known sides?
- Can you factor the resulting expression?



60. The area in square centimeters of a square area rug is $25x^2 - 10x + 1$. What are the dimensions of the rug in terms of x?

Factor each expression completely.

61. $9x^2 - 36$

62. $18z^2 - 8$

63. $12y^2 - 75$

64. $64t^2 - 16$

65. $12x^2 + 36x + 27$

66. $16x^2 - 80x + 100$

67. $2a^2 - 16a + 32$

68. $3x^2 - 24x - 27$

69. $18b^2 + 24b - 10$

70. $4n^2 - 20n + 24$

71. $3y^2 + 24y + 45$

72. $-x^2 + 5x - 4$

73. $4x^2 - 22x + 10$

74. $\frac{1}{2}x^2 - \frac{1}{2}$

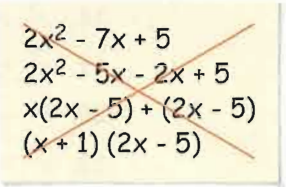
75. $-6z^2 - 600$

76. $2x^2 - 11x + 5$

77. $x^2 - y^2$

78. $-\frac{1}{16}s^2 + 1$

79. Error Analysis Your friend attempted to factor an expression as shown. Find the error in your friend's work. Then factor the expression correctly.



~~$2x^2 - 7x + 5$~~
~~$2x^2 - 5x - 2x + 5$~~
~~$x(2x - 5) + (2x - 5)$~~
~~$(x + 1)(2x - 5)$~~

80. Agriculture The area in square feet of a rectangular field is $x^2 - 120x + 3500$. The width, in feet, is $x - 50$. What is the length, in feet?

Find the GCF of each expression. Then factor the expression.

81. $y^2 - y$

82. $ab^2 - b$

83. $10x^2 - 90$

84. $3t^2 - 24t$

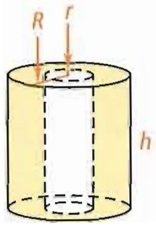
85. $2x^2 - 74x + 12$

86. $x^2y^2 + xy$

87. What is the factored form of $4x^2 + 15x - 4$?

Ⓐ $(2x + 2)(2x - 2)$
Ⓑ $(2x - 4)(2x + 1)$
Ⓒ $(4x + 1)(x - 4)$
Ⓓ $(4x - 1)(x + 4)$

88. Geometry What is the volume of the shaded pipe with outer radius R, inner radius r, and height h as shown? Express your answer in completely factored form.



89. Open-Ended Write a quadratic trinomial that you can factor, where $a \neq 1$, $ac > 0$, and $b < 0$. Factor the expression.

90. Writing Explain how to factor $3x^2 + 6x - 72$ completely.

C Challenge

Factor each expression completely.

91. $0.25t^2 - 0.16$

92. $8100x^2 - 10{,}000$

93. $(x + 3)^2 + 3(x + 3) - 54$

94. $(x - 2)^2 - 15(x - 2) + 56$

95. $6(x + 5)^2 - 5(x + 5) + 1$

96. $3(2a - 3)^2 + 17(2a - 3) + 10$

97. **Reasoning** Explain how to factor $4x^4 + 24x^3 + 32x^2$.

Factor each expression completely.

98. $x^4 - y^4$

99. $16x^4 - 625y^4$

100. $243a^5 - 3a$

101. When the expression $x^2 + bx - 24$ is factored completely, the difference of the factors is 11. Find both factors if it is known that b is negative.

102. Prove that $n^3 - n$ is divisible by 3 for all positive integer values of n. (*Hint:* Factor the expression completely.)

Sunshine State Standards Practice

MA.912.A.4.3

103. How can you write $(m - 5)(m + 4) + 8$ as a product of two binomials?

Ⓐ $(m - 1)(m + 8)$

Ⓑ $(m - 4)(m + 3)$

Ⓒ $(m + 8)(m + 8)$

Ⓓ $(m - 5)(8m + 32)$

MA.912.A.7.6

104. The graph of a quadratic function has vertex (7, 6). What is the axis of symmetry?

Ⓕ $x = 6$

Ⓖ $y = 6$

Ⓗ $x = 7$

Ⓘ $y = 7$

MA.912.A.2.6

105. **Extended Response** Suppose you hit a baseball and its flight takes a parabolic path. The height of the ball at certain times appears in the table below.

Time (s)	0.5	0.75	1	1.25
Height (ft)	10	10.5	9	5.5

a. Find a quadratic model for the ball's height as a function of time.

b. Write the quadratic function in factored form.

Mixed Review

106. Find a quadratic model for the values in the table.

x	0	5	10	15	20
y	17	39	54	61	61

See Lesson 4-3.

107. **Coins** The combined mass of a penny, a nickel, and a dime is 9.8 g. Ten nickels and three pennies have the same mass as 25 dimes. Fifty dimes have the same mass as 18 nickels and 10 pennies. Write and solve a system of equations to find the mass of each type of coin.

See Lesson 3-6.

Get Ready! **To prepare for Lesson 4-5, do Exercises 108–110.**

Graph each function.

See Lesson 4-2.

108. $y = x^2 - 2x - 5$

109. $y = x^2 - 4x + 4$

110. $y = -x^2 - 3x + 8$

4 Mid-Chapter Quiz

Do you know HOW?

Graph each function.

1. $y = 4x^2 + 16x + 7$
2. $y = (x + 8)^2 - 3$
3. $y = -(x + 2)^2 - 7$
4. $y = -3x^2 - 2x + 1$

Identify the axis of symmetry, maximum or minimum value, and the domain and range of each function.

5. $y = -x^2 + 6x + 5$
6. $y = \frac{1}{2}(x - 6)^2 + 7$
7. $y = -3(x + 2)^2 + 1$
8. $y = 4x^2 - 8x$
9. Rewrite the equation $y = -3x^2 - 6x - 8$ in vertex form. Identify the vertex and the axis of symmetry of the graph.

Write each expression in factored form.

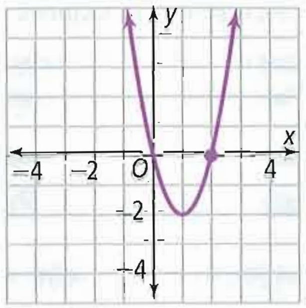
10. $16 - 2m^2$
11. $-x^2 + 3x$
12. $y^2 - 13y + 12$
13. $k^2 - 5k - 24$
14. $4y^2 - 9$
15. $-10n + 25 + n^2$
16. $2x^2 + 7x + 6$

Find a quadratic model in standard form for each set of values.

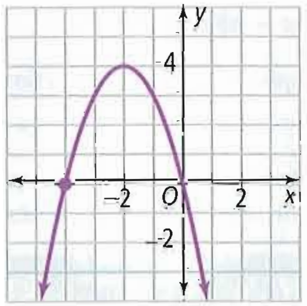
17. (0, 3), (1, 10), (2, 19)
18. (0, 0), (1, −5), (2, 0)

Write the equation of each parabola in vertex form.

19.



20.



Do you UNDERSTAND?

21. Write the expression $3x^4 - 12x^3 - 36x^2$ in factored form. Explain how you know the expression is completely factored.
22. **Open-Ended** Write the equation of a parabola with a vertex at (3, 2). Name the axis of symmetry and the coordinates of two other points on the graph.
23. **Writing** Explain how to factor $25x^2 - 30x + 9$.
24. **Reasoning** Write the equation of two parabolas such that they have a common vertex and are reflections of each other across the x-axis.
25. Write the equation of a parabola in standard form and explain how to convert it to vertex form. How would you reverse the process?
26. What is the relationship between the x-intercepts of the graph of a quadratic function and the x-coordinate of the vertex of that graph? Explain how you determined your answer.

Algebra Review

For Use With Lesson 4-5

Square Roots and Radicals

Sunshine State Standard

Prepares for MA.912.A.6.2 Add, subtract, multiply, and divide radical expressions.

A radical symbol $\sqrt{}$ indicates a square root. In general, $\sqrt{x^2} = |x|$ for all real numbers x.

Square Roots

Multiplication Property of Square Roots

For any numbers $a \geq 0$ and $b \geq 0$,

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$

Division Property of Square Roots

For any numbers $a \geq 0$ and $b > 0$,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Example

Simplify each expression.

A $\sqrt{50}$

$\sqrt{50} = \sqrt{25} \cdot \sqrt{2}$ Multiplication Property of Square Roots

$= 5\sqrt{2}$ Simplify.

B $\sqrt{\frac{5}{11}}$

$\sqrt{\frac{5}{11}} = \frac{\sqrt{5}}{\sqrt{11}}$ Division Property of Square Roots

$= \frac{\sqrt{5}}{\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}}$ Multiply both the numerator and denominator by $\sqrt{11}$.

$= \frac{\sqrt{55}}{\sqrt{121}}$ Multiplication Property of Square Roots.

$= \frac{\sqrt{55}}{11}$ Simplify.

Exercises

Simplify each radical expression.

1. $\sqrt{18}$

2. $\sqrt{75}$

3. $-\sqrt{32}$

4. $\sqrt{\frac{-5}{7}}$

5. $-\sqrt{\frac{7}{13}}$

6. $\sqrt{\frac{3}{15}}$

7. $-\sqrt{200}$

8. $5\sqrt{320}$

9. $(2\sqrt{27})^2$

10. $-\sqrt{10^4}$

11. $\sqrt{x^2y^2}$

12. $\sqrt{\frac{8}{x^2}}$

13. $-\sqrt{\frac{7x^3}{5x}}$

14. $\sqrt{\frac{(-3)^4}{12}}$

15. $\sqrt{\frac{200}{28}}$

16. $\sqrt{120x}$

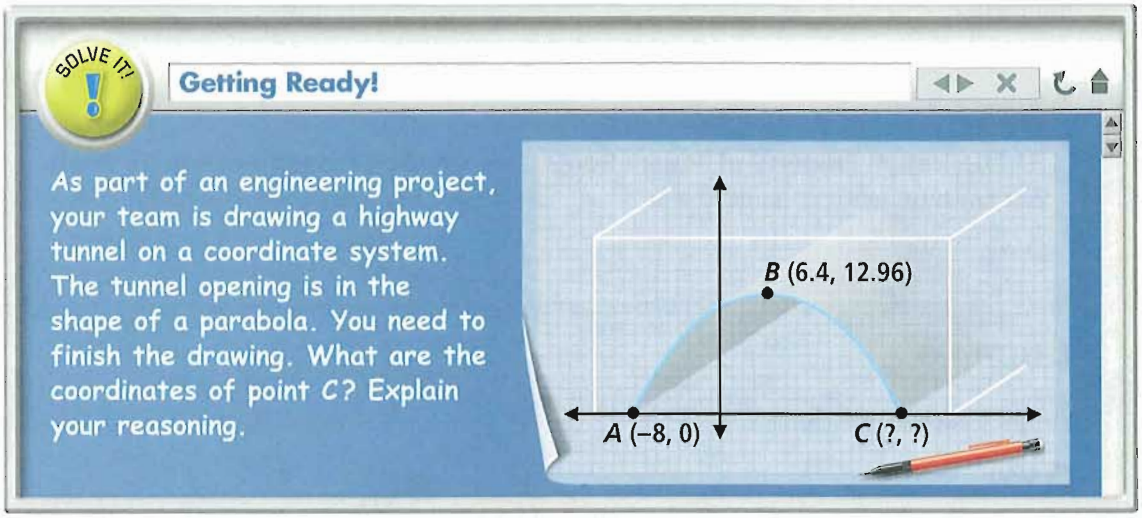
4-5 Quadratic Equations

Sunshine State Standards

MA.912.A.4.8 Describe the relationships among the solutions of an equation, the zeros of a function, the x-intercepts of a graph, and the factors of a polynomial expression, with and without technology.

MA.912.A.7.10 Use graphing technology to approximate solutions of quadratic equations.

Objectives To solve quadratic equations by factoring
To solve quadratic equations by graphing



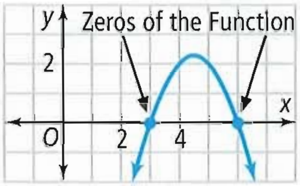
Getting Ready!

As part of an engineering project, your team is drawing a highway tunnel on a coordinate system. The tunnel opening is in the shape of a parabola. You need to finish the drawing. What are the coordinates of point C? Explain your reasoning.

Dynamic Activity
Quadratics in Factored Form

Lesson Vocabulary
- zero of a function
- Zero-Product Property

Wherever the graph of a function $f(x)$ intersects the x-axis, $f(x) = 0$. A value of x for which $f(x) = 0$ is a **zero of the function**.



Essential Understanding To find the zeros of a quadratic function $y = ax^2 + bx + c$, solve the related quadratic equation $0 = ax^2 + bx + c$.

You can solve some quadratic equations in standard form by factoring the quadratic expression and using the **Zero-Product Property**.

take note

Property Zero-Product Property

If $ab = 0$, then $a = 0$ or $b = 0$.

Problem 1 Solving a Quadratic Equation by Factoring

Think

What do you know about the factors of $x^2 - bx + c$?
The product of their constant terms is c. The sum is $-b$.

What are the solutions of the quadratic equation $x^2 - 5x + 6 = 0$?

$(x - 2)(x - 3) = 0$	Factor the quadratic expression.
$x - 2 = 0$ or $x - 3 = 0$	Use the Zero-Product Property.
$x = 2$ or $x = 3$	Solve for x.

The solutions are $x = 2$ and $x = 3$.

Got It? **1.** What are the solutions of the quadratic equation $x^2 - 7x = -12$?

Problem 2 Solving a Quadratic Equation With Tables

Think

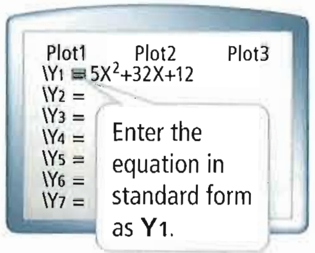
What should you look for in the calculator table?
Look for x-values for which $y = 0$.

What are the solutions of the quadratic equation $5x^2 + 30x + 14 = 2 - 2x$?

$$5x^2 + 30x + 14 = 2 - 2x$$

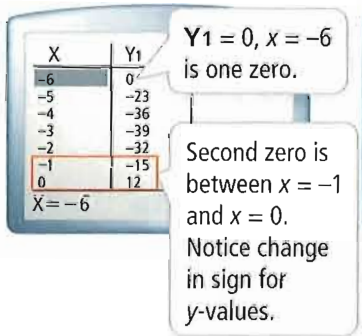
$$5x^2 + 32x + 12 = 0 \qquad \text{Rewrite in standard form.}$$

Use your calculator's **TABLE** feature to find the zeros.



```
Plot1     Plot2     Plot3
\Y1 ■5X²+32X+12
\Y2 =
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

Enter the equation in standard form as **Y1**.

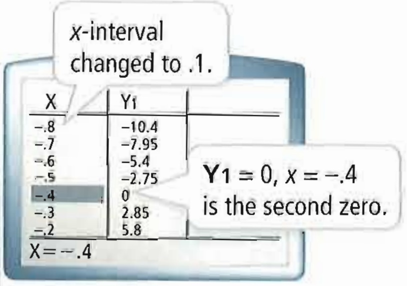


X	Y1
-6	0
-5	-23
-4	-36
-3	-39
-2	-32
-1	-15
0	12

X= -6

Y1 = 0, $x = -6$ is one zero.

Second zero is between $x = -1$ and $x = 0$. Notice change in sign for y-values.



x-interval changed to .1.

X	Y1
-.8	-10.4
-.7	-7.95
-.6	-5.4
-.5	-2.75
-.4	0
-.3	2.85
-.2	5.8

X= -.4

Y1 = 0, $x = -.4$ is the second zero.

The solutions are $x = -6$ and $x = -0.4$.

Got It? **2.** What are the solutions of the quadratic equation $4x^2 - 14x + 7 = 4 - x$?

Problem 3 Solving a Quadratic Equation by Graphing

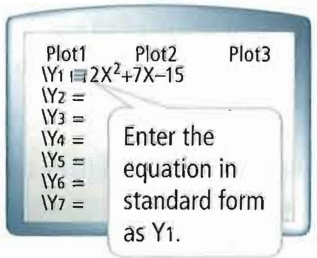
Plan

How can you use a graph to find the solutions?
Find the zeros of the related quadratic function.

What are the solutions of the quadratic equation $2x^2 + 7x = 15$?

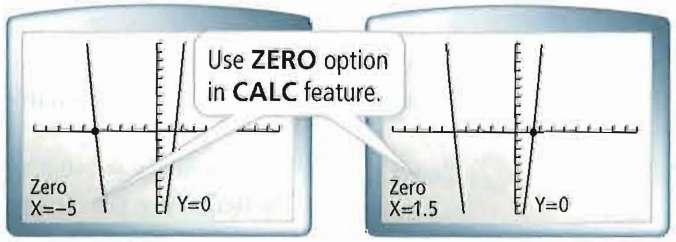
$$2x^2 + 7x = 15$$

$$2x^2 + 7x - 15 = 0 \qquad \text{Rewrite in standard form.}$$



```
Plot1     Plot2     Plot3
\Y1 ■2X²+7X−15
\Y2 =
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

Enter the equation in standard form as Y1.



Zero
X=−5   Y=0

Zero
X=1.5   Y=0

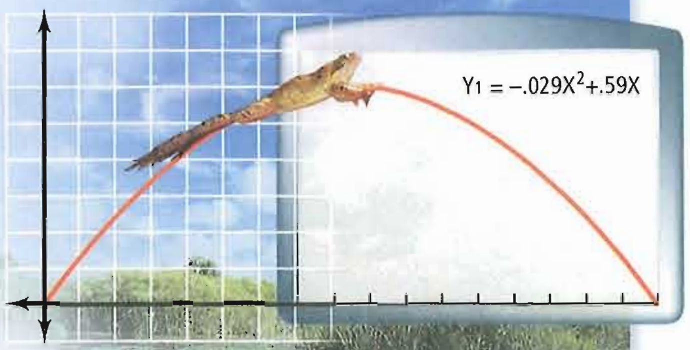
Use **ZERO** option in **CALC** feature.

The solutions are $x = -5$ and $x = 1.5$.

Got It? **3.** What are the solutions of the quadratic equation $x^2 + 2x - 24 = 0$?

Problem 4 Using a Quadratic Equation

Competition From the time Mark Twain wrote *The Celebrated Jumping Frog of Calaveras County* in 1865, frog-jumping competitions have been growing in popularity. The graph shows a function modeling the height of one frog's jump, where x is the distance, in feet, from the jump's start.



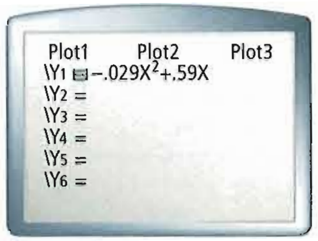
Think

How can you use a graphing calculator to determine the distance?

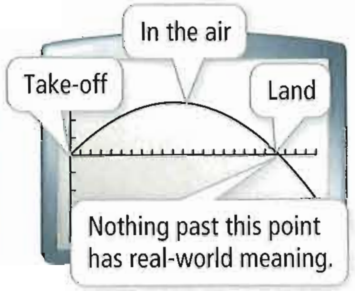
Graph the function and locate the point where the graph crosses the x-axis.

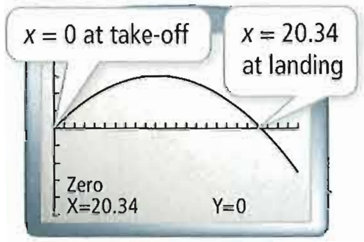
A **How far did the frog jump?**

The height of the jump is 0 at the start and end of the jump. Find the zeros of the function. Use a graphing calculator to find the zeros of the related function $y = -0.029x^2 + 0.59x$.



```
Plot1   Plot2   Plot3
\Y1 ▤–.029X²+.59X
\Y2 =
\Y3 =
\Y4 =
\Y5 =
\Y6 =
```





The frog jumped about 20.34 ft.

B **How high did the frog jump?**

The maximum height of the jump is the maximum value of the function. This occurs midway, at 10.17 ft from the start. Find y for $x = 10.17$.

$$y = -0.029(10.17)^2 + 0.59(10.17) \approx 3.0$$

The frog jumped to a height of about 3.0 ft.

C **What is a reasonable domain and range for such a frog-jumping function?**

While the function $y = -0.029x^2 + 0.59x$ has a domain of all real numbers, actual frog jumping does not allow negative values. So, a reasonable domain for frog-jumping distances is $0 \leq x \leq 30$. A reasonable range is $0 \leq y \leq 5$.

Got It? **4. a.** The function $y = -0.03x^2 + 1.60x$ models the path of a kicked soccer ball. The height is y, the distance is x, and the units are meters. How far does the soccer ball travel? How high does the soccer ball go? Describe a reasonable domain and range for the function.

b. **Reasoning** Are all domains and ranges reasonable for real-world situations? Explain.

Lesson Check

Do you know HOW?

Solve each equation by factoring.

1. $x^2 - 9 = 0$

2. $x^2 + 13x = -36$

3. $3x^2 - x - 2 = 0$

Solve by graphing.

4. $x^2 - 3x = 6$

5. $2x^2 - x = 11$

Do you UNDERSTAND?

6. **Vocabulary** If 5 is a zero of the function $y = x^2 + bx - 20$, what is the value of b? Explain.

7. **Compare and Contrast** When is it easier to solve a quadratic equation by factoring than to solve it using a table?

8. **Reasoning** Using tables, how might you recognize that a quadratic equation likely has exactly one solution? no solutions?

Practice and Problem-Solving Exercises

A Practice

Solve each equation by factoring. Check your answers. **See Problem 1.**

9. $x^2 + 6x + 8 = 0$

10. $x^2 + 18 = 9x$

11. $2x^2 - x = 3$

12. $x^2 - 10x + 25 = 0$

13. $2x^2 + 6x = -4$

14. $3x^2 = 16x + 12$

15. $x^2 - 4x = 0$

16. $6x^2 + 4x = 0$

17. $2x^2 = 8x$

Graphing Calculator **Solve each equation using tables. Give each answer to at most two decimal places.** **See Problem 2.**

18. $x^2 + 5x + 3 = 0$

19. $x^2 - 11x + 24 = 0$

20. $x^2 - 7x = 11$

21. $2x^2 - x = 2$

22. $x^2 - 16x = 36$

23. $x^2 + 6x = 40$

24. $4x^2 = x + 3$

25. $5x^2 + x = 4$

26. $10x^2 + 3 = 11x$

Graphing Calculator **Solve each equation by graphing. Give each answer to at most two decimal places.** **See Problem 3.**

27. $6x^2 = -19x - 15$

28. $3x^2 - 5x - 4 = 0$

29. $5x^2 - 7x - 3 = 8$

30. $6x^2 + 31x = 12$

31. $1 = 4x^2 + 3x$

32. $\frac{1}{2}x^2 - x = 8$

33. $x^2 = 4x + 8$

34. $x^2 + 4x = 6$

35. $2x^2 - 2x - 5 = 0$

36. **Physics** An object is dropped from a height of 1700 ft above the ground. The function $h = -16t^2 + 1700$ gives the object's height h in feet during free fall at t seconds. **See Problem 4.**

a. When will the object be 1000 ft above the ground?

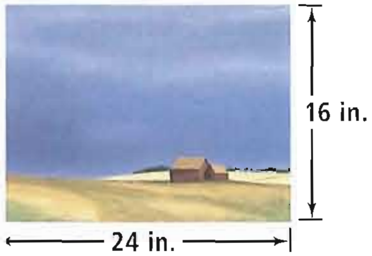
b. When will the object be 940 ft above the ground?

c. What are a reasonable domain and range for the function h?

B Apply

37. Think About a Plan Suppose you want to put a frame around the painting shown at the right. The frame will be the same width around the entire painting. You have 276 in.^{2} of framing material. How wide should the frame be?

- What does 276 in.^{2} represent in this situation?
- How can you write the dimensions of the frame using two binomials?



38. The period of a pendulum is the time the pendulum takes to swing back and forth. The function $L = 0.81t^2$ relates the length L in feet of a pendulum to the time t in seconds that it takes to swing back and forth. A convention center has a pendulum that is 90 feet long. Find the period.

39. Landscaping Suppose you have an outdoor pool measuring 25 ft by 10 ft. You want to add a cement walkway around the pool. If the walkway will be 1 ft thick and you have 304 ft^3 of cement, how wide should the walkway be?

40. Error Analysis A classmate solves the quadratic equation as shown. Find and correct the error. What are the correct solutions?

~~$x^2 + 5x + 6 = 2$~~
~~$(x + 2)(x + 3) = 2$~~
~~$x = -2$ or $x = -3$~~

41. Open-Ended Write an equation with the given solutions.

a. 3 and 5 **b.** -3 and 2 **c.** -1 and -6

Solve each equation by factoring, using tables, or by graphing. If necessary, round your answer to the nearest hundredth.

42. $x^2 + 2x = 6 - 6x$
43. $6x^2 + 13x + 6 = 0$
44. $2x^2 + x - 28 = 0$
45. $2x^2 + 8x = 5x + 20$
46. $3x^2 + 7x = 9$
47. $2x^2 - 6x = 8$
48. $(x + 3)^2 = 9$
49. $x^2 + 4x = 0$
50. $x^2 = 8x - 7$
51. $x^2 - 3x = 6$
52. $4x^2 + 5x = 4$
53. $7x - 3x^2 = -10$

Reasoning The graphs of each pair of functions intersect. Find their points of intersection without using a calculator. (*Hint:* Solve as a system using substitution.)

54. $y = x^2$
$y = -\frac{1}{2}x^2 + \frac{3}{2}x + 3$

55. $y = x^2 - 2$
$y = 3x^2 - 4x - 2$

56. $y = -x^2 + x + 4$
$y = 2x^2 - 6$

C Challenge

57. The equation $x^2 - 10x + 24 = 0$ can be written in factored form as $(x - 4)(x - 6) = 0$. How can you use this fact to find the vertex of the graph of $y = x^2 - 10x + 24$?

58. a. Let $a > 0$. Use algebraic or arithmetic ideas to explain why the lowest point on the graph of $y = a(x - h)^2 + k$ must occur when $x = h$.

b. Suppose that the function in part (*a*) is $y = a(x - h)^3 + k$. Is your reasoning still valid? Explain.

59. Physics When serving in tennis, a player tosses the tennis ball vertically in the air. The height h of the ball after t seconds is given by the quadratic function $h(t) = -5t^2 + 7t$ (the height is measured in meters from the point of the toss).

a. How high in the air does the ball go?

b. Assume that the player hits the ball on its way down when it's 0.6 m above the point of the toss. For how many seconds is the ball in the air between the toss and the serve?

Sunshine State Standards Practice

MA.912.A.4.9 **60.** What are the solutions of the equation $6x^2 + 9x - 15 = 0$?

Ⓐ $1, -15$ Ⓑ $1, -\frac{5}{2}$ Ⓒ $-1, -5$ Ⓓ $3, \frac{5}{2}$

MA.912.A.7.6 **61.** The vertex of a parabola is (3, 2). A second point on the parabola is (1, 7). Which point is also on the parabola?

Ⓕ $(-1, 7)$ Ⓖ $(3, 7)$ Ⓗ $(5, 7)$ Ⓘ $(3, -2)$

MA.912.A.2.6 **62.** For which quadratic function is -3 the constant term?

Ⓐ $y = (3x + 1)(-x - 3)$

Ⓑ $y = x^2 - 3x + 3$

Ⓒ $f(x) = (x - 3)(x - 3)$

Ⓓ $g(x) = -3x^2 + 3x + 9$

MA.912.A.2.10 **63. Short Response** What transformations are needed to go from the parent function $f(x) = x^2$ to the new function $g(x) = -3x^2 + 2$? Graph $g(x)$.

Mixed Review

Factor each expression. See Lesson 4-4.

64. $16x^2 - 1$ **65.** $5x^2 - 26x + 5$ **66.** $2x^2 + 13x - 7$

Solve each system by elimination. Check your answer. See Lesson 3-5.

67. $\begin{cases} 7x - 2y - 5z = 24 \\ -x + 3y + 4z = -10 \\ x - y - z = 4 \end{cases}$

68. $\begin{cases} -2x + 9y - z = 8 \\ 3x - 4y + z = -5 \\ 5x + 5y - z = -10 \end{cases}$

69. $\begin{cases} x - 9y + 8z = -10 \\ x + y - z = 9 \\ -x - 9z = 2 \end{cases}$

Without graphing, identify the vertex, axis of symmetry, and transformations from the parent function $f(x) = |x|$. See Lesson 2-7.

70. $y = |x + 9| + 4$ **71.** $y = |2x - 7|$ **72.** $y = \frac{3}{4}|x| - 1$

Get Ready! **To prepare for Lesson 4-6, do Exercises 73–75.**

Simplify each expression. See Lesson 4-4.

73. $(x + 4)(x + 4) - 3$ **74.** $(2x - 1)(2x - 1)$ **75.** $(x - 3)(x - 3)$

Concept Byte

For Use With Lesson 4-5

Writing Equations From Roots

Sunshine State Standard

Prepares for MA.912.A.4.7 Write a polynomial equation for a given set of real roots.

The **root** of an equation is a value that makes the equation true. You can use the Zero-Product Property to write a quadratic function from its zeros or a quadratic equation from its roots.

Activity 1

1. a. Write a nonzero linear function $f(x)$ that has a zero at $x = 3$.

b. Write a nonzero linear function $g(x)$ that has a zero at $x = 4$.

2. a. For f and g from Exercise 1, write the product function $h(x) = f(x) \cdot g(x)$.

b. What kind of function is $h(x)$?

c. Solve the equation $h(x) = 0$.

Mental Math **Write a quadratic equation with each pair of values as roots.**

3. 5 and 3

4. 2.5 and 4

5. −4 and 4

6. 5 and 10

7. $\frac{3}{2}$ and −2

You can also use zeros or roots to write quadratic expressions in standard form.

Activity 2

8. a. Copy and complete the table. Write the product $(x - a)(x - b)$ in standard form for each pair a and b.

b. Is there a pattern in the table? Explain.

9. a. If you know the roots, you can write a quadratic function or equation in standard form. Explain how.

b. Demonstrate your method for each pair of values in Exercises 3–7.

a	b	a + b	ab	(x − a) (x − b)
4	5	9	20	$x^2 - 9x + 20$
−4	5	1	−20	■
4	−5	■	■	■
−4	−5	■	■	■
−9	−1	■	■	■
−2	7	■	■	■

Exercises

10. Explain how to write a quadratic equation that has −6 as its only root.

11. Describe the family of quadratic functions that have zeros at r and s. Sketch several members of the family in the coordinate plane.

Find the sum and product of the roots for each quadratic equation.

12. $2x^2 + 3x - 2 = 0$

13. $x^2 - 2x + 1 = 0$

14. $x^2 - 5x + 6 = 0$

Given the sum and product of the roots, write a quadratic equation in standard form.

15. sum = −3, product = −18

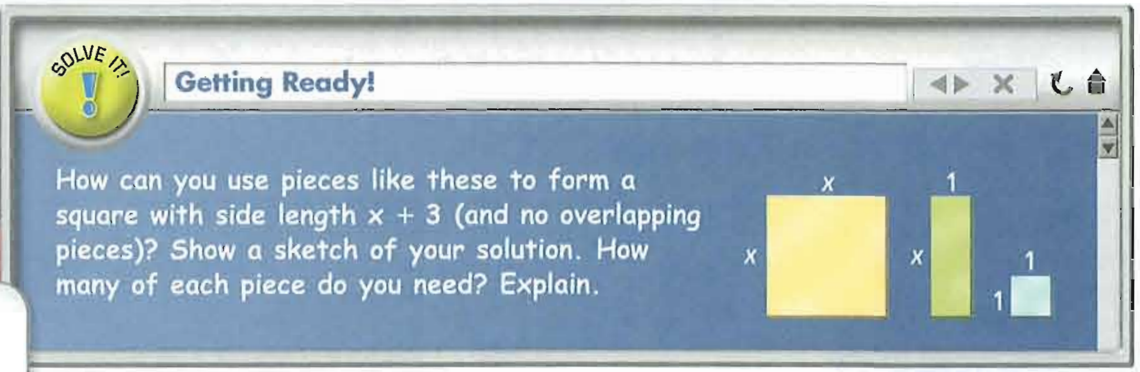
16. sum = 4, product = 3

17. sum = 2, product = $\frac{3}{4}$

4-6 Completing the Square

Sunshine State Standard
MA.912.A.7.3 Solve quadratic equations over the real numbers by completing the square.

Objectives To solve equations by completing the square
To rewrite functions by completing the square



How can you use pieces like these to form a square with side length $x + 3$ (and no overlapping pieces)? Show a sketch of your solution. How many of each piece do you need? Explain.

This suggests another way to write $(x + 3)^2$.

Lesson Vocabulary
- completing the square

Forming a square with model pieces provides a useful geometric image for completing a square algebraically.

Essential Understanding Completing a perfect square trinomial allows you to factor the completed trinomial as the square of a binomial.

You can solve an equation that contains a perfect square by finding square roots. The simplest of this type of equation has the form $ax^2 = c$.

Problem 1 Solving by Finding Square Roots

What is the solution of each equation?

A $4x^2 + 10 = 46$		**B** $3x^2 - 5 = 25$
$4x^2 = 36$	← Rewrite in $ax^2 = c$ form. →	$3x^2 = 30$
$\frac{4x^2}{4} = \frac{36}{4}$	← Isolate x^2. →	$\frac{3x^2}{3} = \frac{30}{3}$
$x^2 = 9$		$x^2 = 10$
$x = \pm 3$	← Find square roots. →	$x = \pm\sqrt{10}$

Plan
How is solving this equation like solving a linear equation?
You isolate the variable term.

Got It? **1.** What is the solution of each equation?
a. $7x^2 - 10 = 25$
b. $2x^2 + 9 = 13$

Problem 2 Determining Dimensions

Dynamic Activity
Completing the Square

Architecture **While designing a house, an architect used windows like the one shown here. What are the dimensions of the window if it has 2766 square inches of glass?**

Step 1 Find the area of the window.

The area of the rectangular part is $(2x)(x) = 2x^2$ in.^{2}.

The area of the semicircular part is

$$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = \frac{1}{2}\pi\frac{x^2}{4} = \frac{\pi}{8}x^2 \text{ in.}^2.$$

So, the total amount of glass used is

$$2x^2 + \frac{\pi}{8}x^2 = 2766 \text{ in.}^2.$$

Step 2 Solve for x.

$$\left(2 + \frac{\pi}{8}\right)x^2 = 2766$$ Write the equation in $ax^2 = c$ form.

$$x^2 = \frac{2766}{2 + \frac{\pi}{8}}$$ Isolate x^2.

$$x \approx \pm 34$$ Find square roots. Use a calculator.

Length cannot be negative. So the rectangular portion of the window is 34 in. wide by 68 in. long. The semicircular top has a radius of 17 in.

Think

Is the answer reasonable?
Yes; the rectangular part is about $30 \times 70 = 2100$ in.^{2}. This leaves enough glass for the semicircle.

Got It? **2.** The lengths of the sides of a rectangular window have the ratio 1.6 to 1. The area of the window is 2822.4 in.^{2}. What are the window dimensions?

Sometimes an equation shows a perfect square trinomial equal to a constant. To solve, factor the perfect square trinomial into the square of a binomial. Then find square roots.

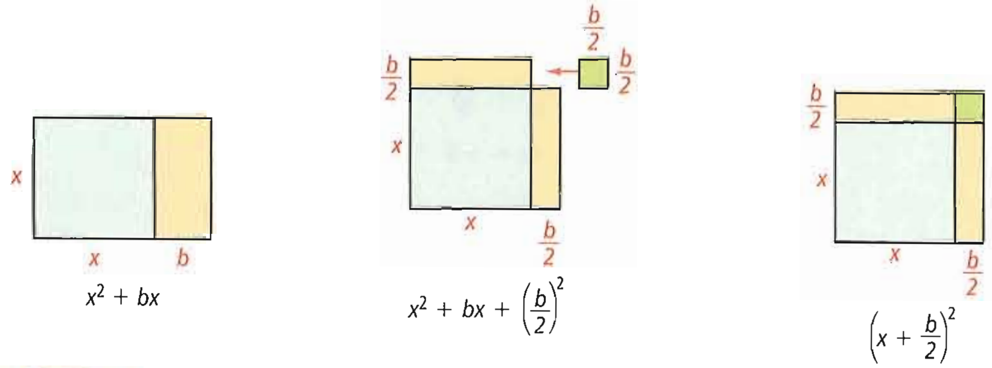
Problem 3 Solving a Perfect Square Trinomial Equation

What is the solution of $x^2 + 4x + 4 = 25$?

Think	Write
Factor the perfect square trinomial.	$x^2 + 4x + 4 = 25$ $(x + 2)^2 = 25$
Find square roots.	$x + 2 = \pm 5$
Rewrite as two equations.	$x + 2 = 5$ or $x + 2 = -5$
Solve for x.	$x = 3$ or $x = -7$

Got It? **3.** What is the solution of $x^2 - 14x + 49 = 25$?

If $x^2 + bx$ is not part of a perfect square trinomial, you can use the coefficient b to find a constant c so that $x^2 + bx + c$ is a perfect square. When you do this, you are **completing the square**. The diagram models this process.



take note

Key Concept Completing the Square

You can form a perfect square trinomial from $x^2 + bx$ by adding $\left(\frac{b}{2}\right)^2$.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Problem 4 Completing the Square

Think

Why do you want a perfect square trinomial?
You can factor a perfect square trinomial into the square of a binomial.

What value completes the square for $x^2 - 10x$? Justify your answer.

$x^2 - 10x$	Identify b; $b = -10$
$\left(\frac{b}{2}\right)^2 = \left(\frac{-10}{2}\right)^2 = (-5)^2 = 25$	Find $\left(\frac{b}{2}\right)^2$.
$x^2 - 10x + 25$	Add the value of $\left(\frac{b}{2}\right)^2$ to complete the square.
$x^2 - 10x + 25 = (x - 5)^2$	Rewrite as the square of a binomial.

Got It? **4. a.** What value completes the square for $x^2 + 6x$?

b. **Reasoning** Is it possible for more than one value to complete the square for an expression? Explain.

take note

Key Concept Solving an Equation by Completing the Square

1. Rewrite the equation in the form $x^2 + bx = c$. To do this, get all terms with the variable on one side of the equation and the constant on the other side. Divide all the terms of the equation by the coefficient of x^2 if it is not 1.
2. Complete the square by adding $\left(\frac{b}{2}\right)^2$ to each side of the equation.
3. Factor the trinomial.
4. Find square roots.
5. Solve for x.

Problem 5 Solving by Completing the Square

What is the solution of $3x^2 - 12x + 6 = 0$?

$$3x^2 - 12x + 6 = 0$$

$$3x^2 - 12x = -6 \qquad \text{Rewrite. Get all terms with } x \text{ on one side of the equation.}$$

$$\frac{3x^2}{3} - \frac{12x}{3} = \frac{-6}{3} \qquad \text{Divide each side by 3 so the coefficient of } x^2 \text{ will be 1.}$$

$$x^2 - 4x = -2 \qquad \text{Simplify.}$$

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-4}{2}\right)^2 = (-2)^2 = 4 \qquad \text{Find } \left(\frac{b}{2}\right)^2 = 4.$$

$$x^2 - 4x + 4 = -2 + 4 \qquad \text{Add 4 to each side.}$$

$$(x - 2)^2 = 2 \qquad \text{Factor the trinomial.}$$

$$x - 2 = \pm\sqrt{2} \qquad \text{Find square roots.}$$

$$x = 2 \pm \sqrt{2} \qquad \text{Solve for } x.$$

Check your results on your calculator. Replace x in the original equation with $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

Think

Is there a way to check without a calculator?
Yes; you can check that your solutions are reasonable by estimating.

Got It? **5.** What is the solution of $2x^2 - x + 3 = x + 9$?

You can complete a square to change a quadratic function to vertex form.

Problem 6 Writing in Vertex Form

Plan

What should be your first step?
Complete the square.

What is $y = x^2 + 4x - 6$ in vertex form? Name the vertex and y-intercept.

$$y = x^2 + 4x - 6$$

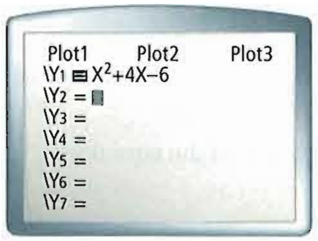
$$y = x^2 + 4x + 2^2 - 6 - 2^2 \qquad \text{Add } \left(\frac{4}{2}\right)^2 = 2^2 \text{ to complete the square. Also, subtract } 2^2 \text{ to leave the function unchanged.}$$

$$y = (x + 2)^2 - 6 - 2^2 \qquad \text{Factor the perfect square trinomial.}$$

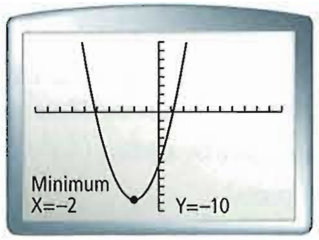
$$y = (x + 2)^2 - 10 \qquad \text{Simplify.}$$

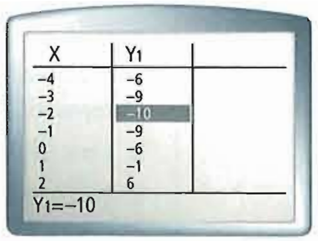
The vertex is $(-2, -10)$. The y-intercept is $(0, -6)$.

Check with a graphing calculator.



```
Plot1    Plot2    Plot3
\Y1 ■X²+4X–6
\Y2 =■
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```





X	Y1
–4	–6
–3	–9
–2	–10
–1	–9
0	–6
1	–1
2	6

Y1=–10

Got It? **6.** What is $y = x^2 + 3x - 6$ in vertex form? Name the vertex and y-intercept.

Lesson Check

Do you know HOW?

Solve each equation by finding square roots.

1. $2x^2 = 72$

2. $6x^2 = 54$

Complete the square.

3. $x^2 + 2x + \blacksquare$

4. $x^2 + 10x + \blacksquare$

5. $x^2 - 4x + \blacksquare$

6. $x^2 + 12x + \blacksquare$

7. $x^2 + 100x + \blacksquare$

8. $x^2 - 32x + \blacksquare$

Do you UNDERSTAND?

9. Vocabulary Explain the process of completing the square.

10. How can you rewrite the equation $x^2 + 12x + 5 = 3$ so the left side of the equation is in the form $(x + a)^2$?

11. Error Analysis Your friend completed the square and wrote the expression shown. Explain your friend's error and write the expression correctly.

$$\xcancel{\begin{array}{l} x^2 - 14x + 36 \\ x^2 - 14x + 49 + 36 \\ (x - 7)^2 + 36 \end{array}}$$

Practice and Problem-Solving Exercises

A Practice

Solve each equation by finding square roots. **See Problem 1.**

12. $5x^2 = 80$

13. $x^2 - 4 = 0$

14. $2x^2 = 32$

15. $9x^2 = 25$

16. $3x^2 - 15 = 0$

17. $5x^2 - 40 = 0$

18. Fitness A rectangular swimming pool is 6 ft deep. One side of the pool is 2.5 times longer than the other. The amount of water needed to fill the swimming pool is 2160 cubic feet. Find the dimensions of the pool. **See Problem 2.**

Solve each equation. **See Problem 3.**

19. $x^2 + 6x + 9 = 1$

20. $x^2 - 4x + 4 = 100$

21. $x^2 - 2x + 1 = 4$

22. $x^2 + 8x + 16 = \frac{16}{9}$

23. $4x^2 + 4x + 1 = 49$

24. $x^2 - 12x + 36 = 25$

25. $25x^2 + 10x + 1 = 9$

26. $x^2 - 30x + 225 = 400$

27. $9x^2 + 24x + 16 = 36$

Complete the square. **See Problem 4.**

28. $x^2 + 18x + \blacksquare$

29. $x^2 - x + \blacksquare$

30. $x^2 - 24x + \blacksquare$

31. $x^2 + 20x + \blacksquare$

32. $m^2 - 3m + \blacksquare$

33. $x^2 + 4x + \blacksquare$

Solve each quadratic equation by completing the square. **See Problem 5.**

34. $x^2 + 6x - 3 = 0$

35. $x^2 - 12x + 7 = 0$

36. $x^2 + 4x + 2 = 0$

37. $x^2 - 2x = 5$

38. $x^2 + 8x = 11$

39. $x^2 + 12 = 10x$

40. $x^2 - 3x = x - 1$

41. $x^2 + 2 = 6x + 4$

42. $2x^2 + 2x - 5 = x^2$

43. $4x^2 + 10x - 3 = 0$

44. $9x^2 - 12x - 2 = 0$

45. $25x^2 + 30x = 12$

Rewrite each equation in vertex form. **See Problem 6.**

46. $y = x^2 + 4x + 1$

47. $y = 2x^2 - 8x + 1$

48. $y = -x^2 - 2x + 3$

49. $y = x^2 + 4x - 7$

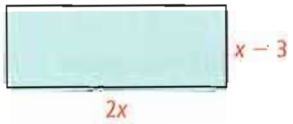
50. $y = 2x^2 - 6x - 1$

51. $y = -x^2 + 4x - 1$

B Apply

52. Think About a Plan The area of the rectangle shown is 80 square inches. What is the value of x?

- How can you write an equation to represent 80 in terms of x?
- How can you find the value of x by completing the square?



Find the value of k that would make the left side of each equation a perfect square trinomial.

53. $x^2 + kx + 25 = 0$

54. $x^2 - kx + 100 = 0$

55. $x^2 - kx + 121 = 0$

56. $x^2 + kx + 64 = 0$

57. $x^2 - kx + 81 = 0$

58. $25x^2 - kx + 1 = 0$

59. $x^2 + kx + \frac{1}{4} = 0$

60. $9x^2 - kx + 4 = 0$

61. $36x^2 - kx + 49 = 0$

62. Geometry The table shows some possible dimensions of rectangles with a perimeter of 100 units. Copy and complete the table.

a. Plot the points (width, area). Find a model for the data set.

b. What is another point in the data set? Use it to verify your model.

c. What is a reasonable domain for this function? Explain.

d. Find the maximum possible area. What dimensions yield this area?

e. Find a function for area in terms of width without using the table. Do you get the same model as in part (a)? Explain.

Width	Length	Area
1	49	49
2	48	■
3	■	■
4	■	■
5	■	■

Solve each quadratic equation by completing the square.

63. $x^2 + 5x - 3 = 0$

64. $x^2 + 3x = 2$

65. $x^2 - x = 5$

66. $x^2 + x - 1 = 0$

67. $3x^2 - 4x = 2$

68. $5x^2 - x = 4$

69. $x^2 + \frac{3}{4}x = \frac{1}{2}$

70. $2x^2 - \frac{1}{2}x = \frac{1}{8}$

71. $3x^2 + x = \frac{2}{3}$

72. $-x^2 + 2x + 4 = 0$

73. $-x^2 - 6x = 2$

74. $-0.25x^2 - 0.6x + 0.3 = 0$

75. Football The quadratic function $h = -0.01x^2 + 1.18x + 2$ models the height of a punted football. The horizontal distance in feet from the point of impact with the kicker's foot is x, and h is the height of the ball in feet.

a. Write the function in vertex form. What is the maximum height of the punt?

b. The nearest defensive player is 5 ft horizontally from the point of impact. How high must the player reach to block the punt?

c. Suppose the ball was not blocked but continued on its path. How far down the field would the ball go before it hit the ground?

Challenge

Solve for x in terms of a.

76. $2x^2 - ax = 6a^2$
77. $3x^2 + ax = a^2$
78. $2a^2x^2 - 8ax = -6$
79. $4a^2x^2 + 8ax + 3 = 0$
80. $3x^2 + ax^2 = 9x + 9a$
81. $6a^2x^2 - 11ax = 10$

82. Solve $x^2 = (6\sqrt{2})x + 7$ by completing the square.

Rewrite each equation in vertex form. Then find the vertex of the graph.

83. $y = -4x^2 - 5x + 3$
84. $y = \frac{1}{2}x^2 - 5x + 12$
85. $y = -\frac{1}{5}x^2 + \frac{4}{5}x + \frac{11}{5}$

Sunshine State Standards Practice

MA.912.A.3.14 **86.** The graph of which inequality has its vertex at $\left(2\frac{1}{2}, -5\right)$?

Ⓐ $y < |2x - 5| + 5$
Ⓒ $y > |2x + 5| - 5$
Ⓑ $y < |2x + 5| - 5$
Ⓓ $y > |2x - 5| - 5$

MA.912.A.3.6 **87.** Which number is a solution of $|9 - x| = 9 + x$?

Ⓕ -3
Ⓖ 0
Ⓗ 3
Ⓘ 6

MA.912.A.7.6 **88.** Joanne tosses an apple seed on the ground. It travels along a parabola with the equation $y = -x^2 + 4$. Assume the seed was thrown from a height of 4 ft. How many feet away from Joanne will the apple seed land?

Ⓐ 1 ft
Ⓑ 2 ft
Ⓒ 4 ft
Ⓓ 8 ft

MA.912.A.7.3 **89. Extended Response** List the steps for solving the equation $x^2 - 9 = -8x$ by the completing the square method. Explain each step.

Mixed Review

Solve each equation by factoring. Check your answers. See Lesson 4-5.

90. $2x^2 - 3x + 1 = 0$
91. $x^2 - 4 = -3x$
92. $16 + 22x = 3x^2$

Determine whether a quadratic model exists for each set of values. If so, write the model. See Lesson 4-3.

93. $(-4, 3), (-3, 3), (-2, 4)$
94. $\left(-1, \frac{1}{2}\right), (0, 2), (2, 2)$
95. $(0, 2), (1, 0), (2, 4)$

Solve each system by elimination. See Lesson 3-2.

96. $\begin{cases} 2x + y = 4 \\ 3x - y = 6 \end{cases}$
97. $\begin{cases} 2x + y = 7 \\ -2x + 5y = -1 \end{cases}$
98. $\begin{cases} 2x + 4y = 10 \\ 3x + 5y = 14 \end{cases}$

Get Ready!

To prepare for Lesson 4-7, do Exercises 99–100.

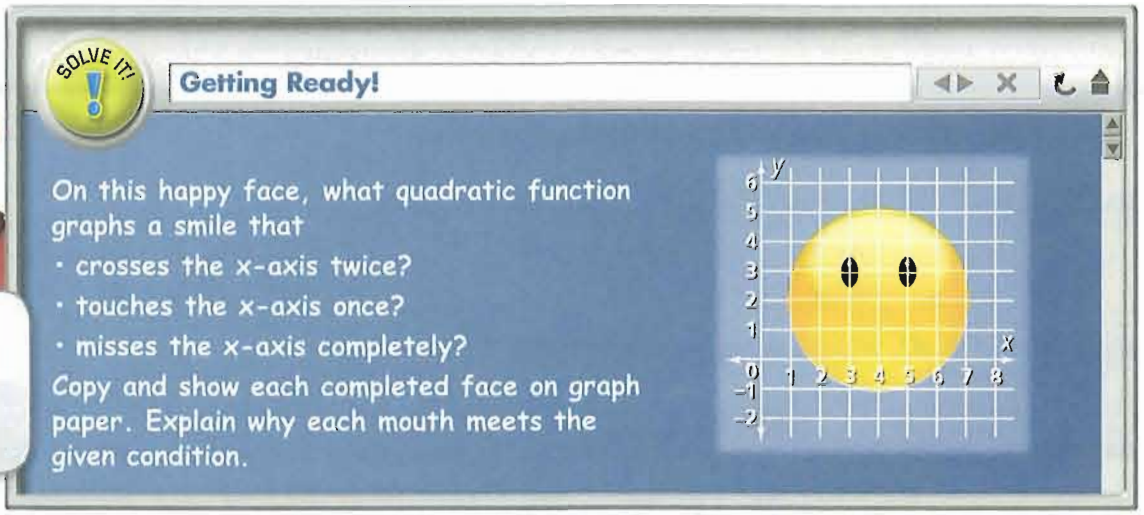
Evaluate each expression for the given values of the variables. See Lesson 1-3.

99. $b^2 - 4ac;\ a = 1, b = 6, c = 3$
100. $b^2 - 4ac;\ a = -5, b = 2, c = 4$

4-7 The Quadratic Formula

Sunshine State Standard
MA.912.A.7.4 Use the discriminant to determine the nature of the roots of a quadratic equation.

Objectives To solve quadratic equations using the Quadratic Formula
To determine the number of solutions by using the discriminant



Getting Ready!

On this happy face, what quadratic function graphs a smile that
- crosses the x-axis twice?
- touches the x-axis once?
- misses the x-axis completely?

Copy and show each completed face on graph paper. Explain why each mouth meets the given condition.

Dynamic Activity
Roots of a Quadratic

Lesson Vocabulary
- Quadratic Formula
- discriminant

Another way to solve a quadratic equation $ax^2 + bx + c = 0$ is by completing a square and then factoring.

Essential Understanding You can solve a quadratic equation $ax^2 + bx + c = 0$ in more than one way. In general, you can find a formula that gives values of x in terms of a, b, and c.

Here's how to solve $ax^2 + bx + c = 0$ to get the *Quadratic Formula.*

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Divide each side by a.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Rewrite so all terms containing x are on one side.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Complete the square.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Factor the perfect square trinomial. Also, simplify.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Find square roots.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Solve for x. Also, simplify the radical.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Simplify.

take note

Key Concept The Quadratic Formula

To solve the quadratic equation $ax^2 + bx + c = 0$, use the **Quadratic Formula**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ONLINE PROBLEMS

Problem 1 Using the Quadratic Formula

What are the solutions? Use the Quadratic Formula.

Plan

Should you write the equation in standard form?
Yes; write the equation in standard form to identify *a*, *b*, and *c*.

A $2x^2 - x = 4$

$2x^2 - x = 4$

$2x^2 - x - 4 = 0$ Write in standard form.

$a = 2, b = -1, c = -4$ Find the values of a, b, and c.

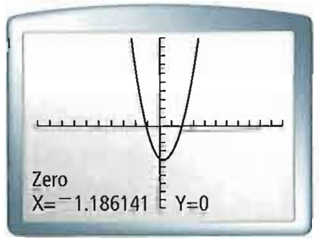
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Write the Quadratic Formula.

$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-4)}}{2(2)}$ Substitute for a, b, and c.

$= \frac{1 \pm \sqrt{33}}{4}$ Simplify.

$= \frac{1 + \sqrt{33}}{4}$ or $\frac{1 - \sqrt{33}}{4}$

Check Use a graphing calculator to graph $y = 2x^2 - x - 4$. The x-intercepts are about $-1.186 \approx \frac{1 - \sqrt{33}}{4}$ and $1.686 \approx \frac{1 + \sqrt{33}}{4}$, as expected.


Zero
X=⁻1.186141 Y=0

B $x^2 + 6x + 9 = 0$

Think

Why is there only one solution?
Because if you add or subtract zero you get the same number.

$x^2 + 6x + 9 = 0$

$a = 1, b = 6, c = 9$ Find the values of a, b, and c.

$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(9)}}{2(1)}$ Substitute into $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$= \frac{-6 \pm \sqrt{36 - 36}}{2}$ Simplify.

$= \frac{-6 \pm \sqrt{0}}{2}$

$= -3$

Got It? **1.** What are the solutions? Use the Quadratic Formula.

a. $x^2 + 4x = -4$

b. $x^2 + 4x - 3 = 0$

Problem 2 Applying the Quadratic Formula

GRIDDED RESPONSE

Fundraising **Your school's jazz band is selling CDs as a fundraiser. The total profit p depends on the amount x that your band charges for each CD. The equation $p = -x^2 + 48x - 300$ models the profit of the fundraiser. What is the least amount, in dollars, you can charge for a CD to make a profit of \$200?**

$$p = -x^2 + 48x - 300$$

$$200 = -x^2 + 48x - 300$$ Substitute 200 for p.

$$0 = -x^2 + 48x - 500$$ Write the equation in standard form.

$$a = -1,\ b = 48,\ c = -500$$ Find the values of a, b, and c.

$$x = \frac{-48 \pm \sqrt{48^2 - 4(-1)(-500)}}{2(-1)}$$ Substitute into $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$x = \frac{-48 \pm \sqrt{304}}{-2}$$ Simplify.

$$x \approx 15.282 \text{ or } x \approx 32.717$$ Use a calculator.

The least amount you can charge is \$15.29 for each CD to make a profit of \$200.

The answer is 15.29.

Think

Does it make sense that two different prices can yield the same profit?

Yes. You can generate a given profit either by selling many CDs at a low price, or fewer CDs at a high price.

Got It? **2. a.** In Problem 2, what is the least amount you can charge for each CD to make a \$100 profit?

b. **Reasoning** Would a negative profit make sense in this problem? Explain.

A quadratic equation can have two real solutions ($x^2 = 4$), one real solution ($x^2 = 0$), or no real solutions ($x^2 = -4$). In the Quadratic Formula, the value under the radical symbol, $b^2 - 4ac$, tells you how many real-number solutions exist.

In Problem 1(a), $b^2 - 4ac > 0$ and there are two real solutions. In Problem 1(b), $b^2 - 4ac = 0$ and there is only one real solution.

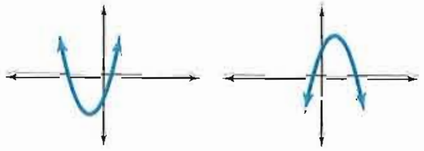
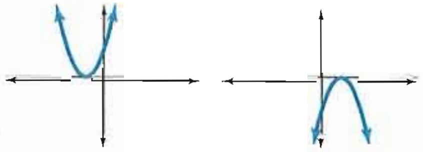
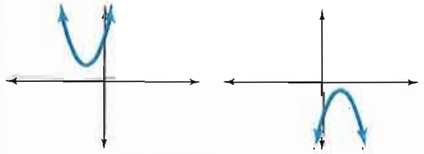
take note

Key Concept Discriminant

The **discriminant** of a quadratic equation in the form $ax^2 + bx + c = 0$ is the value of the expression $b^2 - 4ac$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac \;\leftarrow \text{discriminant}}}{2a}$$

Discriminants and Solutions of Quadratic Equations

Value of the Discriminant	Number of Solutions for $ax^2 + bx + c = 0$	x-intercepts of Graph of Related Function $y = ax^2 + bx + c$
$b^2 - 4ac > 0$	two real solutions	two x-intercepts 
$b^2 - 4ac = 0$	one real solution	one x-intercept 
$b^2 - 4ac < 0$	no real solutions	no x-intercepts 

Problem 3 Using the Discriminant

What is the number of real solutions of $-2x^2 - 3x + 5 = 0$?

Think	Write
Find the values of a, b, and c.	$a = -2,\ b = -3,\ c = 5$
Evaluate $b^2 - 4ac$.	$b^2 - 4ac = (-3)^2 - 4(-2)(5)$ $= 49$
Interpret the discriminant.	The discriminant is positive. The equation has two real solutions.

Got It? **3.** What is the number of real solutions of each equation?

a. $2x^2 - 3x + 7 = 0$

b. $x^2 = 6x + 5$

Problem 4 Using the Discriminant to Solve a Problem

Projectile Motion You hit a golf ball into the air from a height of 1 in. above the ground with an initial vertical velocity of 85 ft/s. The function $h = -16t^2 + 85t + \frac{1}{12}$ models the height, in feet, of the ball at time t, in seconds. Will the ball reach a height of 115 ft?

Plan

What value should you substitute for h?
You are trying to determine whether the ball will reach 115 ft. Replace h with 115.

$$h = -16t^2 + 85t + \frac{1}{12}$$

$$115 = -16t^2 + 85t + \frac{1}{12}$$ Substitute 115 for h.

$$0 = -16t^2 + 85t - 114\tfrac{11}{12}$$ Write the equation in standard form.

$$a = -16,\ b = 85,\ c = -114\tfrac{11}{12}$$ Find the values of a, b, and c.

$$b^2 - 4ac = 85^2 - 4(-16)\left(-114\tfrac{11}{12}\right)$$ Evaluate the discriminant.

$$= 7225 - 7354\tfrac{2}{3}$$ Simplify.

$$= -129\tfrac{2}{3}$$

The discriminant is negative. The equation $115 = -16t^2 + 85t + \frac{1}{12}$ has no real solutions. The golf ball will not reach a height of 115 feet.

Got It? **4. Reasoning** Without solving an equation, will the golf ball in Problem 4 reach a height of 110 ft? Explain.

Lesson Check

Do you know HOW?

Solve each equation using the Quadratic Formula.

1. $x^2 - 5x - 7 = 0$

2. $x^2 + 3x - 13 = 0$

3. $2x^2 - 5x - 3 = 0$

4. $3x^2 - 4x + 3 = 0$

Find the discriminant of each quadratic equation. Determine the number of real solutions.

5. $-x^2 + 2x - 9 = 0$

6. $x^2 + 17x + 4 = 0$

7. $x^2 - 6x + 9 = 0$

Do you UNDERSTAND?

8. Reasoning For what values of k does the equation $x^2 + kx + 9 = 0$ have one real solution? two real solutions?

9. Error Analysis Your friend concluded that because two discriminants are equal, the solutions to the two equations are the same. Explain your friend's error. Give an example of two quadratic equations that disprove this conclusion.

10. Reasoning If one quadratic equation has a positive discriminant, and another quadratic equation has a discriminant equal to 0, can the two quadratic equations share a solution? Explain why or why not. If so, give two quadratic equations that meet this criterion.

Practice and Problem-Solving Exercises

A Practice

Solve each equation using the Quadratic Formula. See Problem 1.

11. $x^2 - 4x + 3 = 0$
12. $x^2 + 8x + 12 = 0$
13. $2x^2 + 5x = 7$
14. $3x^2 + 2x - 1 = 0$
15. $x^2 + 10x = -25$
16. $2x^2 - 5 = -3x$
17. $x^2 = 3x - 1$
18. $6x - 5 = -x^2$
19. $3x^2 = 2(2x + 1)$
20. $2x(x - 1) = 3$
21. $x(x - 5) = -4$
22. $12x + 9x^2 = 5$

23. Fundraising Your class is selling boxes of flower seeds as a fundraiser. The total profit p depends on the amount x that your class charges for each box of seeds. The equation $p = -0.5x^2 + 25x - 150$ models the profit of the fundraiser. What's the smallest amount, in dollars, that you can charge and make a profit of at least $125? See Problem 2.

24. Baking Your local bakery sells more bagels when it reduces prices, but then its profit changes. The function $y = -1000x^2 + 1100x - 2.5$ models the bakery's daily profit in dollars, from selling bagels, where x is the price of a bagel in dollars. What's the highest price the bakery can charge, in dollars, and make a profit of at least $200?

Evaluate the discriminant for each equation. Determine the number of real solutions. See Problem 3.

25. $x^2 + 4x + 5 = 0$
26. $x^2 - 4x - 5 = 0$
27. $-4x^2 + 20x - 25 = 0$
28. $-2x^2 + x - 28 = 0$
29. $2x^2 + 7x - 15 = 0$
30. $6x^2 - 2x + 5 = 0$
31. $-2x^2 + 7x = 6$
32. $x^2 - 12x + 36 = 0$
33. $x^2 + 8x = -16$
34. $3x^2 + x = -3$
35. $x + 2 = -3x^2$
36. $12x(x + 1) = -3$

37. Business The weekly revenue for a company is $r = -3p^2 + 60p + 1060$, where p is the price of the company's product. Use the discriminant to find whether there is a price for which the weekly revenue would be $1500. See Problem 4.

38. Physics The equation $h = 80t - 16t^2$ models the height h in feet reached in t seconds by an object propelled straight up from the ground at a speed of 80 ft/s. Use the discriminant to find whether the object will ever reach a height of 90 ft.

B Apply

39. Think About a Plan The area of a rectangle is 36 in.^{2}. The perimeter of the rectangle is 36 in. What are the dimensions of the rectangle to the nearest hundredth of an inch?

- How can you write an equation using one variable to find the dimensions of the rectangle?
- How can the discriminant of the equation help you solve the problem?

40. Writing Summarize how to use the discriminant to analyze the types of solutions of a quadratic equation.

Solve each equation using any method. When necessary, round real solutions to the nearest hundredth.

41. $6x^2 - 5x - 1 = 0$	**42.** $7x^2 - x - 12 = 0$	**43.** $5x^2 + 8x - 11 = 0$
44. $4x^2 + 4x = 22$	**45.** $2x^2 - 1 = 5x$	**46.** $2x^2 + x = \frac{1}{2}$
47. $x^2 = 11x - 10$	**48.** $5x^2 = 210x$	**49.** $4x^2 + 4x = 3$
50. $2x^2 + 4x = 10$	**51.** $x^2 - 3x - 8 = 0$	**52.** $-3x^2 + 147 = 0$
53. $x^2 + 8x = 4$	**54.** $4x^2 - 4x - 3 = 0$	**55.** $x^2 = 11 - 6x$

56. **Air Pollution** The function $y = 0.4409x^2 - 5.1724x + 99.0321$ models the emissions of carbon monoxide in the United States since 1987, where y represents the amount of carbon monoxide released in a year in millions of tons, and $x = 0$ represents the year 1987.

a. How can you use a graph to estimate the year in which more than 100 million tons of carbon monoxide were released into the air?

b. How can you use the Quadratic Formula to estimate the year in which more than 100 million tons of carbon monoxide were released into the air?

c. Which method do you prefer? Explain why.

57. **Sports** A diver dives from a 10 m springboard. The equation $f(t) = -4.9t^2 + 4t + 10$ models her height above the pool at time t in seconds. At what time does she enter the water?

Without graphing, determine how many x-intercepts each function has.

58. $y = -2x^2 + 3x - 1$	**59.** $y = 0.25x^2 + 2x + 4$	**60.** $y = x^2 + 3x + 5$
61. $y = -x^2 + 3x + 10$	**62.** $y = 3x^2 - 10x + 6$	**63.** $y = 10x^2 + 13x - 3$
64. $y = x^2 + 17x - 2$	**65.** $y = -5x^2 - 4x + 3$	**66.** $y = 7x^2 - 2x + 9$

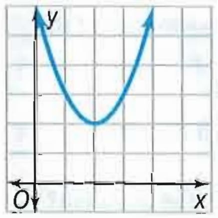
67. **Reasoning** Determine the value(s) of k for which $3x^2 + kx + 12 = 0$ has each type of solution.

a. no real solutions **b.** exactly one real solution **c.** two real solutions

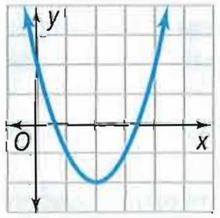
68. Use the discriminant to match each function with its graph.

a. $f(x) = x^2 - 4x + 2$ **b.** $f(x) = x^2 - 4x + 4$ **c.** $f(x) = x^2 - 4x + 6$

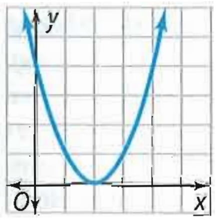
I.



II.



III.



69. a. **Geometry** Write an equation to find the dimensions of a square that has the same area as a circle with a radius of 10 cm.

b. Find the length of a side of the square, to the nearest hundredth centimeter.

Challenge

Write a quadratic equation with the given solutions.

70. $\frac{3 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2}$

71. $\frac{-5 + \sqrt{13}}{2}, \frac{-5 - \sqrt{13}}{2}$

72. $\frac{-5 + \sqrt{17}}{4}, \frac{-5 - \sqrt{17}}{4}$

Solve each equation.

73. $|5 - 2x^2| = 5$

74. $|x^2 - 4x| = 3$

75. $|x^2 + 4x + 3| = 8$

76. Use the Quadratic Formula to prove each statement.
- **a.** The sum of the solutions of the quadratic equation $ax^2 + bx + c = 0$ is $-\frac{b}{a}$.
- **b.** The product of the solutions of the quadratic equation $ax^2 + bx + c = 0$ is $\frac{c}{a}$.

77. Explain the meaning of the value $\frac{\sqrt{b^2 - 4ac}}{2a}$ in terms of the graph of the standard quadratic function $y = ax^2 + bx + c$.

Sunshine State Standards Practice

GRIDDED RESPONSE

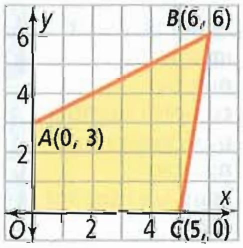
MA.912.A.7.4 **78.** How many different real solutions are there for $2x^2 - 3x + 5 = 0$?

MA.912.A.7.6 **79.** What is the y-value of the y-intercept of the quadratic function $y = 2(x + 2)^2 - 5$?

MA.912.A.3.14 **80.** What is the x-value in the solution to the system $\begin{cases} 3x + y = -7 \\ 2x - 2y = -10 \end{cases}$?

MA.912.A.3.14 **81.** The graph of the system of inequalities $\begin{cases} y \leq \frac{1}{2}x + 3 \\ y \geq 6x - 30 \\ x \geq 0 \\ y \geq 0 \end{cases}$ is shown at the right.

What is the maximum value of the function $P = 3x - 4y$ for the (x, y) pairs in the bounded region shown?



Mixed Review

Solve each equation by completing the square. See Lesson 4-6.

82. $x^2 - 8x - 20 = 0$

83. $2y^2 = 4y - 1$

84. $x^2 - 3x - 8 = 0$

Simplify by combining like terms. See Lesson 1-3.

85. $z^2 + 8z^2 - 2z + 5z$

86. $4k - x - 3k + 5x$

87. $4y - (2y + 3x) - 5x$

Get Ready! **To prepare for Lesson 4-8, do Exercises 88–90.** See p. 683.

Simplify each expression.

88. $\sqrt{(-2)^2 + 8^2}$

89. $\sqrt{3^2 + 4^2}$

90. $\sqrt{5^2 + (-12)^2}$

4-8 Complex Numbers

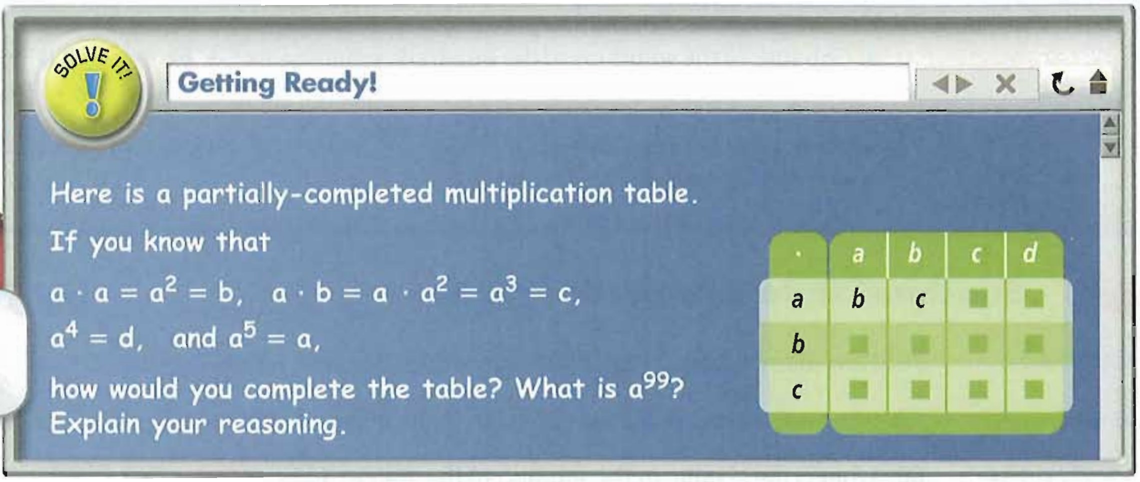
Sunshine State Standards

MA.912.A.7.5 Solve quadratic equations over the complex number system.

MA.912.A.1.6 Identify the real and imaginary parts of complex numbers and perform basic operations.

Objectives To identify, graph, and perform operations with complex numbers
To find complex number solutions of quadratic equations

One of a, b, c, and d acts like an identity element.



Here is a partially-completed multiplication table.

If you know that

$a \cdot a = a^2 = b$, $a \cdot b = a \cdot a^2 = a^3 = c$,

$a^4 = d$, and $a^5 = a$,

how would you complete the table? What is a^{99}? Explain your reasoning.

·	a	b	c	d
a	b	c	■	■
b	■	■	■	■
c	■	■	■	■

Lesson Vocabulary
- imaginary unit
- imaginary number
- complex number
- pure imaginary number
- complex number plane
- absolute value of a complex number
- complex conjugates

In Chapter 1, you learned about different subsets of real numbers. The set of real numbers is itself a subset of a larger set of numbers, the *complex numbers*. Curiously, the complex numbers include a number like a in the Solve It. Its fifth power is itself.

Essential Understanding The complex numbers are based on a number whose square is −1.

The **imaginary unit** i is the complex number whose square is −1. So, $i^2 = -1$, and $i = \sqrt{-1}$.

take note

Key Concept Square Root of a Negative Real Number

Algebra
For any positive number a,
$\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$.

Example
$\sqrt{-5} = i\sqrt{5}$

Note that $(\sqrt{-5})^2 = (i\sqrt{5})^2 = i^2(\sqrt{5})^2 = -1 \cdot 5 = -5$ (not 5).

Problem 1 Simplifying a Number Using *i*

Think

Is $\sqrt{-18}$ a real number?

No. There is no real number that when multiplied by itself gives −18. You must use the imaginary unit *i* to write $\sqrt{-18}$.

How do you write $\sqrt{-18}$ by using the imaginary unit *i*?

$$\sqrt{-18} = \sqrt{-1 \cdot 18}$$

$= \sqrt{-1} \cdot \sqrt{18}$ Multiplication Property of Square Roots

$= i \cdot \sqrt{18}$ Definition of *i*

$= i \cdot 3\sqrt{2}$ Simplify.

$= 3i\sqrt{2}$

Got It? **1.** How do you write each number in parts (a)–(c) by using the imaginary unit *i*?

a. $\sqrt{-12}$ **b.** $\sqrt{-25}$ **c.** $\sqrt{-7}$

d. **Reasoning** Explain why $\sqrt{-64} \neq -\sqrt{64}$.

An **imaginary number** is any number of the form $a + bi$, where a and b are real numbers and $b \neq 0$. Imaginary numbers and real numbers together make up the set of *complex numbers*.

take note Key Concept Complex Numbers

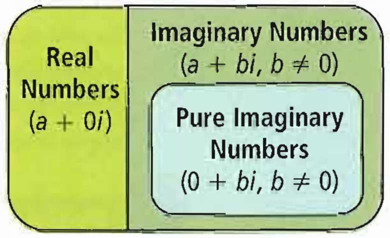
You can write a **complex number** in the form $a + bi$, where a and b are real numbers.

If $b = 0$, the number $a + bi$ is a real number.

If $a = 0$ and $b \neq 0$, the number $a + bi$ is a **pure imaginary number**.

$$\underset{\text{Real part}}{\overset{\uparrow}{a}} + \underset{\text{Imaginary part}}{\overset{\uparrow}{bi}}$$

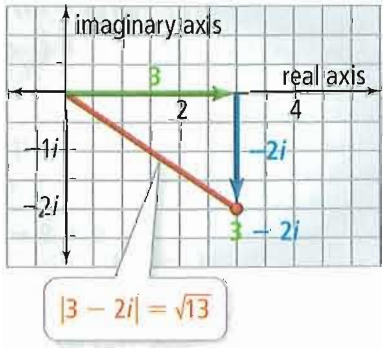
Complex Numbers ($a + bi$)



In the **complex number plane**, the point (a, b) represents the complex number $a + bi$. To graph a complex number, locate the real part on the horizontal axis and the imaginary part on the vertical axis.

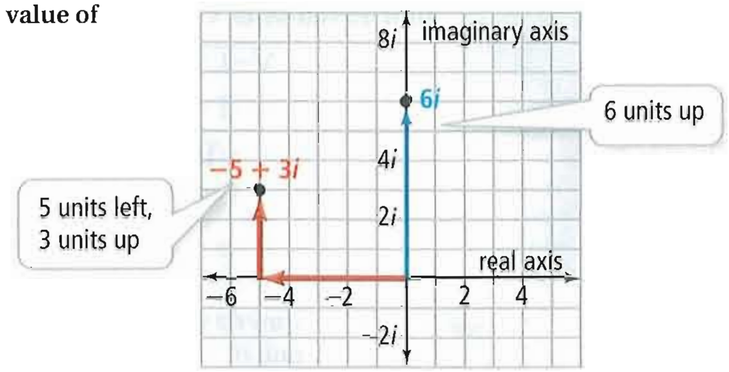
The **absolute value of a complex number** is its distance from the origin in the complex plane.

$$|a + bi| = \sqrt{a^2 + b^2}$$



Problem 2 Graphing in the Complex Number Plane

What are the graph and absolute value of each number?



A $-5 + 3i$

$$\begin{aligned} |-5 + 3i| &= \sqrt{(-5)^2 + 3^2} \\ &= \sqrt{34} \end{aligned}$$

B $6i$

$$\begin{aligned} |6i| &= |0 + 6i| \\ &= \sqrt{0^2 + 6^2} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

Think

Where is a pure imaginary number in the complex plane? The real part of a pure imaginary number is 0. The number must be on the imaginary axis.

Got It? **2.** What are the graph and absolute value of each number?

a. $5 - i$ **b.** $-3 - 2i$ **c.** $1 + 4i$

Essential Understanding You can define operations on the set of complex numbers so that when you restrict the operations to the subset of real numbers, you get the familiar operations on the real numbers.

To add or subtract complex numbers, combine the real parts and the imaginary parts separately. If the sum of two complex numbers is 0, or $0 + 0i$, then each number is the opposite, or additive inverse, of the other. The associative and commutative properties apply to complex numbers as well.

Problem 3 Adding and Subtracting Complex Numbers

What is each sum or difference?

A $(4 - 3i) + (-4 + 3i)$

$4 + (-4) + (-3i) + 3i$ Use the commutative and associative properties.

$0 + 0 = 0$ $4 - 3i$ and $-4 + 3i$ are additive inverses.

B $(5 - 3i) - (-2 + 4i)$

$5 - 3i + 2 - 4i$ To subtract, add the opposite.

$5 + 2 - 3i - 4i$ Use the commutative and associative properties

$7 - 7i$ Simplify.

Plan

How is adding complex numbers similar to adding algebraic expressions? Adding the real parts and imaginary parts separately is like adding like terms.

Got It? **3.** What is each sum or difference?

a. $(7 - 2i) + (-3 + i)$ **b.** $(1 + 5i) - (3 - 2i)$

c. $(8 + 6i) - (8 - 6i)$ **d.** $(-3 + 9i) + (3 + 9i)$

You multiply complex numbers $a + bi$ and $c + di$ as you would multiply binomials. For imaginary parts bi and di, $(bi)(di) = bd(i)^2 = bd(-1) = -bd$.

Problem 4 Multiplying Complex Numbers

Think

How do you multiply two binomials?
Multiply each term of one binomial by each term of the other binomial.

What is each product?

A $(3i)(-5 + 2i)$

$-15i + 6i^2$ Distributive Property

$-15i + 6(-1)$ Substitute -1 for i^2.

$-6 - 15i$ Simplify.

B $(4 + 3i)(-1 - 2i)$

$-4 - 8i - 3i - 6i^2$

$-4 - 8i - 3i - 6(-1)$

$2 - 11i$

Substitute -1 for i^2.

C $(-6 + i)(-6 - i)$

$36 + 6i - 6i - i^2$

$36 + 6i - 6i - (-1)$

37

Got It? **4.** What is each product?

a. $(7i)(3i)$ **b.** $(2 - 3i)(4 + 5i)$ **c.** $(-4 + 5i)(-4 - 5i)$

In Problem 4(c), the product is a real number. Number pairs of the form $a + bi$ and $a - bi$ are **complex conjugates**. The product of complex conjugates is a real number.

$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2$$

You can use complex conjugates to simplify quotients of complex numbers.

Problem 5 Dividing Complex Numbers

Plan

What is the goal?
Write the quotient in the form $a + bi$.

What is each quotient?

A $\frac{9 + 12i}{3i}$

$\frac{9 + 12i}{3i} \cdot \frac{-3i}{-3i}$

$\frac{-27i - 36i^2}{-9i^2}$

$\frac{-27i - 36(-1)}{-9(-1)}$

$\frac{36 - 27i}{9}$

$4 - 3i$

Multiply numerator and denominator by the complex conjugate of the denominator.

Substitute -1 for i^2.

B $\frac{2 + 3i}{1 - 4i}$

$\frac{2 + 3i}{1 - 4i} \cdot \frac{1 + 4i}{1 + 4i}$

$\frac{2 + 8i + 3i + 12i^2}{1 + 4i - 4i - 16i^2}$

$\frac{2 + 8i + 3i + 12(-1)}{1 + 4i - 4i - 16(-1)}$

$\frac{-10 + 11i}{17}$

$-\frac{10}{17} + \frac{11}{17}i$

Got It? **5.** What is each quotient?

a. $\frac{5 - 2i}{3 + 4i}$ **b.** $\frac{4 - i}{6i}$ **c.** $\frac{8 - 7i}{8 + 7i}$

Essential Understanding Every quadratic equation has complex number solutions (that sometimes are real numbers).

Some quadratic equations have pure imaginary solutions.

Problem 6 Finding Pure Imaginary Solutions

Plan

How do you solve a quadratic equation of the form $ax^2 + c = 0$?
Use properties of equality to isolate the variable.

What are the solutions of $2x^2 + 32 = 0$?

$2x^2 + 32 = 0$	
$2x^2 = -32$	Isolate x^2.
$x^2 = -16$	Divide both sides by 2.
$x = \pm\sqrt{-16}$	Find square roots.
$x = \pm 4i$	Simplify.

Got It? **6.** What are the solutions of each equation?
a. $5x^2 + 20 = 0$
b. $x^2 + 15 = 0$

Some quadratic equations have solutions that are imaginary numbers.

Problem 7 Finding Imaginary Solutions

What are the solutions of $2x^2 - 3x + 5 = 0$?

Think	Write
Use the Quadratic Formula with $a = 2, b = -3$, and $c = 5$.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(5)}}{2(2)}$ $= \frac{3 \pm \sqrt{9 - 40}}{4}$
Simplify.	$= \frac{3 \pm \sqrt{-31}}{4}$ $= \frac{3}{4} \pm \frac{\sqrt{31}}{4}i$

Got It? **7.** What are the solutions of each equation?
a. $3x^2 - x + 2 = 0$
b. $x^2 - 4x + 5 = 0$

Lesson Check

Do you know HOW?

1. Simplify $\sqrt{-75}$ by using the imaginary number i.
2. Find the absolute value of $4 - 3i$.
3. Solve the equation $x^2 + 16 = 0$. Check your answers.

Simplify each expression.

4. $(4 - 2i) - (-3 + i)$
5. $(2 + i)(4 - 5i)$

Do you UNDERSTAND?

6. **Vocabulary** Explain the difference between the additive inverse of a complex number and a complex conjugate.
7. **Error Analysis** Describe and correct the error made in simplifying the expression $(4 - 7i)(4 + 7i)$.

~~$(4 - 7i)(4 + 7i) = 16 + 28i - 28i + 49i^2$~~
~~$= 16 - 49$~~
~~$= -33$~~

Practice and Problem-Solving Exercises

A Practice

Simplify each number by using the imaginary number i. See Problem 1.

8. $\sqrt{-4}$
9. $\sqrt{-7}$
10. $\sqrt{-15}$
11. $\sqrt{-81}$
12. $\sqrt{-50}$

Plot each complex number and find its absolute value. See Problem 2.

13. $2i$
14. $5 + 12i$
15. $2 - 2i$
16. $1 - 4i$
17. $3 - 6i$

Simplify each expression. See Problems 3 and 4.

18. $(2 + 4i) + (4 - i)$
19. $(-3 - 5i) + (4 - 2i)$
20. $(7 + 9i) + (-5i)$
21. $(12 + 5i) - (2 - i)$
22. $(-6 - 7i) - (1 + 3i)$
23. $(8 + i)(2 + 7i)$
24. $(-6 - 5i)(1 + 3i)$
25. $(-6i)^2$
26. $(9 + 4i)^2$

Write each quotient as a complex number. See Problem 5.

27. $\frac{3 - 2i}{5i}$
28. $\frac{-2i}{1 + i}$
29. $\frac{4 - 3i}{-1 - 4i}$
30. $\frac{i + 2}{i - 2}$
31. $\frac{4}{2 - 3i}$
32. $\frac{3 + 2i}{(1 + i)^2}$

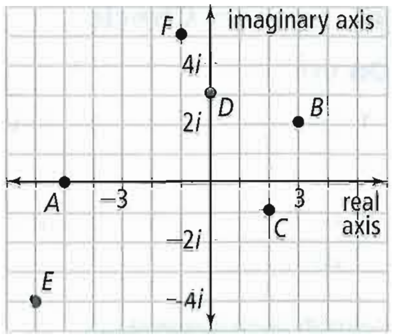
Solve each equation. Check your answer. See Problem 6.

33. $x^2 + 25 = 0$
34. $2x^2 + 1 = 0$
35. $3s^2 + 2 = -62$
36. $x^2 = -7$
37. $x^2 + 36 = 0$
38. $-5x^2 - 3 = 0$

Find all solutions to each quadratic equation. See Problem 7.

39. $x^2 + 2x + 3 = 0$
40. $-3x^2 + x - 3 = 0$
41. $2x^2 - 4x + 7 = 0$
42. $x^2 - 2x + 2 = 0$
43. $x^2 + 5 = 4x$
44. $2x(x - 3) = -5$

B Apply



45. a. Name the complex number represented by each point on the graph at the right.

b. Find the additive inverse of each number.

c. Find the complex conjugate of each number.

d. Find the absolute value of each number.

46. Think About a Plan In the complex number plane, what geometric figure describes the complex numbers with absolute value 10?

- What does the absolute value of a complex number represent?
- How can you use the complex number plane to solve this problem?

47. Solve $(x + 3i)(x - 3i) = 34$.

Simplify each expression.

48. $(8i)(4i)(-9i)$

49. $(2 + \sqrt{-1}) + (-3 + \sqrt{-16})$

50. $(4 + \sqrt{-9}) + (6 - \sqrt{-49})$

51. $(10 + \sqrt{-9}) - (2 + \sqrt{-25})$

52. $(8 - \sqrt{-1}) - (-3 + \sqrt{-16})$

53. $2i(5 - 3i)$

54. $-5(1 + 2i) + 3i(3 - 4i)$

55. $(3 + \sqrt{-4})(4 + \sqrt{-1})$

56. Open-Ended In the equation $x^2 - 6x + c = 0$, find values of c that will give:

a. two real solutions **b.** two imaginary solutions **c.** one real solution

57. A student wrote the numbers 1, 5, $1 + 3i$, and $4 + 3i$ to represent the vertices of a quadrilateral in the complex number plane. What type of quadrilateral has these vertices?

The multiplicative inverse of a complex number z is $\frac{1}{z}$ where $z \neq 0$. Find the multiplicative inverse, or reciprocal, of each complex number. Then use complex conjugates to simplify. Check each answer by multiplying it by the original number.

58. $2 + 5i$

59. $8 - 12i$

60. $a + bi$

Find the sum and product of the roots of each equation.

61. $x^2 - 2x + 3 = 0$

62. $5x^2 + 2x + 1 = 0$

63. $-2x^2 + 3x - 3 = 0$

For $ax^2 + bx + c = 0$, the sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$. Find a quadratic equation for each pair of roots. Assume $a = 1$.

64. $-6i$ and $6i$

65. $2 + 5i$ and $2 - 5i$

66. $4 - 3i$ and $4 + 3i$

Two complex numbers $a + bi$ and $c + di$ are equal when $a = c$ and $b = d$. Solve each equation for x and y.

67. $2x + 3yi = -14 + 9i$

68. $3x + 19i = 16 - 8yi$

69. $-14 - 3i = 2x + yi$

Challenge

70. Show that the product of any complex number $a + bi$ and its complex conjugate is a real number.

71. For what real values of x and y is $(x + yi)^2$ an imaginary number?

72. **Reasoning** True or false: The conjugate of the additive inverse of a complex number is equal to the additive inverse of the conjugate of that complex number. Explain your answer.

Sunshine State Standards Practice

MA.912.A.1.6 **73.** How can you rewrite the expression $(8 - 5i)^2$ in the form $a + bi$?

Ⓐ $39 + 80i$ Ⓑ $39 - 80i$ Ⓒ $69 + 80i$ Ⓓ $69 - 80i$

MA.912.A.7.4 **74.** How many solutions does the quadratic equation $4x^2 - 12x + 9 = 0$ have?

Ⓕ two real solutions Ⓗ two imaginary solutions

Ⓖ one real solution Ⓘ one imaginary solution

MA.912.A.7.5 **75.** What are the solutions of $3x^2 - 2x - 4 = 0$?

Ⓐ $\frac{1 \pm \sqrt{13}}{3}$ Ⓑ $\frac{1 \pm i\sqrt{11}}{3}$ Ⓒ $\frac{-1 \pm \sqrt{13}}{3}$ Ⓓ $\frac{-1 \pm i\sqrt{11}}{3}$

MA.912.A.4.3 **76.** **Short Response** Using factoring, what are all four solutions to $x^4 - 16 = 0$? Show your work.

Mixed Review

Solve each equation using the Quadratic Formula. See Lesson 4-7.

77. $2x^2 + 3x - 4 = 0$ **78.** $4x^2 + x = 1$ **79.** $x^2 = -7x - 8$

Graph each function. Identify the axis of symmetry. See Lesson 4-1.

80. $y = -2(x + 1)^2 - 3$ **81.** $y = \frac{1}{2}(x - 4)^2 + 1$ **82.** $y = 3(x - 1)^2 - 5$

Write an equation for each line. See Lesson 2-3.

83. $m = 3$ and the y-intercept is -4

84. $m = -0.5$ and the y-intercept is -2

85. $m = -7$ and the y-intercept is 10

86. $m = 2$ and the y-intercept is 8

Get Ready! **To prepare for Lesson 4-9, do Exercises 87–89.** See Lesson 3-3.

Solve each system of inequalities by graphing.

87. $\begin{cases} y < 2x + 4 \\ y \geq |x - 3| + 2 \end{cases}$ **88.** $\begin{cases} y > -x \\ y < -|x + 1| \end{cases}$ **89.** $\begin{cases} y \leq |x| + 2 \\ y \geq -\frac{1}{2}x + 4 \end{cases}$

Concept Byte

For Use With Lesson 4-9

Quadratic Inequalities

Sunshine State Standard
Prepares for MA.912.A.7.7 Solve non-linear systems of equations with and without using technology.

To solve some quadratic inequalities, relate the quadratic expression to 0 and factor. To determine the sign of each factor, use what you know about multiplying positive and negative numbers.

Example 1

Solve each inequality algebraically.

a. $2x^2 - 14x < 0$

$2x(x - 7) < 0$	Factor.
$2x > 0$ and $(x - 7) < 0$, or $2x < 0$ and $(x - 7) > 0$	The product is negative, so the two factors must have *different* signs.
$x > 0$ and $x < 7$, or $x < 0$ and $x > 7$	Simplify.
$0 < x < 7$	No value can be both greater than 7 *and* less than 0.

b. $2x^2 - 14x > 0$

$2x(x - 7) > 0$	Factor.
$2x > 0$ and $(x - 7) > 0$, or $2x < 0$ and $(x - 7) < 0$	The product is positive, so the two factors must have the *same* signs.
$x > 0$ and $x > 7$, or $x < 0$ and $x < 7$	Simplify.
$x > 7$ or $x < 0$	A value that is greater than both 0 *and* 7 is always greater than 7. A value that is less than both 0 *and* 7 is always less than 0.

You can use a table to solve inequalities by analyzing the values of y around 0.

Activity

Use a table to find the solutions of $x^2 - 6x + 5 < 0$.

1. What happens to the value of y when $0 \leq x \leq 6$?
2. Does this make sense when you think of the shape of the graph of $y = x^2 - 6x + 5$? Explain.
3. What x-values in the table make the inequality $x^2 - 6x + 5 < 0$ true?
4. What are the solutions of $x^2 - 6x + 5 < 0$?

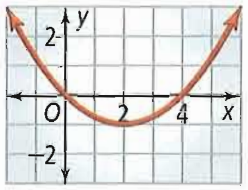
x	y
0	5
1	0
2	−3
3	−4
4	−3
5	0
6	5

You can determine the solution of a quadratic inequality based on how many times and where the graph of the related function crosses the x-axis. The graph could open upward or downward, and could intersect the x-axis at 0, 1, or 2 points.

You can solve inequalities of the form $ax^2 + bx + c > 0$ or $ax^2 + bx + c < 0$ by graphing the corresponding function and seeing where the graph is above or below the x-axis.

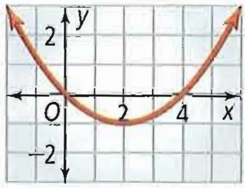
Example 2

Find the solution sets for $\frac{1}{4}(x - 2)^2 - 1 > 0$ and $\frac{1}{4}(x - 2)^2 - 1 < 0$.



The solution set for $\frac{1}{4}(x - 2)^2 - 1 > 0$ is all x-values of points on the parabola that lie above the x-axis.

$x < 0$ or $x > 4$



The solution set for $\frac{1}{4}(x - 2)^2 - 1 < 0$ is all x-values of points on the parabola that lie below the x-axis.

$0 < x < 4$

Example 3

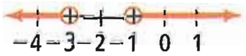
Solve $-2x^2 - 8x - 6 < 0$.

Think: Since the coefficient of x^2 is less than zero, the graph of $y = -2x^2 - 8x - 6$ opens downward.

Solve: Find where $-2x^2 - 8x - 6$ equals 0.

$$-2x^2 - 8x - 6 = 0$$
$$-2(x^2 + 4x + 3) = 0$$
$$-2(x + 3)(x + 1) = 0$$
$$x = -3 \text{ or } x = -1$$

The graph of $y = -2x^2 - 8x - 6$ opens down and crosses the x-axis at $x = -3$ and $x = -1$. The solution of $-2x^2 - 8x - 6 < 0$ is $x < -3$ or $x > -1$.



Exercises

5. Solve each inequality. Graph your solution on a number line.

a. $x^2 < 36$ **b.** $x^2 - 9 > 0$ **c.** $x^2 < -4$ **d.** $x^2 - 3x - 18 > 0$

6. How can you use the graph of $y = 3x - 4$ to solve the linear inequality $3x - 4 < 0$? Graph the solution.

7. How can you solve the absolute value inequality $|-3x + 4| > 0$?

8. Example 2 shows two possible graphs for a quadratic inequality. What other possibilities are there?

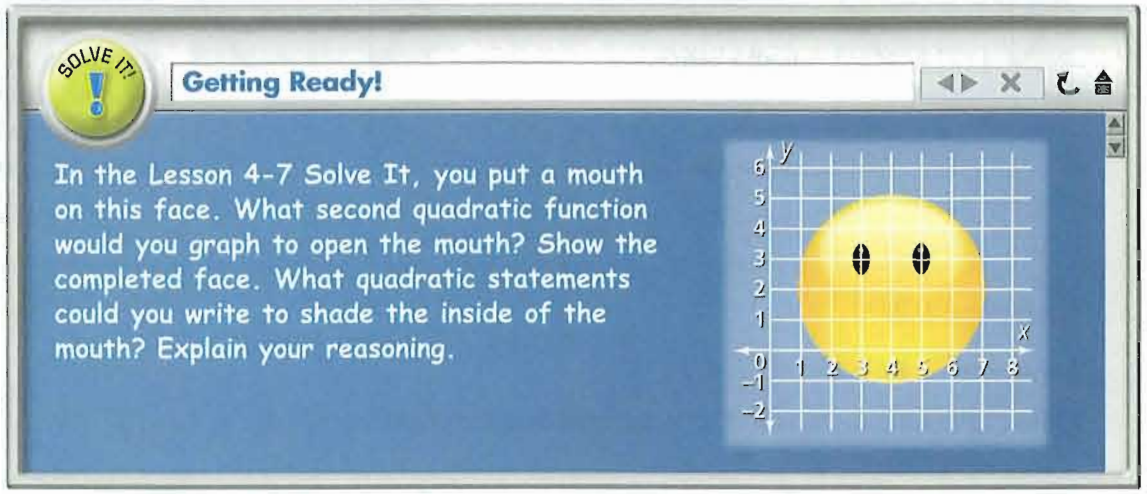
9. Describe the graphs of possible solutions of $ax^3 + bx^2 + cx + d > 0$.

4-9 Quadratic Systems

Sunshine State Standard
MA.912.A.7.7 Solve non-linear systems of equations with and without using technology.

Objectives To solve and graph systems of linear and quadratic equations
To solve and graph systems of quadratic inequalities

Dynamic Activity
Systems of Quadratic Inequalities



Getting Ready!

In the Lesson 4-7 Solve It, you put a mouth on this face. What second quadratic function would you graph to open the mouth? Show the completed face. What quadratic statements could you write to shade the inside of the mouth? Explain your reasoning.

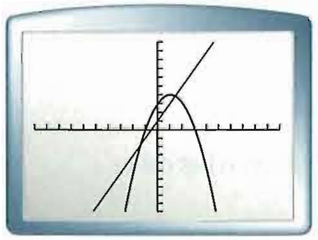
By drawing a second parabola in the Solve It, you created a quadratic system.

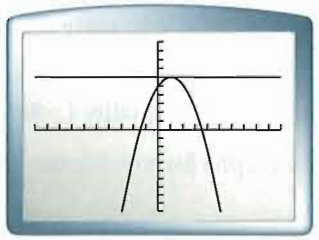
Essential Understanding You can solve systems involving quadratic equations using methods similar to the ones used to solve systems of linear equations.

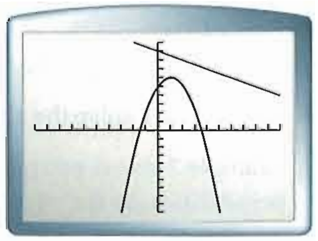
The points where the graphs of the equations intersect represent the solutions of a system.

take note **Key Concept** Solutions of a Linear-Quadratic System

A system of one quadratic equation and one linear equation can have two solutions, one solution, or no solution.


Two solutions


One solution


No solution

Problem 1 Solving a Linear-Quadratic System by Graphing

Multiple Choice Which numbers are *y*-values of the solutions of the system of equations? $\begin{cases} y = -x^2 + 5x + 6 \\ y = x + 6 \end{cases}$

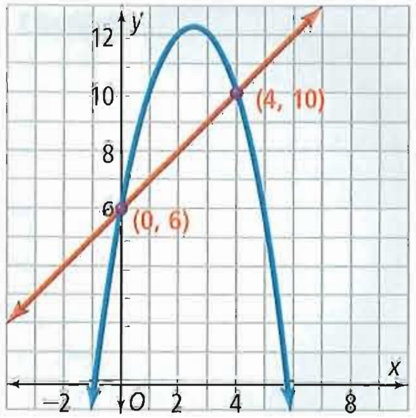
Ⓐ 4 only　Ⓑ 6 only　Ⓒ 4 and 6　Ⓓ 6 and 10

Plan

How can you graph these two equations? Use slope-intercept form to graph the linear equation. Make a table of values to graph the quadratic equation.

Graph the equations. Find their intersections.

The solutions appear to be (0, 6) and (4, 10).



Check

$y = -x^2 + 5x + 6$　　$y = x + 6$

$6 \overset{?}{=} -(0)^2 + 5(0) + 6$　　$6 \overset{?}{=} 0 + 6$

$6 = 6$ ✔　　$6 = 6$ ✔

$y = -x^2 + 5x + 6$　　$y = x + 6$

$10 \overset{?}{=} -(4)^2 + 5(4) + 6$　　$10 \overset{?}{=} 4 + 6$

$10 = 10$ ✔　　$10 = 10$ ✔

The *y*-values of the solutions are 6 and 10, choice D.

Got It? **1.** What is the solution of the system? $\begin{cases} y = x^2 + 6x + 9 \\ y = x + 3 \end{cases}$

Problem 2 Solving a Linear-Quadratic System Using Substitution

What is the solution of the system of equations? $\begin{cases} y = -x^2 - x + 6 \\ y = x + 3 \end{cases}$

Think	Write
Substitute $x + 3$ for y in the quadratic equation.	$x + 3 = -x^2 - x + 6$
Write in standard form.	$x^2 + 2x - 3 = 0$
Factor. Solve for x.	$(x - 1)(x + 3) = 0$ $x = 1$ or $x = -3$
Substitute for x in $y = x + 3$.	$x = 1 \rightarrow y = 1 + 3 = 4$ $x = -3 \rightarrow y = -3 + 3 = 0$ The solutions are (1, 4) and (−3, 0).

Got It? **2.** What is the solution of the system? $\begin{cases} y = -x^2 - 3x + 10 \\ y = x + 5 \end{cases}$

You can solve quadratic–quadratic systems using the same methods you used for linear–quadratic systems.

Problem 3 Solving a Quadratic System of Equations

Plan

Which variable should you substitute for? You can substitute for either variable, but substituting for y results in a simple equation.

What is the solution of the system? $\begin{cases} y = -x^2 - x + 12 \\ y = x^2 + 7x + 12 \end{cases}$

Method 1 Use substitution.

Substitute $y = -x^2 - x + 12$ for y in the second equation. Solve for x.

$-x^2 - x + 12 = x^2 + 7x + 12$ Substitute for y.

$-2x^2 - 8x = 0$ Write in standard form.

$-2x(x + 4) = 0$ Factor.

$x = 0$ or $x = -4$ Solve for x.

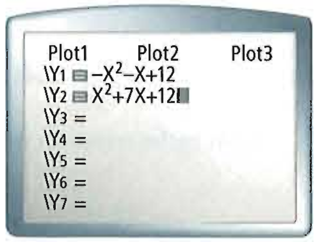
Substitute each value of x into either equation. Solve for y.

$y = x^2 + 7x + 12$	$y = x^2 + 7x + 12$
$y = (0)^2 + 7(0) + 12$	$y = (-4)^2 + 7(-4) + 12$
$y = 0 + 0 + 12 = 12$	$y = 16 - 28 + 12 = 0$

The solutions are (0, 12) and (−4, 0).

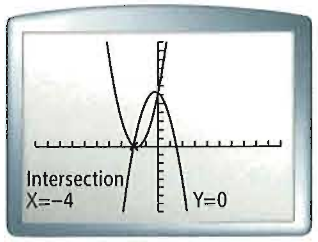
Method 2 Graph the equations.

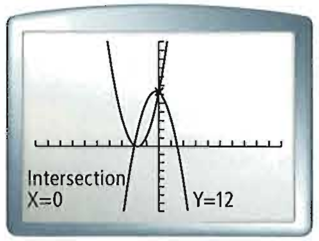
Use a graphing calculator. Define functions $\mathbf{Y_1}$ and $\mathbf{Y_2}$.



```
Plot1   Plot2   Plot3
\Y1 =-X²-X+12
\Y2 =X²+7X+12
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

Use the **INTERSECT** feature to find the points of intersection.





The solutions are (−4, 0) and (0, 12).

Got It? **3.** What is the solution of each system of equations?

a. $\begin{cases} y = x^2 - 4x + 5 \\ y = -x^2 + 5 \end{cases}$

b. $\begin{cases} y = x^2 - 4x + 5 \\ y = -x^2 - 5 \end{cases}$

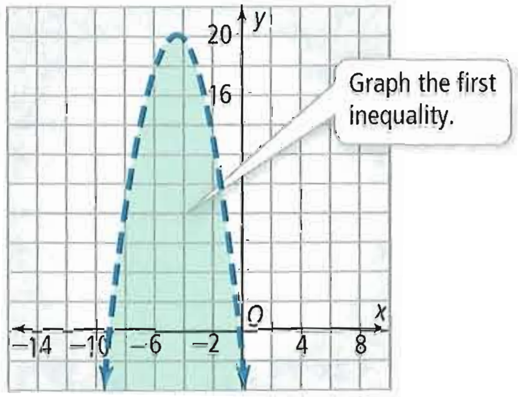
You can use the techniques for solving a linear system of inequalities to solve a quadratic system of inequalities.

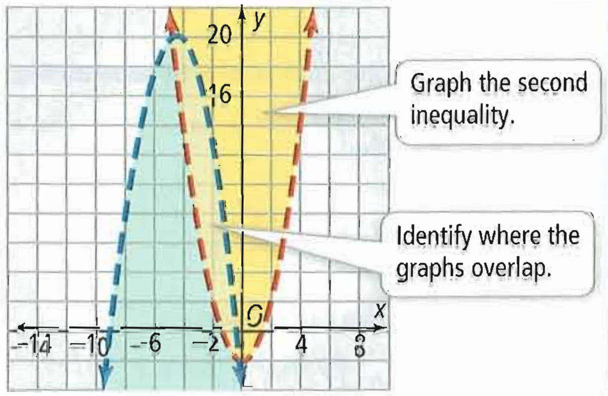
Problem 4 Solving a Quadratic System of Inequalities

Plan

How can you find the solution?
Graph each inequality and find the region where the graphs overlap.

What is the solution of the system of inequalities? $\begin{cases} y < -x^2 - 9x - 2 \\ y > x^2 - 2 \end{cases}$





The solution of this system is the region where the graphs overlap. The region contains no boundary points.

Got It? **4. a.** What is the solution of this system of inequalities? $\begin{cases} y \leq -x^2 - 4x + 3 \\ y > x^2 + 3 \end{cases}$

b. Reasoning How many solutions can a system of inequalities have?

Lesson Check

Do you know HOW?

Solve each system by substitution.

1. $\begin{cases} y = x^2 - 2x + 3 \\ y = x + 1 \end{cases}$

2. $\begin{cases} y = 2x^2 - 5x + 2 \\ y = x - 2 \end{cases}$

3. $\begin{cases} y = x^2 - 3x - 3 \\ y = -2x^2 - x + 5 \end{cases}$

Solve each system by graphing.

4. $\begin{cases} y > 2x^2 + x + 3 \\ y < -x^2 - 4x + 1 \end{cases}$

5. $\begin{cases} y > -3x^2 - 6x + 1 \\ y < -2x^2 - 3x + 5 \end{cases}$

Do you UNDERSTAND?

6. Compare and Contrast How are solving systems of two linear equations or inequalities and solving systems of two quadratic equations or inequalities alike? How are they different?

7. Reasoning How many points of intersection can graphs of the following types of functions have? Draw graphs to justify your answers.

a. a linear function and a quadratic function

b. two quadratic functions

c. a quadratic function and an absolute value function (*Hint:* Graph $y = x^2$ and $y = |x|$ together. Can you transform one of the graphs slightly to increase the number of intersections?)

Practice and Problem-Solving Exercises

A Practice

Solve each system by graphing. Check your answers.

See Problem 1.

8. $\begin{cases} y = -x^2 + 2x + 1 \\ y = 2x + 1 \end{cases}$

9. $\begin{cases} y = x^2 - 2x + 1 \\ y = 2x + 1 \end{cases}$

10. $\begin{cases} y = x^2 - x + 3 \\ y = -2x + 5 \end{cases}$

11. $\begin{cases} y = 2x^2 + 3x + 1 \\ y = -2x + 1 \end{cases}$

12. $\begin{cases} y = -x^2 - 3x + 2 \\ y = x + 6 \end{cases}$

13. $\begin{cases} y = -x^2 - 2x - 2 \\ y = x - 4 \end{cases}$

Solve each system by substitution. Check your answers.

See Problem 2.

14. $\begin{cases} y = x^2 + 4x + 1 \\ y = x + 1 \end{cases}$

15. $\begin{cases} y = -x^2 + 2x + 10 \\ y = x + 4 \end{cases}$

16. $\begin{cases} y = -x^2 + x - 1 \\ y = -x - 1 \end{cases}$

17. $\begin{cases} y = 2x^2 - 3x - 1 \\ y = x - 3 \end{cases}$

18. $\begin{cases} y = x^2 - 3x - 20 \\ y = -x - 5 \end{cases}$

19. $\begin{cases} y = -x^2 - 5x - 1 \\ y = x + 2 \end{cases}$

Solve each system.

See Problem 3.

20. $\begin{cases} y = x^2 + 5x + 1 \\ y = x^2 + 2x + 1 \end{cases}$

21. $\begin{cases} y = x^2 - 2x - 1 \\ y = -x^2 - 2x - 1 \end{cases}$

22. $\begin{cases} y = -x^2 - 3x - 2 \\ y = x^2 + 3x + 2 \end{cases}$

23. $\begin{cases} y = -x^2 - x - 3 \\ y = 2x^2 - 2x - 3 \end{cases}$

24. $\begin{cases} y = -3x^2 - x + 2 \\ y = x^2 + 2x + 1 \end{cases}$

25. $\begin{cases} y = x^2 + 2x + 1 \\ y = x^2 + 2x - 1 \end{cases}$

Solve each system by graphing.

See Problem 4.

26. $\begin{cases} y > x^2 + 2x \\ y > x^2 - 1 \end{cases}$

27. $\begin{cases} y > x^2 - 3x \\ y < 2x^2 - 3x \end{cases}$

28. $\begin{cases} y < -x^2 - 3x \\ y > x^2 - 1 \end{cases}$

B Apply

29. Think About a Plan A manufacturer is making cardboard boxes by cutting out four equal squares from the corners of the rectangular piece of cardboard and then folding the remaining part into a box. The length of the cardboard piece is 1 in. longer than its width. The manufacturer can cut out either 3 × 3 in. squares, or 4 × 4 in. squares. Find the dimensions of the cardboard for which the volume of the boxes produced by both methods will be the same.

- How can you represent the volume of the box using one variable?
- What system of equations can you write?
- Which method can you use to solve the system?

30. Open-Ended Can you solve the system of equations shown by graphing? Justify your answer. Can you solve this system using another method? If so, solve the system and explain why you chose that method.

$$\begin{cases} x = y^2 + 2y + 1 \\ y = x - 4 \end{cases}$$

Solve each system by substitution.

31. $\begin{cases} x + y = 3 \\ y = x^2 - 8x - 9 \end{cases}$

32. $\begin{cases} y - 2x = x + 5 \\ y + 1 = x^2 + 5x + 3 \end{cases}$

33. $\begin{cases} y - \frac{1}{2}x^2 = 1 + 3x \\ y + \frac{1}{2}x^2 = x \end{cases}$

34. $\begin{cases} x + y - 2 = 0 \\ x^2 + y - 8 = 0 \end{cases}$

35. $\begin{cases} x^2 - y = x + 4 \\ x - 1 = y + 3 \end{cases}$

36. $\begin{cases} 2y = y - x^2 + 1 \\ y = x^2 - 5x - 2 \end{cases}$

Graph the solution to each set of inequalities.

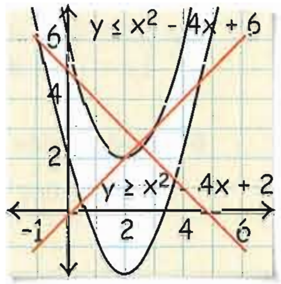
37. $\begin{cases} y < -3x^2 + 1 \\ y > x^2 - x - 5 \end{cases}$

38. $\begin{cases} y < 3x^2 + 2x + 1 \\ y > 2x^2 - x + 1 \end{cases}$

39. $\begin{cases} y > x^2 - 5x + 4 \\ y > x^2 + 3x + 2 \end{cases}$

40. **Error Analysis** A classmate graphed the system of inequalities and concluded that because the shaded regions do not intersect, there are no solutions to the system. Describe and correct the error.

$\begin{cases} y \leq x^2 - 4x + 6 \\ y \geq x^2 - 4x + 2 \end{cases}$



Solve each system.

41. $\begin{cases} y = 3x^2 - 2x - 1 \\ y = x - 1 \end{cases}$

42. $\begin{cases} y = -x^2 + x - 5 \\ y = x - 5 \end{cases}$

43. $\begin{cases} y = x^2 - 3x - 2 \\ y = 4x + 28 \end{cases}$

44. $\begin{cases} y = \frac{1}{2}x^2 + 4x + 4 \\ y = -4x + 12\frac{1}{2} \end{cases}$

45. $\begin{cases} y = -\frac{3}{4}x^2 - 4x \\ y = 3x + 8 \end{cases}$

46. $\begin{cases} y = -\frac{1}{4}x^2 + x + 1 \\ y = x - 4 \end{cases}$

47. **Business** A company's weekly revenue R is given by the formula $R = -p^2 + 30p$, where p is the price of the company's product. The company is considering hiring a distributor, which will cost the company $4p + 25$ per week.

a. Use a system of equations to find the values of the price p for which the product will still remain profitable if they hire this distributor.

b. Which value of p will maximize the profit after including the distributor cost?

Solve each system.

48. $\begin{cases} y = 5x^2 + 9x + 4 \\ y = -5x + 3 \end{cases}$

49. $\begin{cases} y = -7x^2 - 9x + 6 \\ y = \frac{1}{2}x + 11 \end{cases}$

50. $\begin{cases} y = x^2 + 3x + 6 \\ y = -x + 2 \end{cases}$

51. $\begin{cases} y = -4x^2 + 7x + 1 \\ y = 3x + 2 \end{cases}$

52. **Reasoning** Sketch the graphs of $y = 2x^2 + 4x - 5$ and $y = x^2 + 2x - 3$. Change these equations into inequalities so the system has solutions that comprise:

a. two non-overlapping regions

b. one bounded region

Solve the systems by graphing. For each system indicate one point in the solution set.

53. $\begin{cases} y < x^2 - 1 \\ y > 3x^2 - 3 \end{cases}$

54. $\begin{cases} y > x^2 \\ y < -x^2 + 1 \end{cases}$

55. $\begin{cases} y > (x - 3)^2 + 4 \\ y < -(x - 3)^2 + 5 \end{cases}$

Challenge

56. Find a value of a for which the line $y = x + a$ separates the parabolas $y = x^2 - 3x + 2$ and $y = -x^2 + 8x - 15$.

Determine whether the following systems *always, sometimes,* or *never* have solutions. (Assume that different letters refer to unequal constants.) Explain.

57. $\begin{cases} y = x^2 + c \\ y = x^2 + d \end{cases}$

58. $\begin{cases} y = ax^2 + c \\ y = bx^2 + c \end{cases}$

59. $\begin{cases} y = (x + a)^2 \\ y = (x + b)^2 \end{cases}$

60. $\begin{cases} y = a(x + m)^2 + c \\ y = b(x + n)^2 + d \end{cases}$

61. Find the side of the square with vertical and horizontal sides inscribed in the region representing the solution of the system $\begin{cases} y \leq -x^2 + 1 \\ y \geq x^2 - 1 \end{cases}$.

Sunshine State Standards Practice

MA.912.A.7.4 **62.** How many solutions does the system have? $\begin{cases} y = -\frac{1}{4}x^2 - 2x \\ y = x^2 + \frac{3}{4} \end{cases}$

(A) 0 (B) 1 (C) 2 (D) 3

MA.912.A.1.6 **63.** Which expression is equivalent to $(-3 + 2i)(2 - 3i)$?

(F) $13i$ (G) 12 (H) $12 + 13i$ (I) -12

MA.912.A.5.3 **64.** Which expression is equivalent to $(2 - 7i) \div (2i)^3$?

(A) $\frac{7}{8} - \frac{1}{4}i$ (B) $\frac{1}{4} - \frac{7}{8}i$ (C) $\frac{7}{8} + \frac{1}{4}i$ (D) $\frac{1}{4} + \frac{7}{8}i$

MA.912.A.7.5 **65. Short Response** Solve the equation $-3x^2 + 5x + 4 = 0$. Show your work.

Mixed Review

Find the sum or difference. See Lesson 4-8.

66. $(1 - i) + (-5 + 4i)$ **67.** $(3 + 4i) - (-4 - 3i)$ **68.** $(1 + i) + (2 + 2i)$

Solve each equation using the Quadratic Formula. See Lesson 4-7.

69. $2m^2 + 5m + 3 = 0$ **70.** $p^2 - 4p + 3 = 0$ **71.** $25x^2 - 30x + 9 = 0$

Rewrite each equation in vertex form. See Lesson 4-6.

72. $y = -k^2 + 4k + 6$ **73.** $y = x^2 + 6x + 1$ **74.** $y = 2n^2 - 8n - 3$

Get Ready! **To prepare for Lesson 5-1, do Exercises 75–77.**

Simplify by combining like terms. See Lesson 1-3.

75. $3q + 9q - q$ **76.** $-2ab^2 + 2a^2b + 3ab^2$ **77.** $-4y^2 + 2y + 3y^2$

Concept Byte
EXTENSION

Powers of Complex Numbers

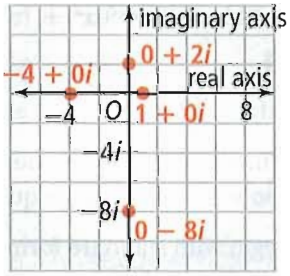
Sunshine State Standard
MA.912.A.1.6 Identify the real and imaginary parts of complex numbers and perform basic operations.

You can use the rules for multiplying complex numbers to find powers of complex numbers.

Example 1

Compute and graph $(2i)^n$, for $n = 0, 1, 2,$ and 3.

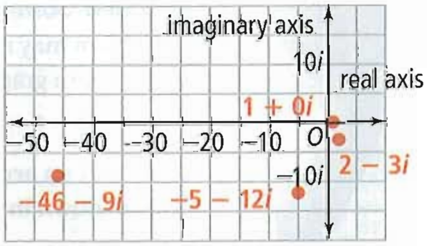
n	$(2i)^n$
0	$(2i)^0 = 1$
1	$(2i)^1 = 2i$
2	$(2i)^2 = 4i^2 = 4(-1) = -4$
3	$(2i)^3 = 8i^3 = 8(i^2 \cdot i) = 8(-1 \cdot i) = -8i$



Example 2

Compute and graph $(2 - 3i)^n$, for $n = 0, 1, 2,$ and 3.

n	$(2 - 3i)^n$
0	$(2 - 3i)^0 = 1$
1	$(2 - 3i)^1 = 2 - 3i$
2	$(2 - 3i)^2 = 4 - 6i - 6i + 9i^2 = 4 - 12i + 9(-1) = -5 - 12i$
3	$(2 - 3i)^3 = -10 - 24i + 15i + 36i^2 = -10 - 9i + 36(-1) = -46 - 9i$



Exercises

1. Based on the graph in Example 1, predict the location of $(2i)^5$.

2. Compute and graph $(-3i)^n$ for $n = 0, 1, 2,$ and 3.

3. a. Connect the points from the graph in Example 1 with a smooth curve. Estimate $(2i)^{\frac{1}{2}}$.
b. Use a graphing calculator to compute $(2i)^{\frac{1}{2}}$. Does it fall on the curve? Was it close to your estimate?

4. Use a graphing calculator to find values of $(2 - 3i)^n$ for $n = 0.5, 1.5,$ and 2.5. Copy the graph and add these points.

5. Compute and graph $(3 - 4i)^n$ for $n = 0, 1, 2,$ and 3.

4

Pull It All Together

To solve these problems, you will pull together many concepts and skills related to solving quadratic equations. Be sure to show your work and justify your reasoning.

BIG idea Equivalence and Function

The parameters a, b, c, h, and k in the standard and vertex forms of a quadratic function give information on how the graph of the function relates to the graph of the parent function $y = x^2$.

Standard form: $y = ax^2 + bx + c$ Vertex form: $y = a(x - h)^2 + k$

Task 1

Refer to the two forms shown above.

a. What information do the parameters, or combinations of parameters, provide about the graph of the quadratic function?

b. Begin with standard form. Transform it to vertex form. What are the values of h and k in terms of a, b, and c?

c. Show how the Quadratic Formula follows from your result in part (b). *Hint:* Set the expression in your vertex form equal to 0. Then solve by factoring.

BIG idea Solving Equations and Inequalities

A problem may require different types of equation solving. You should know when and how to use a graphing calculator to help you with your work.

Task 2

You shoot an arrow at a target. The parabolic path of your arrow passes through the points shown in the table.

x	y
30	6
60	7
100	4

a. Find a quadratic function in standard form that models the path of your arrow. *Hint:* The three points are (x, y)-values that satisfy $y = ax^2 + bx + c$.

b. If the y-value represents height above the ground, for what value of x would your arrow hit the ground if you miss the target?

c. If the target bull's-eye is at $x = 100$, at what height should the bull's-eye be for your arrow to hit it?

d. If the target bull's-eye is at height $y = 2.98$, at what value of x should the bull's-eye be for the arrow to hit it?

4 Chapter Review

Connecting BIG ideas and Answering the Essential Questions

1 Equivalence
Vertex form of a quadratic function shows the vertex of the parabola. Standard form is "calculator ready." Both forms give additional information.

2 Function
Any quadratic function is possibly a stretch or compression, a reflection, and a translation of $y = x^2$.

3 Solving Equations and Inequalities
The real solutions of a quadratic equation show the zeros of the related quadratic function and the x-intercepts of its graph.

The Different Forms of a Quadratic Function (Lessons 4-1 and 4-2)

$y = 2(x - 1)^2 + 3$ has vertex $(1, 3)$ and opens upward $(2 > 0)$.

$y = -2x^2 + 4x + 1$ has vertex with x-coordinate $-\frac{4}{2(-2)} = 1$ and opens downward $(-2 < 0)$.

Each has axis of symmetry $x = 1$.
Each is a stretch of $y = x^2$ by the factor 2.

Helpful Aids for Solving Quadratic Equations (Lessons 4-4, 4-6, 4-8)

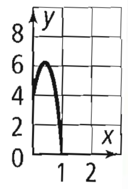
Factor a quadratic: $-16x^2 + 12x + 4$
$= -4(4x + 1)(x - 1)$

Complete the square: $x^2 + 4x + 1$
$= x^2 + 4x + \left(\frac{4}{2}\right)^2 + 1 - \left(\frac{4}{2}\right)^2$
$= (x + 2)^2 - 3$

Complex numbers: $x^2 + 1 = (x + i)(x - i)$
where $i = \sqrt{-1}$.

Modeling With Quadratics (Lessons 4-3 and 4-9)

$y = -16x^2 + 12x + 4$ can model the height y in feet reached by the coin tossed by the referee before the game. x represents time in seconds.



Solving Quadratic Equations (Lessons 4-5, 4-7)

$-16x^2 + 12x + 4 = 0 \rightarrow$
$-4(4x + 1)(x - 1) = 0 \rightarrow$
$x = -\frac{1}{4}$ or $x = 1$.

$-2x^2 + 4x + 1 = 0 \rightarrow$
$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(1)}}{2(-2)} \rightarrow$
$x = 1 + \frac{\sqrt{6}}{2}$ or $x = 1 - \frac{\sqrt{6}}{2}$.

Chapter Vocabulary

- absolute value of a complex number (p. 249)
- axis of symmetry (p. 194)
- completing the square (p. 235)
- complex conjugate (p. 251)
- complex number (p. 249)
- complex number plane (p. 249)
- difference of two squares (p. 220)
- discriminant (p. 242)
- factoring (p. 216)
- greatest common factor (p. 218)
- imaginary number (p. 249)
- imaginary unit (p. 248)
- maximum value (p. 195)
- minimum value (p. 195)
- parabola (p. 194)
- perfect square trinomial (p. 219)
- pure imaginary number (p. 249)
- quadratic formula (p. 240)
- quadratic function (p. 194)
- standard form (p. 202)
- vertex form (p. 194)
- vertex of the parabola (p. 194)
- zero of a function (p. 226)
- zero product property (p. 226)

Choose the correct term to complete each sentence.

1. To solve an equation by factoring, the equation should first be written in (standard form/vertex form).

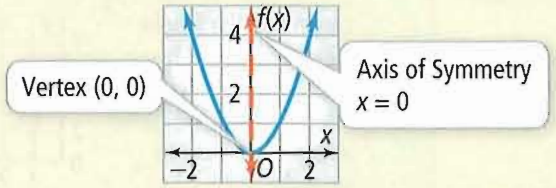
2. The value of $b^2 - 4ac$ for the equation $ax^2 + bx + c = 0$ is called the (discriminant/difference of two squares).

3. The number $a + bi$, where $b = 0$, is an example of a(n)(imaginary/complex) number.

4-1 Quadratic Functions and Transformations

Quick Review

You can write every **quadratic function** in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. A **parabola** is the graph of a quadratic function. Every parabola has a vertex and an axis of symmetry. Shown below is the graph of the quadratic parent function $f(x) = x^2$.



The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The vertex of the parabola formed by a quadratic function is (h, k).

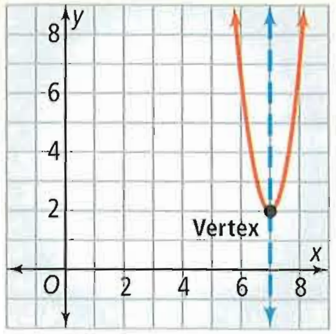
If $a > 0$, k is the **minimum value** of the function.

If $a < 0$, k is the **maximum value** of the function. The axis of symmetry is given by $x = h$.

Example

What is the vertex, axis of symmetry, maximum or minimum, and domain and range of the function $f(x) = 5(x - 7)^2 + 2$?

$a = 5, h = 7, k = 2$	Identify a, h, and k.
vertex: (7, 2)	Find the vertex: (h, k).
axis of symmetry: $x = 7$	The axis of symmetry is at $x = h$.
$k = 2$ is a minimum	Since $a > 0$, k is a minimum.
domain: all real numbers	There are no restrictions on x.
range: $y \geq 2$.	Since the minimum is 2, $y \geq 2$.



Exercises

Identify the vertex, axis of symmetry, maximum or minimum, and domain and range of each function.

4. $f(x) = 4(x + 2)^2 - 6$

5. $f(x) = -(x - 3)^2 + 2$

6. $f(x) = 10(x - 1)^2 + 5$

7. $f(x) = 2(x + 9)^2 - 4$

Graph each function. Describe each transformation of the parent function $f(x) = x^2$.

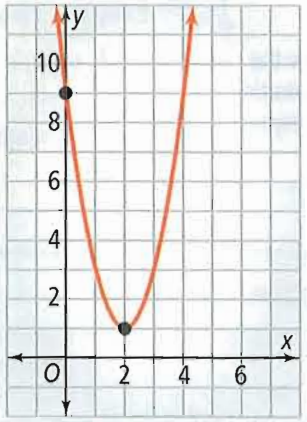
8. $f(x) = x^2 + 4$

9. $f(x) = (x - 9)^2 + 2$

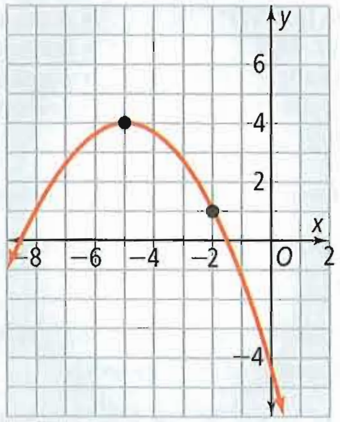
10. $f(x) = \frac{1}{2}(x + 1)^2 - 5$

Write the equation of each parabola in vertex form.

11.



12.



4-2 Standard Form of a Quadratic Function

Quick Review

The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$. When $a > 0$, the parabola opens up. When $a < 0$, the parabola opens down.

The axis of symmetry is the line $x = -\frac{b}{2a}$. The vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, and the y-intercept is $(0, c)$.

Example

What are the vertex, the axis of symmetry and y-intercept of the graph of the function $f(x) = x^2 - 6x + 8$?

axis of symmetry: $x = -\left(\frac{-6}{2(1)}\right) = 3$

vertex: $(3, -1)$

y-intercept: $(0, 8)$

Exercises

Graph each function.

13. $f(x) = x^2 + 6x + 5$

14. $f(x) = x^2 - 7x - 18$

15. $f(x) = x^2 - 7x + 12$

16. $f(x) = x^2 - 9$

Write each function in vertex form.

17. $f(x) = 4x^2 - 8x + 2$

18. $f(x) = x^2 - 8x + 12$

19. $f(x) = 8x^2 + 8x - 12$

20. $f(x) = -2x^2 - 6x + 10$

21. Physics The equation $h = -16t^2 + 32t + 9$ gives the height of a ball, h, in feet above the ground, at t seconds after the ball is thrown upward. How many seconds after the ball is thrown will it reach its maximum height? What is its maximum height?

4-3 Modeling With Quadratic Functions

Quick Review

You can use quadratic functions to model real world data. You can find a quadratic function to model data that passes through any three non-collinear points given that no two of the points lie on a vertical line.

Example

Find the equation of the parabola that passes through the points $(-2, 8)$, $(0, -2)$, and $(1, 2)$.

$y = ax^2 + bx + c$

Use the standard form of a quadratic function.

$$\begin{cases} 8 = a(-2)^2 + b(-2) + c \\ -2 = a(0)^2 + b(0) + c \\ 2 = a(1)^2 + b(1) + c \end{cases}$$

Substitute the (x, y) values to write a system of equations.

$$\begin{cases} 4a - 2b + c = 8 \\ c = -2 \\ a + b + c = 2 \end{cases}$$

$a = 3, b = 1, c = -2$

$y = 3x^2 + x - 2$

Solve the system of equations. Substitute a, b, and c to find the quadratic function.

Exercises

Find the equation of the parabola that passes through each set of points.

22. $(0, 5)$, $(2, -3)$, $(-1, 12)$

23. $(2, 0)$, $(3, -2)$, $(1, -2)$

24. $(4, 10)$, $(0, -18)$, $(-2, -20)$

25. $(0, -7)$, $(7, -14)$, $(-3, -19)$

26. Track and Field The table shows the height of a javelin as it is thrown and travels across a horizontal distance. Use your calculator to find a quadratic model to represent the path of the javelin.

Distance (m)	Height (m)
5	2
18	5
33	8
55	6
68	4
74	3

4-4 Factoring Quadratic Expressions

Quick Review

To factor an expression of the form $ax^2 + bx + c$, when $a \neq 1$, you find numbers that have the product ac and sum b. You can also factor an expression using the FOIL method in reverse or by finding the greatest common factor (GCF).

Example

Factor the expression $5x^2 + 13x + 6$.

$ac = (5)(6) = 30$	Find *ac*.
$30 = 1 \cdot 30 = 2 \cdot 15 = 3 \cdot 10 = 5 \cdot 6$	Find the factors of *ac*.
$b = 13 = 3 + 10$	Find two factors that sum to *b*.
$5x^2 + 10x + 3x + 6$	Rewrite *bx*.
$5x(x + 2) + 3(x + 2)$	Find the common factors.
$(5x + 3)(x + 2)$	Rewrite using the Distributive Property.

Exercises

Factor each expression.

27. $x^2 - 8x + 12$

28. $3x^2 + 11x - 20$

29. $-4x^2 + 14x - 6$

30. $x^2 + 14x + 40$

Factor each perfect square trinomial.

31. $x^2 - 14x + 49$

32. $9x^2 + 30x + 25$

Factor each difference of two squares.

33. $36x^2 - 16$

34. $25x^2 - 4$

Find the GCF of each expression. Then factor each expression.

35. $6x^2 - 24x$

36. $-14x^2 - 49$

4-5 Solving Quadratic Equations

Quick Review

The **zeros** of a quadratic function are the solutions of the related quadratic equation. You can find the zeros from a table or from the x-intercepts of the parabola that is the graph of the function. You can also find them by **factoring** the **standard form of a quadratic equation**, $ax^2 + bx + c = 0$, and using the **Zero-Product Property**.

Example

Solve $2x^2 + 6x = 8$ by factoring.

$2x^2 + 6x - 8 = 0$	Rewrite the equation in standard form.
$2(x^2 + 3x - 4) = 0$	Factor out the GCF, 2.
$2(x + 4)(x - 1) = 0$	Factor the quadratic expression.
$2(x + 4) = 0$ or $x - 1 = 0$	Use the Zero-Product Property.
$x = -4$ or $x = 1$	Solve.

Exercises

Solve each equation by factoring.

37. $x^2 = 4x + 12$

38. $2x^2 - 3x - 14 = 0$

39. $x^2 + 2x = 8$

40. $x^2 + 7x = 18$

Solve each equation by graphing.

41. $5x^2 + 8x - 13 = 0$

42. $9 - 4x = 2x^2$

43. $x^2 - x = 1$

44. $x^2 - 2x - 4 = 0$

Solve each equation by using a table.

45. $x^2 - 6x + 8 = 0$

46. $9x - 14 = 3x^2$

47. $x^2 - 5x + 2 = 0$

48. $2x^2 - 12x = -16$

4-6 Completing the Square

Quick Review

If you cannot solve a quadratic equation by factoring, you can use **completing the square.** You write one side as a perfect square trinomial and then take square roots. You can also convert a quadratic function from standard form to vertex form by completing the square.

Example

Solve $x^2 + 6x - 7 = 0$ by completing the square.

$x^2 + 6x = 7$	Rewrite the equation so the constant is by itself.
$\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$	Find $\left(\frac{b}{2}\right)^2$.
$x^2 + 6x + 9 = 7 + 9$	Add $\left(\frac{b}{2}\right)^2$ to each side.
$(x + 3)^2 = 16$	Factor and simplify.
$x + 3 = \pm 4$	Take the square root of each side.
$x = 1$ or $x = -7$	Solve for x.

Exercises

Solve each equation by finding square roots.

49. $4x^2 = 16$

50. $4x^2 - 20 = 0$

51. $5x^2 - 45 = 0$

52. $3x^2 = 36$

What values complete each square?

53. $x^2 - 6x$

54. $x^2 + 3x$

Solve each equation by completing the square.

55. $x^2 + 8x + 6 = 0$

56. $x^2 - 10x = 13$

57. $9x^2 + 6x + 1 = 4$

58. $x^2 - 2x + 4 = 0$

59. $x^2 + 3x = -25$

60. $4x^2 - x - 3 = 0$

4-7 The Quadratic Formula

Quick Review

You can solve a quadratic equation in the form $ax^2 + bx + c = 0$ by using the **Quadratic Formula,** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The **discriminant** of a quadratic equation in standard form is the value of the expression $b^2 - 4ac$. You can use it to find the quantity and type of solutions of a quadratic equation.

Example

Use the Quadratic Formula to solve $2x^2 - 6x = -3$.

$2x^2 - 6x + 3 = 0$	Write the equation in standard form.
$a = 2, b = -6, c = 3$	Identify a, b, and c.
$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(3)}}{2(2)}$	Substitute a, b, and c into the quadratic formula.
$x = \frac{6 \pm \sqrt{12}}{4} = \frac{3 \pm \sqrt{3}}{2}$	Simplify.

Exercises

Solve each equation using the quadratic formula.

61. $3x^2 + 5x = 8$

62. $x^2 = 6x - 9$

63. $x(x - 3) = 4$

64. $5x^2 - 7x - 3 = 0$

Determine the discriminant of each equation. How many real solutions does each equation have?

65. $4x^2 - 2x = 10$

66. $x^2 - 5x + 7 = 0$

67. $3x^2 + 3 = 6x$

68. $7 - 3x = 8x^2$

69. Gardening Margaret is planning a rectangular garden. Its length is 4 ft less than twice its width. Its area is 170 ft^2. What are the dimensions of the garden?

4-8 Complex Numbers

Quick Review

A **complex number** is written in the form $a + bi$, where a and b are real numbers, and i is equal to $\sqrt{-1}$. If $b = 0$, $a + bi$ is a real number. If $b \neq 0$, $a + bi$ is an **imaginary number**. You can use the Quadratic Formula or completing the square to find the imaginary solutions of quadratic equations.

Example

Use the Quadratic Formula to solve $3x^2 - 4x + 2 = 0$.

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(2)}}{2(3)}$	Enter *a*, *b*, and *c* into the quadratic formula.
$x = \frac{4 \pm \sqrt{16 - 24}}{6} = \frac{4 \pm \sqrt{-8}}{6}$	Simplify.
$x = \frac{2}{3} \pm \frac{\sqrt{2}}{3}i$	Write the solutions.

Exercises

Simplify each expression using the imaginary unit i.

70. $\sqrt{-24}$

71. $\sqrt{-2} - 3$

72. $(4 + \sqrt{-25})(\sqrt{-100})$

73. $2\sqrt{-24} + 6$

Simplify each expression.

74. $(9 + 7i) - (6 - 2i)$

75. $(3 + 11i) + (10 + 9i)$

76. $(1 - 9i)(3 + 2i)$

77. $(3i)^2 - 3(1 + 5i)$

78. $\frac{4 - 6i}{2i}$

79. $\frac{2 - 3i}{1 + 5i}$

Solve each equation.

80. $x^2 + 9 = 0$

81. $5x^2 - 2x + 1 = 0$

82. $-x^2 + 4x = 10$

83. $7x^2 + 8x = -6$

4-9 Quadratic Systems

Quick Review

A system of quadratic equations can be solved by substitution or by graphing. You can use these methods to solve a linear-quadratic system or a quadratic-quadratic system. Use graphing to solve a quadratic system of inequalities.

Example

Use substitution to solve $\begin{cases} y = 2x^2 + 2x - 10 \\ y = x^2 + 5x - 6 \end{cases}$.

$2x^2 + 2x - 10 = x^2 + 5x - 6$	Substitute for *y*.
$x^2 - 3x - 4 = 0$	Rewrite in standard form.
$(x + 1)(x - 4) = 0$	Factor.
$x = -1$ or $x = 4$	Solve for *x*.
$y = (-1)^2 + 5(-1) - 6 = -10$	Substitute for *x* then solve for *y*.
$y = (4)^2 + 5(4) - 6 = 30$	
$(-1, -10)$ and $(4, 30)$	Write solutions as ordered pairs.

Exercises

Solve each system by substitution.

84. $\begin{cases} y = x^2 - 7x - 6 \\ y = 8 - 2x \end{cases}$

85. $\begin{cases} y = -x^2 - 2x + 8 \\ y = x^2 - 8x - 12 \end{cases}$

Solve each system by graphing.

86. $\begin{cases} y = -x^2 - 10x + 12 \\ y = x^2 - 6x - 18 \end{cases}$

87. $\begin{cases} y = x^2 - x - 18 \\ y = 2x + 3 \end{cases}$

Solve each system of inequalities.

88. $\begin{cases} y < x + 4 \\ y \geq x^2 + 2x + 2 \end{cases}$

89. $\begin{cases} y > 3x^2 - 10x - 8 \\ y < x^2 - 5x + 4 \end{cases}$

4 Chapter Test

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

Sketch a graph of the quadratic function with the given vertex and through the given point. Then write the equation of the parabola in vertex form and describe how the function was transformed from the parent function $y = x^2$.

1. vertex (0, 0), point (−3, 3)

2. vertex (1, 5), point (2, 1)

Graph each quadratic function. Identify the axis of symmetry, the vertex, and the domain and the range of each function.

3. $y = x^2 - 7$

4. $y = x^2 + 2x + 6$

5. $y = -x^2 + 5x - 3$

Simplify each expression.

6. $\sqrt{-16}$

7. $4\sqrt{-9} - 2$

8. $(2 + 3i)(8 - 5i)$

9. $(-3 + 2i) - (6 + i)$

10. $\frac{4 + 2i}{2 - i}$

Factor each expression completely.

11. $2y^2 - 8y$

12. $3x^2 + 8x - 3$

13. $9w^2 - 30w + 25$

Solve each quadratic equation.

14. $x^2 - 25 = 0$

15. $x^2 - 2x + 3 = 0$

16. $x^2 - 8x = -6$

Solve the following systems of equations.

17. $\begin{cases} y = 3x^2 - x + 1 \\ y = 3x^2 + x - 1 \end{cases}$

18. $\begin{cases} y = -x^2 + 2x - 3 \\ y = 4x - 3 \end{cases}$

Solve the following systems of inequalities.

19. $\begin{cases} y > 2x^2 + 5x + 1 \\ y < -2x^2 - 5x - 1 \end{cases}$

20. $\begin{cases} y < x^2 - x + 2 \\ y > x^2 - 1 \end{cases}$

Evaluate the discriminant of each equation. How many real and imaginary solutions does each have?

21. $x^2 + 6x - 7 = 0$

22. $3x^2 - x + 3 = 0$

23. $-4x^2 - 4x + 1 = 0$

Do you UNDERSTAND?

24. Writing Compare graphing a number on the complex plane to graphing a point on the coordinate plane. How are they similar? How are they different?

25. Open-Ended Sketch the graph of a quadratic function $f(x) = ax^2 + bx + c$ that has no real zeros. How does this relate to the solutions of the related equation $ax^2 + bx + c = 0$?

26. Physics A model for the path of a toy rocket is given by $h = 68t - 4.9t^2$, where h is the altitude in meters and t is the time in seconds. Explain how to find both the maximum altitude of the rocket and how long it takes to reach that altitude.

27. How many solutions are possible for:
- **a.** a system of two quadratic equations?
- **b.** a system of two quadratic inequalities?

Explain your answers.

4 Sunshine State Standards Practice

TIPS FOR SUCCESS

Some questions on tests require that you model a word problem with a quadratic function.

Roy has a 400 foot roll of wire. He wants to use it to fence in a rectangular area. What is the maximum area of the enclosed space?

Ⓐ 20,000 square feet

Ⓑ 10,000 square feet

Ⓒ 200 square feet

Ⓓ 100 square feet

TIP 1

To identify the function, use what you already know. You know that the perimeter of a rectangle is $2(\ell + w)$ and the area is $\ell \cdot w$.

TIP 2

Use the information from the problem. The perimeter is 400, so $2(\ell + w) = 400$. Solve for w: $w = 200 - \ell$.

Think It Through

Substitute for w in the area formula:

$$\begin{aligned} A = f(\ell) &= \ell \cdot (200 - \ell) \\ &= -\ell^2 + 200\ell. \end{aligned}$$

The maximum value is the y-coordinate of the vertex, $f\left(-\frac{b}{2a}\right) = f(100) = 10{,}000$. So, the maximum area Roy can enclose is 10,000 square feet.

The correct answer is B.

Vocabulary Builder

As you solve test items, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. axis of symmetry

B. discriminant

C. imaginary number

D. Zero-Product Property

E. parabola

F. perfect square trinomial

G. completing the square

I. value of $b^2 - 4ac$ for the equation $ax^2 + bx + c = 0$

II. If $ab = 0$, then $a = 0$ or $b = 0$.

III. line that divides a parabola into two parts that are mirror images

IV. $a + bi$, a and b are real numbers and $b \neq 0$

V. square of a binomial

VI. process of finding the last term to make a perfect square trinomial

VII. graph of a quadratic function

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

1. Which equation is equivalent to $x^2 + 24x + 100 = -46$?

Ⓐ $(x + 12)^2 = -2$

Ⓑ $(x - 12)^2 = -2$

Ⓒ $(x - 12)^2 = 2$

Ⓓ $(x + 12)^2 = 2$

2. What is the solution of the following system of equations?

$$\begin{cases} x + y + z = 13 \\ 2x - y = 4 \\ x + z = -3 \end{cases}$$

Ⓕ $x = 10, y = 16, z = 13$

Ⓖ $x = 16, y = 28, z = -19$

Ⓗ $x = 10, y = 16, z = -13$

Ⓘ $x = -16, y = -28, z = 19$

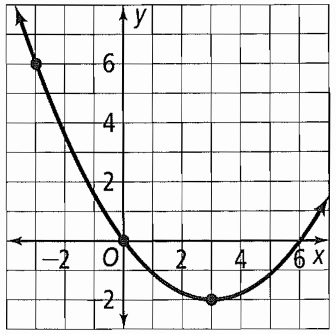
3. What are the factors of the quadratic function graphed below?

Ⓐ $(x + 3)$ and $(x + 2)$

Ⓑ x and $(x - 6)$

Ⓒ x and $(x + 6)$

Ⓓ $(x - 3)$ and $(x + 2)$



4. Which equation has $-1 \pm i$ as its solution?

Ⓕ $x^2 - 2x - 2 = 0$

Ⓖ $2x^2 - 2x - 1 = 0$

Ⓗ $2x^2 + 2x + 1 = 0$

Ⓘ $x^2 + 2x + 2 = 0$

5. Which number is equivalent to $\sqrt{-169} - 64$?

Ⓐ $8 - 13i$

Ⓑ $-64 + 13i$

Ⓒ $-64 - 13i$

Ⓓ $64 - 13i$

6. The graph of which quadratic function includes the points $(-5, 0)$ and $(-1, 0)$?

Ⓕ $y = (x + 5)^2 - 1$

Ⓖ $y = x^2 + 6x + 5$

Ⓗ $y = x^2 - 6x + 5$

Ⓘ $y = (x - 5)^2 - 1$

7. What is the transformation of the graph of $y = (x + 3)^2 - 2$ from its parent function $y = x^2$?

Ⓐ 3 units left and 2 units down

Ⓑ 3 units right and 2 units up

Ⓒ 6 units right and 2 units up

Ⓓ 2 units left and 3 units down

8. What is the axis of symmetry for the graph of the quadratic equation $y = -3x^2 - 12 + 12x$?

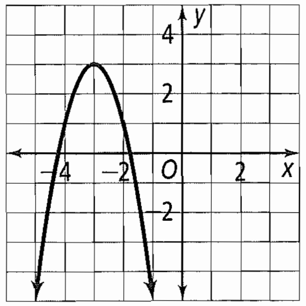
Ⓕ $x = -2$

Ⓖ $x = 2$

Ⓗ $x = 12$

Ⓘ $x = -12$

9. What are the domain and range of the function graphed below?



Ⓐ Domain: All real numbers
Range:
All real numbers ≤ 3

Ⓑ Domain: All real numbers
Range:
All real numbers ≥ 3

Ⓒ Domain: All real numbers between −5 and −1
Range: All real numbers ≤ 3

Ⓓ Domain: All real numbers between −5 and −1
Range: All real numbers ≥ 3

10. Which equation DOES NOT represent a direct variation?

Ⓕ $x - 5y = 0$

Ⓖ $\frac{y}{x} = \frac{4}{3}$

Ⓗ $y + 5 = 3x - 5$

Ⓘ $x = \frac{y}{-5}$

11. What is the solution of $\begin{cases} -y = 3x - 1 \\ 2y = -x - 2 \end{cases}$?

Ⓐ $x = 20, y = -11$

Ⓑ $x = \frac{4}{5}, y = -\frac{7}{5}$

Ⓒ $x = -20, y = 11$

Ⓓ $x = -\frac{4}{5}, y = \frac{7}{5}$

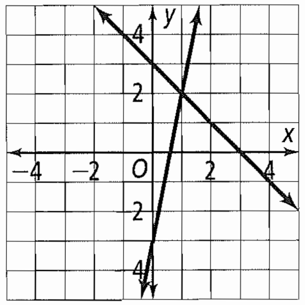
12. Which system of equations is graphed below?

Ⓕ $\begin{cases} y - 3 = 5x \\ y - x = 3 \end{cases}$

Ⓖ $\begin{cases} y + 3 = 5x \\ y + x = 3 \end{cases}$

Ⓗ $\begin{cases} -y + 3 = -5x \\ y + x = -3 \end{cases}$

Ⓘ $\begin{cases} y + 3 = -5x \\ -y + x = 3 \end{cases}$



13. What is the vertex of $y = -2|x + 4| - 5$?

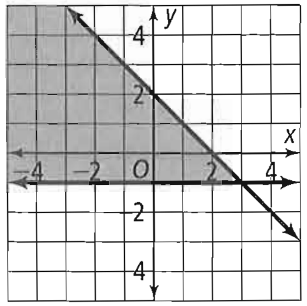
Ⓐ $(-2, -5)$

Ⓑ $(-4, -5)$

Ⓒ $(4, -5)$

Ⓓ $(2, -5)$

14. Which system of inequalities is graphed below?



(F) $\begin{cases} y \leq -1 \\ y + x \geq 2 \end{cases}$

(G) $\begin{cases} y \geq -1 \\ y + x \leq 2 \end{cases}$

(H) $\begin{cases} y < 1 \\ y + x > -2 \end{cases}$

(I) $\begin{cases} y > 1 \\ y + x < -2 \end{cases}$

15. What are the solutions of $|3x - 5| = 2$?

(A) $x = -1$ and $x = \frac{7}{3}$ (C) $x = 1$ and $x = \frac{1}{5}$

(B) $x = 1$ and $x = \frac{7}{3}$ (D) $x = -1$ and $x = \frac{1}{5}$

16. The formula for the total surface area of a regular right pentagonal prism is $A = ap + pH$. Solve this equation for p.

(F) $p = \frac{a + H}{A}$ (H) $p = A - \frac{a}{H}$

(G) $p = \frac{A}{a + H}$ (I) $p = \frac{H - a}{A}$

17. How could you graph $y + 4 < 2|x - 3|$ on a coordinate grid?

(A) Stretch the graph of $y < |x|$ vertically by the factor 2, then translate 3 units to the left and 4 units up.

(B) Translate the graph of $y < 2|x|$ by 3 units to the left and 4 units down.

(C) Stretch the graph of $y < |x|$ vertically by the factor 2, then translate 3 units to the right and 4 units up.

(D) Translate the graph of $y < 2|x|$ by 3 units to the right and 4 units down.

18. A hat company is designing a one-size-fits-all hat with a strap in the back that makes the hat smaller or larger. Head sizes normally range from 51 to 64 centimeters. What absolute value inequality best models the different sizes of the hat?

(F) $-6.5 \leq |s - 57.5| \leq 6.5$

(G) $|s - 57.5| \leq 6.5$

(H) $51 \leq |s| \leq 64$

(I) $|s| \leq 13$

19. Robby decided to earn extra money by making and selling brownies and cookies. He had space in his oven to make at most 80 brownies and cookies. Each brownie cost \$.10 to make and each cookie cost \$.05 to make. He had \$6 to spend in all. If Robby makes a profit of \$.25 on each brownie and \$.20 on each cookie, how many of each dessert could he make and sell to maximize his profit?

(A) 42 brownies, 42 cookies

(B) 38 brownies, 42 cookies

(C) 42 brownies, 38 cookies

(D) 40 brownies, 40 cookies

GRIDDED RESPONSE

20. What is the discriminant of the equation $1.5x^2 - 2.5x - 1.5 = 0$?

21. Find the positive value of k that would make the expression $4x^2 + kx + 4$ a perfect square trinomial.

22. When $y = -3x^2 - 18x - 23$ is written in vertex form $y = a(x - h)^2 + k$, what is the value of k?

23. What is the value of y in the solution to the system of equations $\begin{cases} 2y = x - 2 \\ y - x = -3 \end{cases}$?

24. Line A is perpendicular to $x + 3y = 5$ and passes through the point $(-1, 1)$. What is the y-intercept of Line A?

25. What is the slope of the line that passes through $(5, 1)$ and $(-3, -2)$? Express answer as a fraction.

26. What is the sum of the zeros of $f(x) = x^2 - 2x - 8$?

27. A piggy bank contains \$2.40 in nickels and dimes. If there are 33 coins in all, how many nickels are there?

28. What is the slope of the line parallel to $3y - 7x = 15$? Express answer as a fraction.

29. Claudia has a rectangular flowerbed. She decided that the original width w, in feet, was too small, so she increased the width by 3 feet. She also changed the length to be 1 foot less than twice the original width. The new area of her flowerbed is 72 square feet. How many feet wide was the original flowerbed?

Get Ready!

Lesson 4-2

Graphing Quadratic Functions

Graph each function.

1. $f(x) = x^2 - 8x + 7$ **2.** $f(x) = -\frac{1}{2}x^2 - 4x - 4$ **3.** $f(x) = x^2 + 4x + 4$

Lesson 4-3

Writing Equations of Parabolas

Write in standard form the equation of the parabola passing through the given points.

4. $(-1, -6), (-3, -4), (2, 6)$ **5.** $(3, 4), (-2, 9), (2, 1)$ **6.** $(-5, -8), (4, -8), (-3, 6)$

Lesson 4-5

Solving Quadratic Equations by Graphing

Solve each equation by graphing. Round to the nearest hundredth.

7. $1 = 4x^2 + 3x$ **8.** $\frac{1}{2}x^2 + x - 14 = 0$ **9.** $5x^2 + 30x = 12$

Lesson 4-5

Solving Quadratic Equations by Factoring

Solve each equation by factoring.

10. $x^2 - x - 20 = 0$ **11.** $x^2 + 6x - 27 = 0$ **12.** $3x^2 - 9x + 6 = 0$

Lesson 4-7

Finding the Number and Type of Solutions

Evaluate the discriminant of each equation. Tell how many solutions each equation has and whether the solutions are real or imaginary.

13. $x^2 - 12x + 30 = 0$ **14.** $-4x^2 + 20x - 25 = 0$ **15.** $2x^2 = 8x - 8$

Looking Ahead Vocabulary

16. A *turning point* is a place where a graph changes direction. Suppose you start hiking north on a winding trail, and the trail makes a turn and heads south, and then north again. If you make a total of 3 of these 180 degree turns, in which direction will you be hiking after the last turn?

17. A *relative maximum* is the greatest value in a region. The highest point in Maine is Mt. Katahdin at 5267 ft. How might that compare to the highest point in the United States? What might the relative maximum of a graph be?

18. A contraction is a shortened form of a word or phrase. The expanded form of the contraction "don't" is "do not." You can *expand* a math phrase by multiplying it out. For example, $(x - 2)^2 = (x - 2)(x - 2) = x^2 - 4x + 4$. Expand $(2x + 1)^2$.

CHAPTER

5

Polynomials and Polynomial Functions

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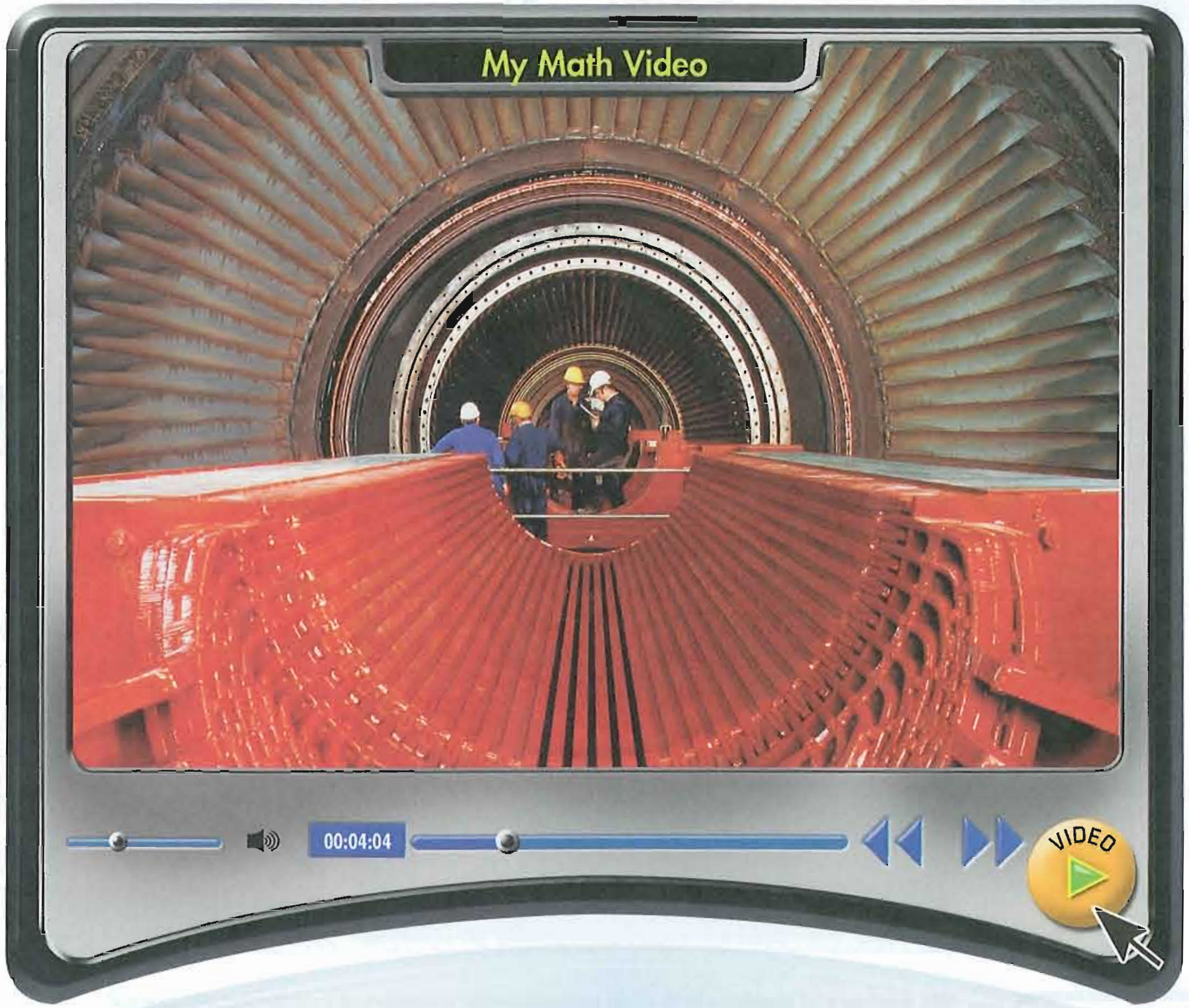
Polynomial functions are used to model all kinds of real-world situations, like the energy produced by a turbine.

In this chapter, you will also learn theorems that will help you when working with polynomial functions and equations.

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
end behavior, *p. 282*	comportamiento extremo
monomial, *p. 280*	monomio
multiplicity, *p. 291*	multiplicidad
Pascal's Triangle, *p. 327*	Triángulo de Pascal
polynomial function, *p. 280*	función polinomial
relative maximum, *p. 291*	máximo relativo
relative minimum, *p. 291*	mínimo relativo
standard form of a polynomial function, *p. 281*	forma normal de una función polinomial
synthetic division, *p. 306*	división sintética
turning point, *p. 282*	punto de giro



BIG ideas

1 Function

Essential Question What does the degree of a polynomial tell you about its related polynomial function?

2 Equivalence

Essential Question For a polynomial function, how are factors, zeros, and *x*-intercepts related?

3 Solving Equations and Inequalities

Essential Question For a polynomial equation, how are factors and roots related?

Chapter Preview

5-1 Polynomial Functions

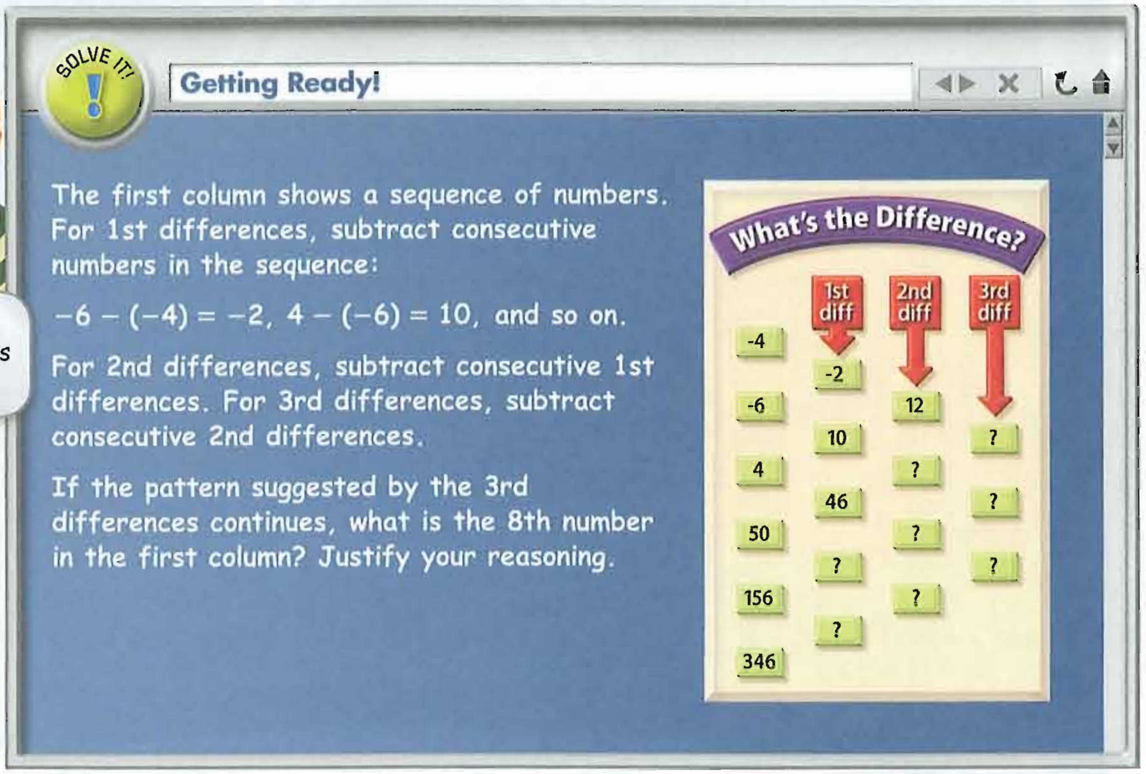
Sunshine State Standards

MA.912.A.2.6 Identify and graph quadratic, cubic, and other polynomial functions.

MA.912.A.4.5 Graph polynomial functions with and without technology and describe end behavior.

Objectives To classify polynomials

To graph polynomial functions and describe end behavior



Getting Ready!

The first column shows a sequence of numbers. For 1st differences, subtract consecutive numbers in the sequence:

$-6-(-4)=-2$, $4-(-6)=10$, and so on.

For 2nd differences, subtract consecutive 1st differences. For 3rd differences, subtract consecutive 2nd differences.

If the pattern suggested by the 3rd differences continues, what is the 8th number in the first column? Justify your reasoning.

What's the Difference?

	1st diff	2nd diff	3rd diff
-4	-2	12	?
-6	10	?	?
4	46	?	?
50	?	?	
156	?		
346			

Lesson Vocabulary

- monomial
- degree of a monomial
- polynomial
- degree of a polynomial
- polynomial function
- standard form of a polynomial function
- turning point
- end behavior

The sequence of numbers in the first column above are values of a particular *polynomial function.* For such a sequence, you can use patterns of 1st differences, 2nd differences, 3rd differences, and so on, to learn more about the polynomial function.

Essential Understanding A polynomial function has distinguishing "behaviors." You can look at its algebraic form and know something about its graph. You can look at its graph and know something about its algebraic form.

A **monomial** is a real number, a variable, or a product of a real number and one or more variables with whole-number exponents. The **degree of a monomial** in one variable is the exponent of the variable. A **polynomial** is a monomial or a sum of monomials. The **degree of a polynomial** in one variable is the greatest degree among its monomial terms.

A polynomial with the variable x defines a **polynomial function** of x. The degree of the polynomial function is the same as the degree of the polynomial.

take note

Key Concept Standard Form of a Polynomial Function

The **standard form of a polynomial function** arranges the terms by degree in descending numerical order.

A polynomial function $P(x)$ in standard form is

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

where n is a nonnegative integer and $a_n, \ldots, a_0$ are real numbers.

$$P(x) = \underbrace{4x^3}_{\text{Cubic term}} + \underbrace{3x^2}_{\text{Quadratic term}} + \underbrace{5x}_{\text{Linear term}} \underbrace{- 2}_{\text{Constant term}}$$

You can classify a polynomial by its degree or by its number of terms. Polynomials of degrees zero through five have specific names, as shown in this table.

Degree	Name Using Degree	Polynomial Example	Number of Terms	Name Using Number of Terms
0	constant	5	1	monomial
1	linear	$x + 4$	2	binomial
2	quadratic	$4x^2$	1	monomial
3	cubic	$4x^3 - 2x^2 + x$	3	trinomial
4	quartic	$2x^4 + 5x^2$	2	binomial
5	quintic	$-x^5 + 4x^2 + 2x + 1$	4	polynomial of 4 terms

Problem 1 Classifying Polynomials

Write each polynomial in standard form. What is the classification of each polynomial by degree? by number of terms?

Think

How do you write a polynomial in standard form?
Combine like terms if possible. Then, write the terms with their degrees in descending order.

A $3x + 9x^2 + 5$

$9x^2 + 3x + 5$

The polynomial has degree 2 and 3 terms. It is a quadratic trinomial.

B $4x - 6x^2 + x^4 + 10x^2 - 12$

$x^4 + 4x^2 + 4x - 12$

The polynomial has degree 4 and 4 terms. It is a quartic polynomial of 4 terms.

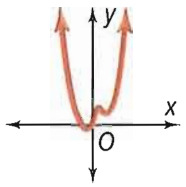
Got It? **1.** Write each polynomial in standard form. What is the classification of each by degree? by number of terms?

a. $3x^3 - x + 5x^4$ **b.** $3 - 4x^5 + 2x^2 + 10$

The degree of a polynomial function affects the shape of its graph and determines the maximum number of **turning points**, or places where the graph changes direction. It also affects the **end behavior**, or the directions of the graph to the far left and to the far right.

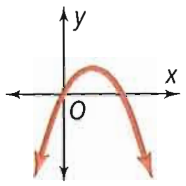
For polynomial functions of degree one or greater, there are four types of end behavior as you move to the left and move to the right, away from the origin: *up and up*, *down and down*, *down and up*, and *up and down*.

Up and Up



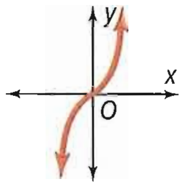
$y = x^4 - 3x^3 + 5x$

Down and Down



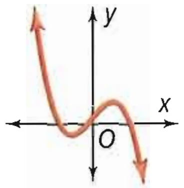
$y = -x^2 + 6x$

Down and Up



$y = x^3$

Up and Down



$y = -0.3x^3 + 4x + 2$

You can determine the end behavior of a polynomial function of degree n from the leading term ax^n of the standard form.

End Behavior of a Polynomial Function of Degree n With Leading Term ax^n (Moving Away From the Origin)

	n Even	n Odd
a Positive	Up and Up	Down and Up
a Negative	Down and Down	Up and Down

Problem 2 Describing End Behavior of Polynomial Functions

Consider the leading term of each polynomial function. What is the end behavior of the graph? Check your answer with a graphing calculator.

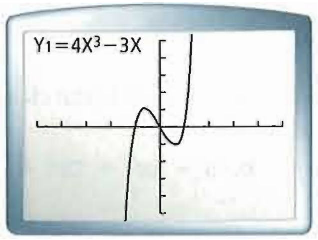
Think

What do a and n represent?
a is the coefficient of the leading term. n is the exponent of the leading term.

A $y = 4x^3 - 3x$

The leading term is $4x^3$. Since n is odd and a is positive, the end behavior is down and up.

Check

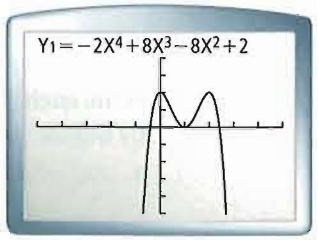


The solution checks.

B $y = -2x^4 + 8x^3 - 8x^2 + 2$

The leading term is $-2x^4$. Since n is even and a is negative, the end behavior is down and down.

Check

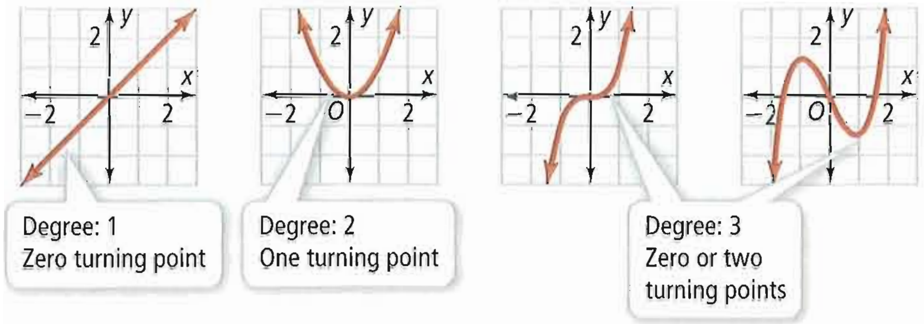


The solution checks.

Got It? **2.** Consider the leading term of $y = -4x^3 + 2x^2 + 7$. What is the end behavior of the graph?

In general, the graph of a polynomial function of degree n ($n \geq 1$) has at most $n - 1$ turning points. The graph of a polynomial function of odd degree has an even number of turning points. The graph of a polynomial function of even degree has an odd number of turning points.

This information, combined with end behavior, determines possible shapes that the graph of a polynomial function can have.



Degree: 1
Zero turning point

Degree: 2
One turning point

Degree: 3
Zero or two turning points

Problem 3 Graphing Cubic Functions

What is the graph of each cubic function? Describe the graph.

Plan

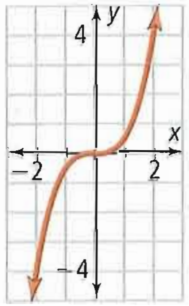
How can you graph a polynomial function? Make a table of values to help you sketch the middle part of the graph. Use what you know about end behavior to sketch the ends of the graph.

A $y = \frac{1}{2}x^3$

Step 1

x	y
−2	−4
−1	−0.5
0	0
1	0.5
2	4

Step 2



Step 3

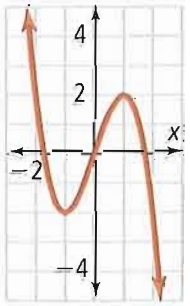
The end behavior is down and up. There are no turning points.

B $y = 3x - x^3$

Step 1

x	y
−2	2
−1	−2
0	0
1	2
2	−2

Step 2



Step 3

The end behavior is up and down. There are two turning points.

Got It? **3.** What is the graph of each cubic function? Describe the graph.

a. $y = -x^3 + 2x^2 - x - 2$

b. $y = x^3 - 1$

Suppose you are given a set of polynomial function outputs. You know that their inputs are an ordered set of x-values in which consecutive x-values differ by a constant. By analyzing the differences of consecutive y-values, it is possible to determine the least-degree polynomial function that could generate the data.

If the first differences are constant, the function is linear. If the second differences (but not the first) are constant, the function is quadratic. If the third differences (but not the second) are constant, the function is cubic, and so on.

Problem 4 Using Differences to Determine Degree

What is the degree of the polynomial function that generates the data shown at the right?

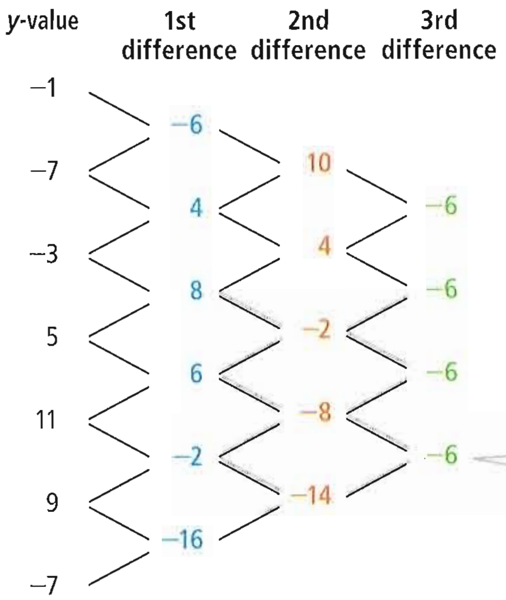
x	y
−3	−1
−2	−7
−1	−3
0	5
1	11
2	9
3	−7

Know
A set of polynomial function values

Need
Degree of the polynomial function

Plan
Check first differences of y-values. Then check second differences, third differences, and so on until they are constant.

Think
How do you find the second differences?
Subtract the consecutive first differences.



The third differences are constant.

The degree of the polynomial function is 3.

Got It? 4. a. What is the degree of the polynomial function that generates the data shown at the right?

b. **Reasoning** What is an example of a polynomial function whose fifth differences are constant but whose fourth differences are not constant?

x	y
−3	23
−2	−16
−1	−15
0	−10
1	−13
2	−12
3	29

Lesson Check

Do you know HOW?

Classify each polynomial by degree and by number of terms.

1. $5x^3$

2. $6x^2 + 4x - 2$

Write each polynomial in standard form.

3. $7x + 3 + 5x^2$

4. $-3 + 9x$

Do you UNDERSTAND?

5. **Vocabulary** Describe the end behavior of the graph of $y = -2x^7 - 8x$.

6. **Reasoning** Can the graph of a polynomial function be a straight line? If so, give an example.

7. **Error Analysis** Your friend claims the graph of the function $y = 4x^3 + 4$ has only one turning point. Describe the error your friend made and give the correct number of turning points.

Practice and Problem-Solving Exercises

A Practice

Write each polynomial in standard form. Then classify it by degree and by number of terms.

See Problem 1.

8. $7x + 3x + 5$

9. $5 - 3x$

10. $2m^2 - 3 + 7m$

11. $-x^3 + x^4 + x$

12. $-4p + 3p + 2p^2$

13. $5a^2 + 3a^3 + 1$

14. $-x^5$

15. $3 + 12x^4$

16. $6x^3 - x^3$

17. $7x^3 - 10x^3 + x^3$

18. $4x + 5x^2 + 8$

19. $x^2 - x^4 + 2x^2$

Determine the end behavior of the graph of each polynomial function.

See Problem 2.

20. $y = -7x^3 + 8x^2 + x$

21. $y = -3x + 6x^2 - 1$

22. $y = 1 - 4x - 6x^3 - 15x^6$

23. $y = 8x^{11} - 2x^9 + 3x^6 + 4$

24. $y = -x^5 - 15x^7 - 4x^9$

25. $y = -3 - 6x^5 - 9x^8$

26. $y = x^4 - 7x^2 + 3$

27. $y = -8x^7 + 16x^6 + 9$

28. $y = -14x^6 + 11x^5 - 11$

29. $y = -x^3 - x^2 + 3$

30. $y = x^3 - 14x - 4$

31. $y = 5 - 17x^7 + 9x^{10}$

Describe the shape of the graph of each cubic function by determining the end behavior and number of turning points.

See Problem 3.

32. $y = 3x^3 - x - 3$

33. $y = -9x^3 - 2x^2 + 5x + 3$

34. $y = 10x^3 + 9$

35. $y = 3x^3$

36. $y = -4x^3 - 5x^2$

37. $y = 8x^3$

Determine the degree of the polynomial function with the given data.

See Problem 4.

38.

x	−2	−1	0	1	2
y	16	7	2	1	4

39.

x	−2	−1	0	1	2
y	−15	−9	−9	−9	−3

B Apply

40. **Think About a Plan** The data shows the power generated by a wind turbine. The x column gives the wind speed in meters per second. The y column gives the power generated in kilowatts. What is the degree of the polynomial function that models the data?

- What are the first differences of the y-values?
- What are the second differences of the y-values?
- When are the differences constant?

x	y
5	10
6	17.28
7	27.44
8	40.96
9	58.32

Classify each polynomial by degree and by number of terms. Simplify first if necessary.

41. $a^2 + a^3 - 4a^4$

42. 7

43. $2x(3x)$

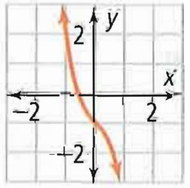
44. $(2a - 5)(a^2 - 1)$

45. $(-8d^3 - 7) + (-d^3 - 6)$

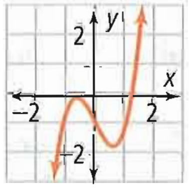
46. $b(b - 3)^2$

Determine the sign of the leading coefficient and the least possible degree of the polynomial function for each graph.

47.



48.



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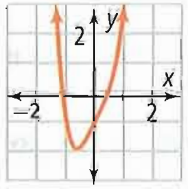


50. **Open-Ended** Write an equation for a polynomial function that has three turning points and end behavior up and up.

51. Show that the third differences of a polynomial function of degree 3 are nonzero and constant. First, use $f(x) = x^3 - 3x^2 - 2x - 6$. Then show third differences are nonzero and constant for $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$.

52. **Reasoning** Suppose that a function pairs elements from set A with elements from set B. A function is called *onto* if it pairs every element in B with at least one element in A. For each type of polynomial function, and for each set B, determine whether the function is *always*, *sometimes*, or *never* onto.

a. linear; $B =$ all real numbers
b. quadratic; $B =$ all real numbers
c. quadratic; $B =$ all real numbers greater than or equal to 4
d. cubic; $B =$ all real numbers

53. Make a table of second differences for each polynomial function. Using your tables, make a conjecture about the second differences of quadratic functions.

a. $y = 2x^2$
b. $y = 5x^2$
c. $y = 5x^2 - 2$
d. $y = 7x^2$
e. $y = 7x^2 + 1$
f. $y = 7x^2 + 3x + 1$

Challenge

54. Copy and complete the table, which shows the first and second differences in y-values for consecutive x-values for a polynomial function of degree 2.

x	y	1st diff.	2nd diff.
−3	14	−8	2
−2	6	■	2
−1	■	−4	2
0	−4	−2	2
1	■	0	2
2	−6	■	
3	■		

55. The outputs for a certain function are 1, 2, 4, 8, 16, 32, and so on.

a. Find the first differences of this function.

b. Find the second differences of this function.

c. Find the tenth difference of this function.

d. Can you find a polynomial function that matches the original outputs? Explain your reasoning.

56. **Reasoning** A cubic polynomial function f has leading coefficient 2 and constant term 7. If $f(1) = 7$ and $f(2) = 9$, what is $f(-2)$? Explain how you found your answer.

Sunshine State Standards Practice

MA.912.A.2.6 **57.** Which expression is a cubic polynomial?

Ⓐ x^3 Ⓑ $3x + 3$ Ⓒ $2x^2 + 3x - 1$ Ⓓ $3x$

MA.912.A.4.7 **58.** Which equation has $-3 \pm 5i$ as its solutions?

Ⓕ $x^2 + 6x = -34$ Ⓖ $x^2 + 6x = -14$ Ⓗ $x^2 + 3x = 4$ Ⓘ $x^2 + 3x = 2$

MA.912.A.7.4 **59.** What is the discriminant of $qx^2 + rx + s = 0$?

Ⓐ qrs Ⓑ $q^2 - 4rs$ Ⓒ $r^2 - 4qs$ Ⓓ $s^2 - 4qr$

MA.912.A.4.2 **60.** **Short Response** What is a simpler form of $x^2(3x^2 - 2x) - 3x^4$? Classify the polynomial by degree and by number of terms.

Mixed Review

Solve each system of equations. See Lesson 4-9.

61. $\begin{cases} y = x^2 - 3x - 7 \\ y = -2x + 3 \end{cases}$ **62.** $\begin{cases} y = x^2 + x - 20 \\ y = x^2 + 2x \end{cases}$ **63.** $\begin{cases} y = 3x^2 + 8x - 3 \\ y = x^2 - 9 \end{cases}$

Write an equation of each line in standard form with integer coefficients. See Lesson 2-4.

64. $y = 7x + 0.4$ **65.** $y = -3x - 2.5$ **66.** $y = -\frac{2}{7}x + 4$ **67.** $y = 1.2x - 0.5$

Get Ready! **To prepare for Lesson 5-2, do Exercises 68–70.**

Factor each quadratic expression. See Lesson 4-4.

68. $x^2 + 7x + 12$ **69.** $x^2 + 8x - 20$ **70.** $x^2 - 14x + 24$

5-2 Polynomials, Linear Factors, and Zeros

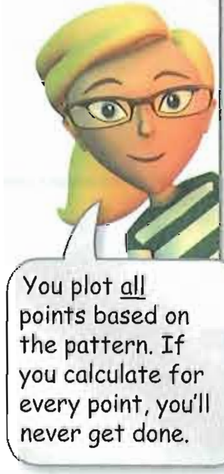
Sunshine State Standards

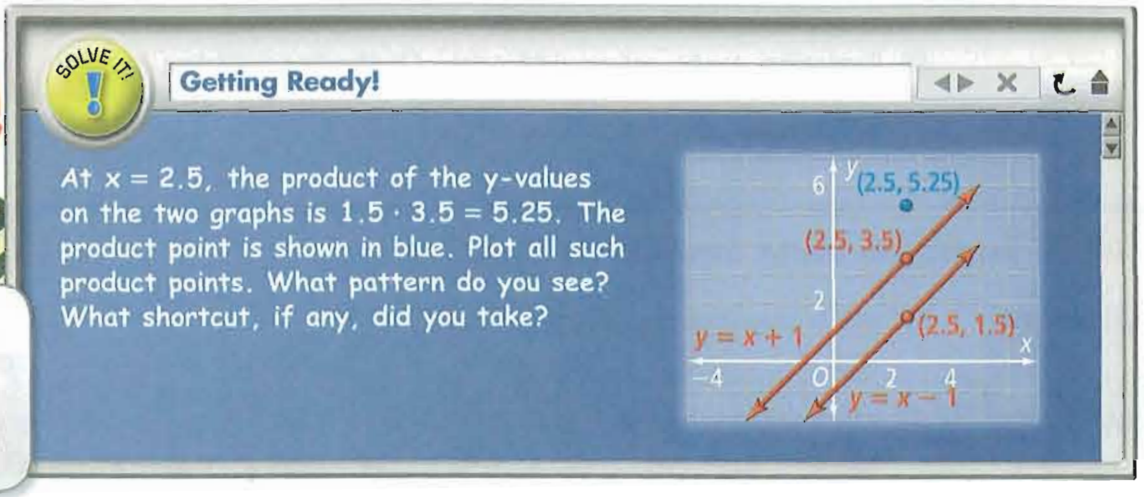
MA.912.A.4.7 Write a polynomial equation for a given set of real roots.

MA.912.A.4.8 Describe the relationships among solutions, zeros, x-intercepts, and factors.

MA.912.A.4.3 Factor polynomial expressions.

Objectives To analyze the factored form of a polynomial
To write a polynomial function from its zeros





Getting Ready!

At $x = 2.5$, the product of the y-values on the two graphs is $1.5 \cdot 3.5 = 5.25$. The product point is shown in blue. Plot all such product points. What pattern do you see? What shortcut, if any, did you take?

Lesson Vocabulary
- factor theorem
- multiple zero
- multiplicity
- relative maximum
- relative minimum

If $P(x)$ is a polynomial function, the solutions of the related polynomial equation $P(x) = 0$ are the zeros of the function.

Essential Understanding Finding the zeros of a polynomial function will help you factor the polynomial, graph the function, and solve the related polynomial equation.

In Chapter 4, you solved a quadratic equation of the form $x^2 + bx + c = 0$ by factoring. You wrote it using *linear factors* in the form $(x - r_1)(x - r_2) = 0$. Then you applied the Zero-Product Property to find the solutions $x = r_1$ and $x = r_2$. You can solve some polynomial equations $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ in much the same way.

Plan

How do you write the factored form of a polynomial?
Write the polynomial as a product of factors. Make sure each factor cannot be factored any further.

Problem 1 Writing a Polynomial in Factored Form

What is the factored form of $x^3 - 2x^2 - 15x$?

$x^3 - 2x^2 - 15x = x(x^2 - 2x - 15)$	Factor out the GCF, x.
$= x(x - 5)(x + 3)$	Factor $x^2 - 2x - 15$.
Check $x(x - 5)(x + 3) = x(x^2 - 2x - 15)$	Multiply $(x - 5)(x + 3)$.
$= x^3 - 2x^2 - 15x$ ✔	Distributive Property

Got It? **1.** What is the factored form of $x^3 - x^2 - 12x$?

take note

Key Concepts Roots, Zeros, and x-intercepts

The following are equivalent statements about a real number b and a polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.

- $x - b$ is a linear factor of the polynomial $P(x)$.
- b is a zero of the polynomial function $y = P(x)$.
- b is a root (or solution) of the polynomial equation $P(x) = 0$.
- b is an x-intercept of the graph of $y = P(x)$.

Dynamic Activity
Polynomials and Linear Factors

Problem 2 Finding Zeros of a Polynomial Function

What are the zeros of $y = (x + 2)(x - 1)(x - 3)$? Graph the function.

Know	Need	Plan
Polynomial function	• Zeros • Additional points • End behavior	• Use the Zero-Product Property to find zeros. • Find points between the zeros. • Sketch the graph.

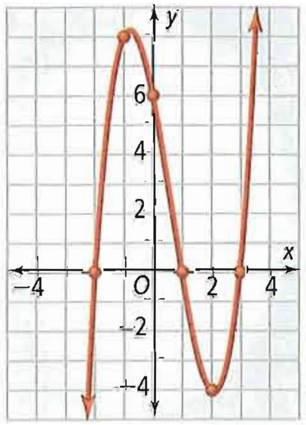
Think

Does knowing the zeros of a function give you enough information to sketch it?
No; several different cubic functions could pass through (−2, 0), (1, 0), and (3,0).

Step 1 Use the Zero-Product Property to find the zeros.

$$(x + 2)(x - 1)(x - 3) = 0$$

so $x + 2 = 0$ or $x - 1 = 0$ or $x - 3 = 0$.
The zeros of the function are -2, 1, and 3.



Step 2 Find points for x-values between the zeros.
Evaluate $y = (x + 2)(x - 1)(x - 3)$ for $x = -1, 0$, and 2.

$(-1 + 2)(-1 - 1)(-1 - 3) = 8$	$(-1, 8)$
$(0 + 2)(0 - 1)(0 - 3) = 6$	$(0, 6)$
$(2 + 2)(2 - 1)(2 - 3) = -4$	$(2, -4)$

Step 3 Determine the end behavior.
The function $y = (x + 2)(x - 1)(x - 3)$ is cubic. The coefficient of x^3 is $+1$, so the end behavior is *down and up*.

Step 4 Use the zeros: $(-2, 0)$, $(1, 0)$, $(3, 0)$; the additional points: $(-1, 8)$, $(0, 6)$, $(2, -4)$; and end behavior to sketch the graph.

Got It? **2.** What are the zeros of $y = x(x - 3)(x + 5)$? Graph the function.

The Factor Theorem describes the relationship between the linear factors of a polynomial and the zeros of a polynomial.

take note

Theorem Factor Theorem

The expression $x - a$ is a factor of a polynomial if and only if the value a is a zero of the related polynomial function.

Problem 3 Writing a Polynomial Function From Its Zeros

Plan

How can you use the zeros to find the function?
By the Factor Theorem, a is a zero means that $x - a$ is a factor of the related polynomial.

A What is a cubic polynomial function in standard form with zeros −2, 2, and 3?

$\begin{matrix} -2 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \end{matrix}$	−2, 2, and 3 are zeros.
$f(x) = (x + 2)(x - 2)(x - 3)$	Write a linear factor for each zero.
$= (x + 2)(x^2 - 5x + 6)$	Multiply $(x-2)$ and $(x-3)$.
$= x(x^2 - 5x + 6) + 2(x^2 - 5x + 6)$	Distributive Property
$= x^3 - 5x^2 + 6x + 2x^2 - 10x + 12$	Distributive Property
$= x^3 - 3x^2 - 4x + 12$	Simplify.

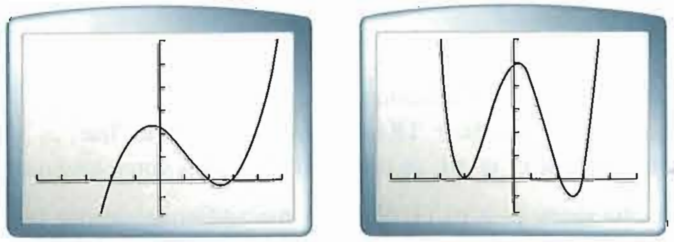
The cubic polynomial $f(x) = x^3 - 3x^2 - 4x + 12$ has zeros −2, 2, and 3.

B What is a quartic polynomial function in standard form with zeros −2, −2, 2, and 3?

$\begin{matrix} -2 & -2 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{matrix}$	−2, −2, 2, and 3 are zeros.
$g(x) = (x + 2)(x + 2)(x - 2)(x - 3)$	Write a linear factor for each zero.
$= x^4 - x^3 - 10x^2 + 4x + 24$	Simplify.

The quartic polynomial $g(x) = x^4 - x^3 - 10x^2 + 4x + 24$ has zeros −2, −2, 2, and 3.

C Graph both functions. How do the graphs differ? How are they similar?



Both screens:
x-scale: 1
y-scale: 5

Both graphs have x-intercepts at −2, 2, and 3. The cubic has down-and-up end behavior. The quartic has up-and-up end behavior.

The cubic function has two turning points, and it crosses the x-axis at −2. The quartic function touches the x-axis at −2 but does not cross it. The quartic function has three turning points.

Got It? **3. a.** What is a quadratic polynomial function with zeros 3 and −3?
b. What is a cubic polynomial function with zeros 3, 3, and −3?
c. Reasoning Graph both functions. How do the graphs differ? How are they similar?

You can write the polynomial functions in Problem 3 in factored form as $f(x) = (x + 2)(x - 2)(x - 3)$ and $g(x) = (x + 2)^2(x - 2)(x - 3)$. In $g(x)$ the repeated linear factor $x + 2$ makes -2 a **multiple zero**.

In particular, since the linear factor $x + 2$ appears twice, you can say that -2 is a zero of **multiplicity** 2. In general, *a is a zero of multiplicity n* means that $x - a$ appears n times as a factor.

take note

Key Concept How Multiple Zeros Affect a Graph

If a is a zero of multiplicity n in the polynomial function $y = P(x)$, then the behavior of the graph at the x-intercept a will be close to linear if $n = 1$, close to quadratic if $n = 2$, close to cubic if $n = 3$, and so on.

Problem 4 Finding the Multiplicity of a Zero

What are the zeros of $f(x) = x^4 - 2x^3 - 8x^2$? What are their multiplicities? How does the graph behave at these zeros?

$f(x) = x^4 - 2x^3 - 8x^2$

$= x^2(x^2 - 2x - 8)$ Factor out the GCF, x^2.

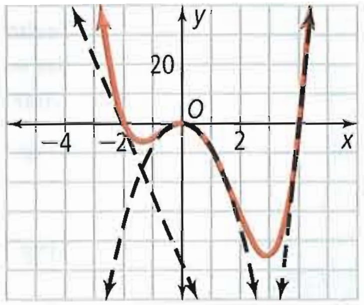
$= x^2(x + 2)(x - 4)$ Factor $(x^2 - 2x - 8)$.

Think

How can you find the multiplicities?
Factor the polynomial. Find the number of times each linear factor appears.

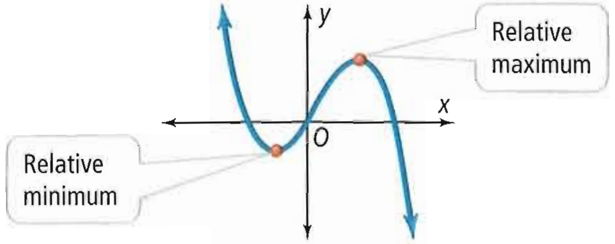
Since $x^2 = (x - 0)^2$, the number 0 is a zero of multiplicity 2. The numbers -2 and 4 are zeros of multiplicity 1.

The graph looks close to linear at the x-intercepts -2 and 4. It resembles a parabola at the x-intercept 0.



Got It? **4.** What are the zeros of $f(x) = x^3 - 4x^2 + 4x$? What are their multiplicities? How does the graph behave at these zeros?

If the graph of a polynomial function has several turning points, the function can have a **relative maximum** and a **relative minimum**. A relative maximum is the value of the function at an up-to-down turning point. A relative minimum is the value of the function at a down-to-up turning point.



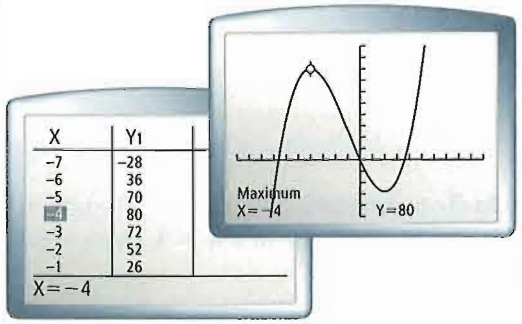
Problem 5 Identifying a Relative Maximum and Minimum

Think

How is a relative maximum different from a maximum at the vertex of a parabola?
A relative maximum is the greatest *y*-value in the "neighborhood" of its *x*-value. The maximum at the vertex of a parabola is the greatest *y*-value for *all* *x*-values.

What are the relative maximum and minimum of $f(x) = x^3 + 3x^2 - 24x$?

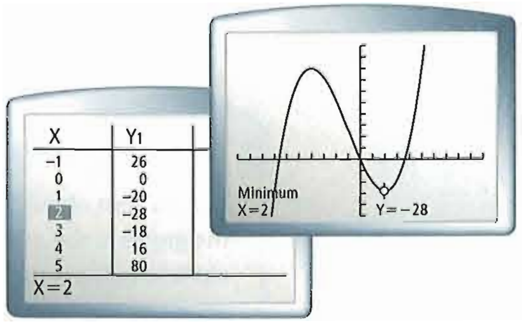
Use a graphing calculator to find a relative maximum and a relative minimum.



X	Y1
−7	−28
−6	36
−5	70
−4	80
−3	72
−2	52
−1	26

X=−4

Relative maximum



X	Y1
−1	26
0	0
1	−20
2	−28
3	−18
4	16
5	80

X=2

Relative minimum

The relative maximum is 80 at $x = -4$ and the relative minimum is -28 at $x = 2$.

Got It? **5.** What are the relative maximum and minimum of $f(x) = 3x^3 + x^2 - 5x$?

Problem 6 Using a Polynomial Function to Maximize Volume

Technology **The design of a digital box camera maximizes the volume while keeping the sum of the dimensions at 6 inches. If the length must be 1.5 times the height, what should each dimension be?**

Think

What is the formula for the volume of a "box"?
$V = \ell wh$

Step 1 Define a variable x.

Let $x =$ the height of the camera.

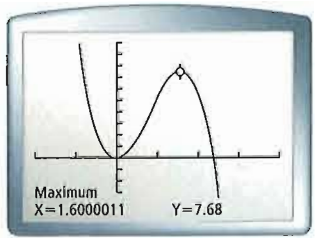
Step 2 Determine length and width.

length $= 1.5x$; width $= 6 - (x + 1.5x) = 6 - 2.5x$

Step 3 Model the volume.

$$V = (\text{length})(\text{width})(\text{height}) = (1.5x)(6 - 2.5x)(x)$$
$$= -3.75x^3 + 9x^2$$

Step 4 Graph the polynomial function. Use the **MAXIMUM** feature to find that the maximum volume is 7.68 in.^{3} for a height of 1.6 in.



height $= x = 1.6$

length $= 1.5x = 1.5(1.6) = 2.4$

width $= 6 - 2.5x = 6 - 2.5(1.6) = 2$

The dimensions of the camera should be 2.4 in. long by 2 in. wide by 1.6 in. high.

Got It? **6.** What is the maximum volume of the camera in Problem 6, if the sum of the dimensions is at most 4 inches?

Lesson Check

Do you know HOW?

Find the zeros of each function.

1. $y = x(x - 6)$

2. $y = (x + 4)(x - 5)$

3. $y = (x + 12)(x - 9)(x - 7)$

4. Write a polynomial function in standard form with zeros -1, 1, and 0.

Do you UNDERSTAND?

5. **Vocabulary** Write a polynomial function h in standard form that has 3 and -5 as zeros of multiplicity 2.

6. **Error Analysis** Your friend says that to write a function that has zeros 3 and -1, you should multiply the two factors $(x + 3)$ and $(x - 1)$ to get $f(x) = x^2 + 2x - 3$. Describe and correct your friend's error.

Practice and Problem-Solving Exercises

A Practice

Write each polynomial in factored form. Check by multiplication. **See Problem 1.**

7. $x^3 + 7x^2 + 10x$ **8.** $x^3 - 7x^2 - 18x$ **9.** $x^3 - 4x^2 - 21x$

10. $x^3 - 36x$ **11.** $x^3 + 8x^2 + 16x$ **12.** $9x^3 + 6x^2 - 3x$

Find the zeros of each function. Then graph the function. **See Problem 2.**

13. $y = (x - 1)(x + 2)$ **14.** $y = (x - 2)(x + 9)$ **15.** $y = x(x + 5)(x - 8)$

16. $y = (x + 1)(x - 2)(x - 3)$ **17.** $y = (x + 1)(x - 1)(x - 2)$ **18.** $y = x(x + 2)(x + 3)$

Write a polynomial function in standard form with the given zeros. **See Problem 3.**

19. $x = 5, 6, 7$ **20.** $x = -2, 0, 1$ **21.** $x = -5, -5, 1$ **22.** $x = 3, 3, 3$

23. $x = 1, -1, -2$ **24.** $x = 0, 4, -\frac{1}{2}$ **25.** $x = 0, 0, 2, 3$ **26.** $x = -1, -2, -3, -4$

Find the zeros of each function. State the multiplicity of multiple zeros. **See Problem 4.**

27. $y = (x + 3)^3$ **28.** $y = x(x - 1)^3$

29. $y = 2x^3 + x^2 - x$ **30.** $y = 3x^3 - 3x$

31. $y = (x - 4)^2$ **32.** $y = (x - 2)^2(x - 1)$

33. $y = (2x + 3)(x - 1)^2$ **34.** $y = (x + 1)^2(x - 1)(x - 2)$

Find the relative maximum and relative minimum of the graph of each function. **See Problem 5.**

35. $f(x) = x^3 + 4x^2 - 5x$ **36.** $f(x) = -x^3 + 16x^2 - 76x + 96$

37. $f(x) = -4x^3 + 12x^2 + 4x - 12$ **38.** $f(x) = x^3 - 7x^2 + 7x + 15$

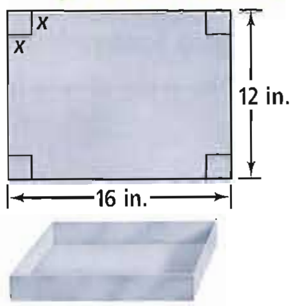
See Problem 6.

39. **Metalwork** A metalworker wants to make an open box from a sheet of metal, by cutting equal squares from each corner as shown.

a. Write expressions for the length, width, and height of the open box.

b. Use your expressions from part (a) to write a function for the volume of the box. (*Hint:* Write the function in factored form.)

c. Graph the function. Then find the maximum volume of the box and the side length of the cut-out squares that generates this volume.



B Apply

Write each function in factored form. Check by multiplication.

40. $y = 3x^3 - 27x^2 + 24x$

41. $y = -2x^3 - 2x^2 + 40x$

42. $y = x^4 + 3x^3 - 4x^2$

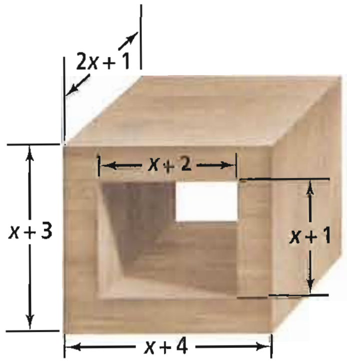
43. **Think About a Plan** A storage company needs to design a new storage box that has twice the volume of its largest box. Its largest box is 5 ft long, 4 ft wide, and 3 ft high. The new box must be formed by increasing each dimension by the same amount. Find the increase in each dimension.

- How can you write the dimensions of the new storage box as polynomial expressions?
- How can you use the volume of the current largest box to find the volume of the new box?

44. **Carpentry** A carpenter hollowed out the interior of a block of wood as shown at the right.

a. Express the volume of the original block and the volume of the wood removed as polynomials in factored form.

b. What polynomial represents the volume of the wood remaining?



45. **Geometry** A rectangular box is $2x + 3$ units long, $2x - 3$ units wide, and $3x$ units high. What is its volume, expressed as a polynomial?

46. **Measurement** The volume in cubic feet of a CD holder can be expressed as $V(x) = -x^3 - x^2 + 6x$, or, when factored, as the product of its three dimensions. The depth is expressed as $2 - x$. Assume that the height is greater than the width.

a. Factor the polynomial to find linear expressions for the height and the width.

b. Graph the function. Find the x-intercepts. What do they represent?

c. What is a realistic domain for the function?

d. What is the maximum volume of the CD holder?

Find the relative maximum, relative minimum, and zeros of each function.

47. $y = 2x^3 - 23x^2 + 78x - 72$

48. $y = 8x^3 - 10x^2 - x - 3$

49. $y = (x + 1)^4 - 1$

50. **Open-Ended** Write a polynomial function with the following features: it has three distinct zeros; one of the zeros is 1; another zero has a multiplicity of 2.

51. **Writing** Explain how the graph of a polynomial function can help you factor the polynomial.

For each function, determine the zeros. State the multiplicity of any multiple zeros.

52. $f(x) = x^3 - 36x$

53. $y = (x + 1)(x - 4)(3 - 2x)$

54. $y = (x + 7)(5x + 2)(x - 6)^2$

Challenge

55. Find a fourth-degree polynomial function with zeros 1, -1, i, and $-i$. Write the function in factored form.

56. **a.** Compare the graphs of $y = (x + 1)(x + 2)(x + 3)$ and $y = (x - 1)(x - 2)(x - 3)$. What transformation could you use to describe the change from one graph to the other?

b. Compare the graphs of $y = (x + 1)(x + 3)(x + 7)$ and $y = (x - 1)(x - 3)(x - 7)$. Does the transformation that you chose in part (a) still hold true? Explain.

c. **Make a Conjecture** What transformation could you use to describe the effect of changing the signs of the zeros of a polynomial function?

Sunshine State Standards Practice

MA.912.A.3.15

57. The three most frequent letters in the English language are E, T and A. They represent on average 30% of all letters. The most frequent letter E, is 4% more frequent than the second most frequent letter T. The combined frequency of T and A is 4% higher than the frequency of E. Approximately, how many letters E can you expect to encounter in a 500 letter paragraph?

Ⓐ 49 Ⓑ 65 Ⓒ 72 Ⓓ 88

MA.912.A.4.3

58. Which expression is the factored form of $x^3 + 2x^2 - 5x - 6$?

Ⓕ $(x + 1)(x + 1)(x - 6)$
Ⓗ $(x + 2)(2x - 5)(x - 6)$
Ⓖ $(x + 3)(x + 1)(x - 2)$
Ⓘ $(x - 3)(x - 1)(x + 2)$

MA.912.A.2.12

59. A ball with a 3 in. radius has volume V_1. A second ball has a 9 in. radius and volume V_2. Which equation represents the volume of the second ball in terms of the first?

Ⓐ $V_2 = 3V_1$ Ⓑ $V_2 = 27V_1$ Ⓒ $V_2 = V_1^2$ Ⓓ $V_2 = 9V_1^2$

MA.912.A.4.7

60. **Extended Response** What is the polynomial function, in factored form, whose zeros are -2, 5, and 6, and whose leading coefficient is -2? Graph this function and find any relative minimums or maximums.

Mixed Review

Write each polynomial in standard form. Then classify it by degree and by number of terms. See Lesson 5-1.

61. $x^2 - 1 - 3x^5 + 2x^2$ **62.** $-2x^3 - 7x^4 + x^3$ **63.** $6x + x^3 - 6x - 2$

Factor each expression. See Lesson 4-4.

64. $x^2 + 5x + 4$ **65.** $x^2 - 2x - 15$ **66.** $x^2 - 12x + 36$

Get Ready! **To prepare for Lesson 5-3, do Exercises 67–69.**

Solve each quadratic equation using any method. See Lesson 4-7.

67. $x^2 + x - 6 = 0$ **68.** $2x^2 - 7x + 3 = 0$ **69.** $4x^2 - 25 = 0$

5-3 Solving Polynomial Equations

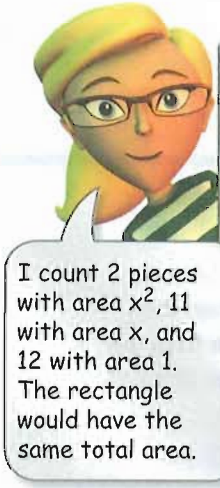
Sunshine State Standards

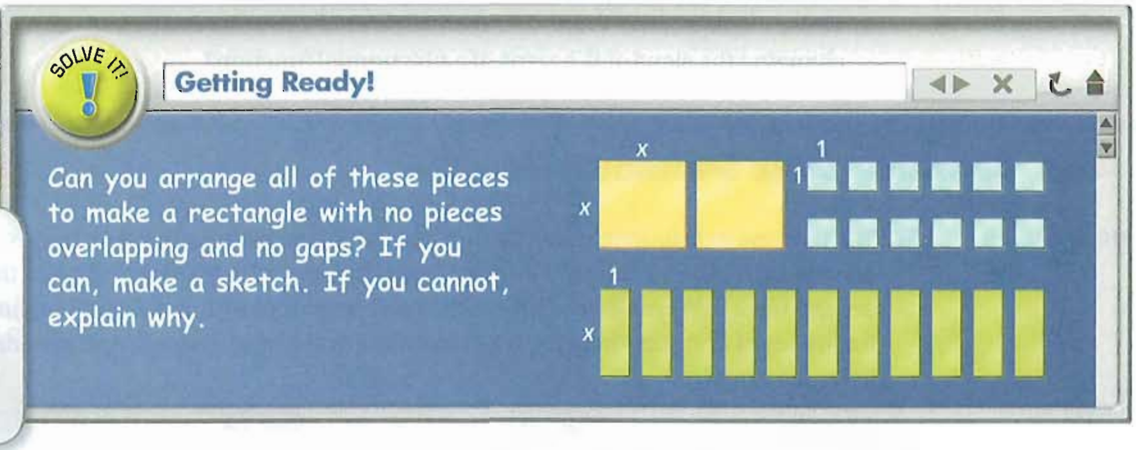
MA.912.A.4.9 Use graphing technology to find approximate solutions for polynomial equations.

MA.912.A.4.3 Factor polynomial expressions.

MA.912.A.4.10 Use polynomial equations to solve real-world problems.

Objectives To solve polynomial equations by factoring
To solve polynomial equations by graphing





Getting Ready!

Can you arrange all of these pieces to make a rectangle with no pieces overlapping and no gaps? If you can, make a sketch. If you cannot, explain why.

Lesson Vocabulary
- sum of cubes
- difference of cubes

Factoring a polynomial like $ax^2 + bx + c$ can help you solve a polynomial equation like $ax^2 + bx + c = 0$.

Essential Understanding If $(x - a)$ is a factor of a polynomial, then the polynomial has value 0 when $x = a$. If a is a real number, then the graph of the polynomial has $(a, 0)$ as an x-intercept.

To solve a polynomial equation by factoring:

1. Write the equation in the form $P(x) = 0$ for some polynomial function P.
2. Factor $P(x)$. Use the Zero Product Property to find the roots.

Problem 1 Solving Polynomial Equations Using Factors

What are the real or imaginary solutions of each polynomial equation?

A $2x^3 - 5x^2 = 3x$

$2x^3 - 5x^2 - 3x = 0$	Rewrite in the form $P(x) = 0$.
$x(2x^2 - 5x - 3) = 0$	Factor out the GCF, x.
$x(2x + 1)(x - 3) = 0$	Factor $2x^2 - 5x - 3$.
$x = 0 \quad \text{or} \quad 2x + 1 = 0 \quad \text{or} \quad x - 3 = 0$	Zero Product Property
$x = 0 \qquad x = -\frac{1}{2} \qquad x = 3$	Solve each equation for x.

The solutions are 0, $-\frac{1}{2}$, and 3.

Plan

What does it mean if x is a common factor of every term in $P(x)$?
You can write $P(x)$ as $xQ(x)$, so 0 will be a solution of $P(x) = 0$.

Think

How will the solution be similar to the solution of the equation in part (a)? Both equations have 0 as a solution, but here it will have a multiplicity of 2.

B $3x^4 + 12x^2 = 6x^3$

$3x^4 - 6x^3 + 12x^2 = 0$ Rewrite in the form $P(x) = 0$.

$x^4 - 2x^3 + 4x^2 = 0$ Multiply by $\frac{1}{3}$ to simplify.

$x^2(x^2 - 2x + 4) = 0$ Factor out the GCF, x^2.

$x^2 = 0$ or $x^2 - 2x + 4 = 0$ Zero Product Property

$$x = 0 \quad \left| \quad \begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} \\ x &= \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3} \end{aligned} \right.$$

Use the Quadratic Formula to solve $x^2 - 2x + 4 = 0$. Substitute $a = 1$, $b = -2$, and $c = 4$.

The solutions are 0, $1 + i\sqrt{3}$, and $1 - i\sqrt{3}$.

Got It? **1.** What are the real or imaginary solutions of each equation?

a. $(x^2 - 1)(x^2 + 4) = 0$ **b.** $x^5 + 4x^3 = 5x^4 - 2x^3$

take note

Concept Summary Polynomial Factoring Techniques

Techniques	**Examples**
Factoring out the GCF Factor out the greatest common factor of all the terms.	$15x^4 - 20x^3 + 35x^2$ $= 5x^2(3x^2 - 4x + 7)$
Quadratic Trinomials For $ax^2 + bx + c$, find factors with product ac and sum b.	$6x^2 + 11x - 10$ $= (3x - 2)(2x + 5)$
Perfect Square Trinomials $a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$	$x^2 + 10x + 25 = (x + 5)^2$ $x^2 - 10x + 25 = (x - 5)^2$
Difference of Squares $a^2 - b^2 = (a + b)(a - b)$	$4x^2 - 15 = (2x + \sqrt{15})(2x - \sqrt{15})$
Factoring by Grouping $ax + ay + bx + by$ $= a(x + y) + b(x + y)$ $= (a + b)(x + y)$	$x^3 + 2x^2 - 3x - 6$ $= x^2(x + 2) + (-3)(x + 2)$ $= (x^2 - 3)(x + 2)$
Sum or Difference of Cubes $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	$8x^3 + 1 = (2x + 1)(4x^2 - 2x + 1)$ $8x^3 - 1 = (2x - 1)(4x^2 + 2x + 1)$

The sum and difference of cubes is a new factoring technique.

Here's Why It Works Factoring $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$:

$$\begin{aligned} a^3 + b^3 &= a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3 && \text{Add 0.} \\ &= a^2(a + b) - ab(a + b) + b^2(a + b) && \text{Factor out } a^2, -ab, \text{ and } b^2. \\ &= (a + b)(a^2 - ab + b^2) && \text{Factor out } (a + b). \end{aligned}$$

For $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, you can follow steps similar to those above, or you can factor $a^3 - b^3$ as the sum of cubes $a^3 + (-b)^3$.

Problem 2 Solving Polynomial Equations by Factoring

What are the real or imaginary solutions of each polynomial equation?

Think

How can you write the polynomial in quadratic form?
Write in terms of x^2: $(x^2)^2 - 3(x^2) - 4 = 0$, which shows the factorable quadratic form $a^2 - 3a - 4 = 0$.

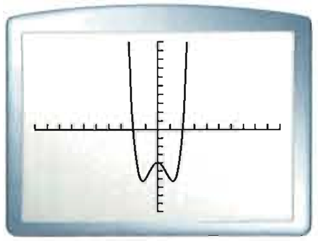
A $x^4 - 3x^2 = 4$

$$\begin{aligned} x^4 - 3x^2 - 4 &= 0 && \text{Rewrite in the form } P(x) = 0. \\ a^2 - 3a - 4 &= 0 && \text{Let } a = x^2. \\ (a - 4)(a + 1) &= 0 && \text{Factor.} \\ (x^2 - 4)(x^2 + 1) &= 0 && \text{Replace } a \text{ with } x^2. \\ (x + 2)(x - 2)(x^2 + 1) &= 0 && \text{Factor } x^2 - 4 \text{ as a difference of squares.} \end{aligned}$$

It follows from the Zero Product Property that $x = 2$, $x = -2$, or $x^2 = -1$. Solving $x^2 = -1$ yields two imaginary roots: $x = i$ or $x = -i$.

Check Graph the related function $y = x^4 - 3x^2 - 4$.

The graph shows zeros at $x = 2$ and $x = -2$. It also shows three turning points. This means that there are imaginary roots, which do not appear on the graph.



B $x^3 = 1$

$$\begin{aligned} x^3 - 1 &= 0 && \text{Rewrite in the form } P(x) = 0. \\ (x - 1)(x^2 + x + 1) &= 0 && \text{Factor the difference of cubes.} \end{aligned}$$

It follows from the Zero Product Property that $x = 1$ or $x^2 + x + 1 = 0$. Use the Quadratic Formula to solve $x^2 + x + 1 = 0$.

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

The three solutions of $x^3 = 1$ are 1, $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Got It? **2.** What are the real or imaginary solutions of each polynomial equation?
a. $x^4 = 16$ **b.** $x^3 = 8x - 2x^2$ **c.** $x(x^2 + 8) = 8(x + 1)$

While factoring is an effective way to solve a polynomial equation, you can also find the real roots quickly by using a graphing calculator.

Problem 3 Finding Real Roots by Graphing

What are the real solutions of the equation $x^3 + 5 = 4x^2 + x$?

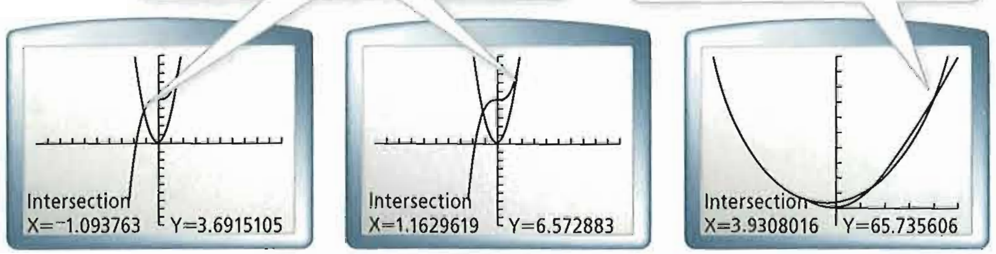
Plan

Why is it helpful to graph Y_1 and Y_2?
The values of x for which $Y_1 = Y_2$ are the solutions of the original equation.

Method 1 Graph $Y_1 = x^3 + 5$ and $Y_2 = 4x^2 + x$. Use the **INTERSECT** feature to find the x values of the points of intersection.

These two intersection points are visible in the standard viewing window.

You must adjust the window to obtain this point of intersection.



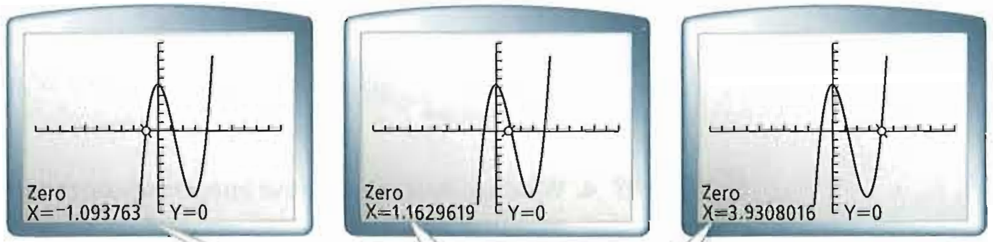
Intersection
X=⁻1.093763 Y=3.6915105

Intersection
X=1.1629619 Y=6.572883

Intersection
X=3.9308016 Y=65.735606

Approximate solutions are $x = -1.09$, $x = 1.16$, and $x = 3.93$.

Method 2 Rewrite the equation as $x^3 - 4x^2 - x + 5 = 0$. Graph the related function $y = x^3 - 4x^2 - x + 5$. Use the **ZERO** feature.



Zero
X=⁻1.093763 Y=0

Zero
X=1.1629619 Y=0

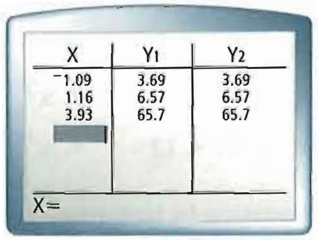
Zero
X=3.9308016 Y=0

The solutions are the same as identified using Method 1.

Approximate solutions are $x = -1.09$, $x = 1.16$, and $x = 3.93$.

Check Verify the solutions by showing that they satisfy the original equation. Show values of $y_1 = x^3 + 5$ and $y_2 = 4x^2 + x$ in a table.

The solution checks.



X	Y1	Y2
⁻1.09	3.69	3.69
1.16	6.57	6.57
3.93	65.7	65.7

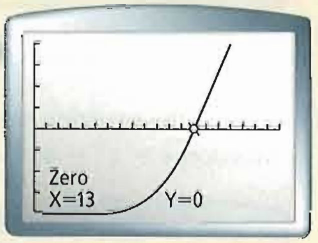
X=

Got It? **3. a.** What are the real solutions of the equation $x^3 + x^2 = x - 1$?

b. Reasoning In Problem 3, which method seems to be an easier and more reliable way to find the solutions of an equation? Explain.

Problem 4 Modeling a Problem Situation

Close friends Stacy, Una, and Amir were all born on July 4. Stacy is one year younger than Una. Una is two years younger than Amir. On July 4, 2010, the product of their ages was 2300 more than the sum of their ages. How old was each friend on that day?

Think	Write
Define variables.	Let x = Una's age on July 4, 2010. Stacy's age = x − 1. Amir's age = x + 2.
Write an equation. Simplify and change it to $P(x) = 0$ form.	$\underbrace{x + (x - 1) + (x + 2)}_{}\text{Sum of ages} + 2300 = \underbrace{x(x - 1)(x + 2)}_{}\text{Product of ages}$ $3x + 2301 = x(x^2 + x - 2)$ $3x + 2301 = x^3 + x^2 - 2x$ $x^3 + x^2 - 5x - 2301 = 0$
Only real solutions make sense, so graphing $y_1 = P(x)$ should show any real solution that exists. Use the zero feature.	 x = 13
Write the answer.	Una was 13, Stacy 12, and Amir 15.

Got It? **4.** What are three consecutive integers whose product is 480 more than their sum?

Lesson Check

Do you know HOW?

Factor each polynomial.

1. $x^2 - 3x - 18$

2. $x^3 - 27$

3. $x^3 + 3x^2 + 4x + 12$

4. $x^4 - 2x^2 - 8$

Solve each equation by factoring.

5. $2x^2 + 7x - 4 = 0$

6. $2x^3 + 2x^2 - 4x = 0$

Do you UNDERSTAND?

7. Vocabulary Identify each expression as a sum of cubes, difference of cubes, or difference of squares.

a. $x^2 - 64$

b. $x^3 + 8$

c. $x^3 - 125$

d. $x^2 - 81$

8. Reasoning Which method of solving polynomial equations will not identify the imaginary roots? Explain.

9. Reasoning Show two different ways to find the real roots of the polynomial equation $0 = x^6 - x^2$. Show your steps.

Practice and Problem-Solving Exercises

A Practice

Find the real or imaginary solutions of each equation by factoring. **See Problems 1 and 2.**

10. $x^3 + 64 = 0$ **11.** $x^3 - 1000 = 0$ **12.** $125x^3 - 27 = 0$

13. $64x^3 - 1 = 0$ **14.** $x^3 + 2x^2 + 5x + 10 = 0$ **15.** $6x^2 + 13x - 5 = 0$

16. $0 = x^3 - 27$ **17.** $0 = x^3 - 64$ **18.** $8x^3 = 1$

19. $64x^3 = -8$ **20.** $x^4 - 10x^2 = -9$ **21.** $x^4 - 8x^2 = -16$

22. $x^4 - 12x^2 = 64$ **23.** $x^4 + 7x^2 = 18$ **24.** $x^4 + 4x^2 = 12$

Find the real solutions of each equation by graphing. **See Problem 3.**

25. $x^3 - 4x^2 - 7x = -10$ **26.** $3x^3 - 6x^2 - 9x = 0$ **27.** $4x^3 - 8x^2 + 4x = 0$

28. $6x^2 = 48x$ **29.** $x^3 + 3x^2 + 2x = 0$ **30.** $2x^3 + 5x^2 = 7x$

31. $4x^3 = 4x^2 + 3x$ **32.** $2x^4 - 5x^3 - 3x^2 = 0$ **33.** $x^2 - 8x + 7 = 0$

34. $x^4 - 4x^3 - x^2 + 16x = 12$ **35.** $x^3 - x^2 - 16x = 20$ **36.** $3x^3 + 12x^2 - 3x = 12$

Graphing Calculator **Write an equation to model each situation. Then solve each equation by graphing.** **See Problem 4.**

37. The Johnson twins were born two years after their older sister. This year, the product of the three siblings ages is exactly 4558 more than the sum of their ages. How old are the twins?

38. The product of three consecutive integers is 210. What are the numbers?

B Apply

Solve each equation.

39. $x^3 + 13x = 10x^2$ **40.** $x^3 - 6x^2 + 6x = 0$ **41.** $12x^3 = 60x^2 + 75x$

42. $125x^3 + 216 = 0$ **43.** $81x^3 - 192 = 0$ **44.** $x^4 - 64 = 0$

45. $-2x^4 - 100 = 0$ **46.** $27 = -x^4 - 12x^2$ **47.** $x^5 - 5x^3 + 4x = 0$

48. $5x^3 = 5x^2 + 12x$ **49.** $x^3 + x^2 + x + 1 = 0$ **50.** $x^3 + 1 = x^2 + x$

51. **Think About a Plan** The width of a plastic storage box is 1 ft longer than the height. The length is 4 ft longer than the height. The volume is 36 ft^3. What are the dimensions of the box?

- What is the formula for the volume of a rectangular prism?
- What variable expressions represent the length, height, and width?
- What equation represents the volume of the plastic storage box?

52. **Error Analysis** A student claims that 1, 2, 3, and 4 are the zeros of a cubic polynomial function. Explain why the student is mistaken.

53. **Geometry** The width of a box is 2 m less than the length. The height is 1 m less than the length. The volume is 60 m^3. What is the length of the box?

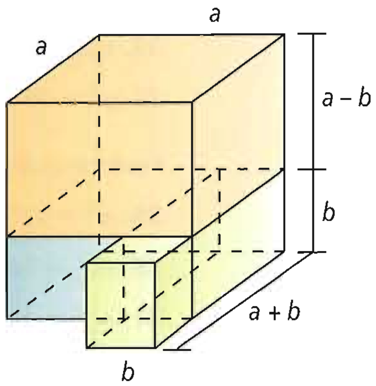
Graph each function to find the zeros. Rewrite the function with the polynomial in factored form.

54. $y = 2x^2 + 3x - 5$ **55.** $y = x^4 - 10x^2 + 9$ **56.** $y = x^3 - 3x^2 + 4$

57. **Open-Ended** To solve a polynomial equation, you can use any combination of graphing, factoring, and the Quadratic Formula. Write and solve an equation to illustrate each method.

Challenge

58. The geometric figure at the right has volume $a^3 + b^3$. You can split it into three rectangular blocks (including the long one with side $a + b$). Explain how to use this figure to prove the factoring formula for the sum of cubes, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.



59. **Open-Ended** Find equations for two different polynomial functions whose zeros include $-12, 0, \frac{1}{4}$, and $\frac{1}{6}$.

60. What are the complex solutions of $x^5 + x^3 + 2x = 2x^4 + x^2 + 1$?

Sunshine State Standards Practice

MA.912.A.4.8

61. Which value is NOT a solution to the equation $x^4 - 3x^2 - 54 = 0$?

Ⓐ -3 Ⓑ 3 Ⓒ $-3i$ Ⓓ $-i\sqrt{6}$

MA.912.A.2.12

62. Ava drove 3 hours at 45 miles per hour. How many miles did she drive?

Ⓕ 45 miles Ⓖ 48 miles Ⓗ 90 miles Ⓘ 135 miles

MA.912.A.4.7

63. Which polynomial has the complex roots $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$?

Ⓐ $x^2 + 2x + 3$ Ⓑ $x^2 - 2x + 3$ Ⓒ $x^2 + 2x - 3$ Ⓓ $x^2 - 2x - 3$

MA.912.A.3.15

64. **Short Response** Sam has only quarters and dimes in his pocket. He has a total of 12 coins, totaling $1.95. How many of each coin does Sam have?

Mixed Review

Write each polynomial in factored form. Check by multiplication. **See Lesson 5-2.**

65. $3x^2 - 18x + 24$ **66.** $2x^4 + 6x^3 - 18x^2 - 54x$ **67.** $x^4 - 4x^3 - 5x^2$

Solve each equation by factoring. Check your answers. **See Lesson 4-5.**

68. $x^2 - 4x = 12$ **69.** $x^2 + 1 = 37$ **70.** $2x^2 - 5x - 3 = 0$

Get Ready! **To prepare for Lesson 5-4, do Exercises 71 and 72.**

Evaluate each expression for the given values of the variables. **See Lesson 1-3.**

71. $\frac{16(x - 4)(y - 2)}{4(x - 3)y}$; $x = 1$ and $y = -2$ **72.** $\frac{2(x + 5)y}{10(x - 4)(y - 2)}$; $x = 1$ and $y = -2$

5-4 Dividing Polynomials

Sunshine State Standards

MA.912.A.4.3 Factor polynomial expressions.
MA.912.A.4.4 Divide polynomials by polynomials with various techniques, including synthetic division.
MA.912.A.4.6 Use the Remainder Theorem of Algebra.

Objectives To divide polynomials using long division
To divide polynomials using synthetic division

In how many ways is it possible to replace the squares with single digit numbers to complete a correct division problem? Justify your answer.

Dynamic Activity
Synthetic Division

Lesson Vocabulary
- synthetic division
- Remainder Theorem

Long division is one of many methods you can use to divide whole numbers.

Essential Understanding You can divide polynomials using steps that are similar to the long-division steps that you use to divide whole numbers.

When you try to factor a polynomial, you are trying to find a divisor of the polynomial that gives a quotient (the other factor) and remainder 0. This suggests that being able to divide one polynomial by another could help you factor polynomials.

Numerical long division and polynomial long division are similar.

Numerical Long Division	
32	
$21\overline{)672}$	21 divides into
$\underline{63}$	67 3 times
42	21 divides into
$\underline{42}$	42 2 times
0	

Polynomial Long Division	
$3x + 2$	
$2x + 1\overline{)6x^2 + 7x + 2}$	$(2x + 1)$ divides into
$\underline{6x^2 + 3x}$	$(6x^2 + 7x)$ $3x$ times
$4x + 2$	$(2x + 1)$ divides into
$\underline{4x + 2}$	$(4x + 2)$ 2 times
0	

The remainder from each division above is 0, so 21 is a factor of 672 and $2x + 1$ is a factor of $6x^2 + 7x + 2$.

Problem 1 Using Polynomial Long Division

Use polynomial long division to divide $4x^2 + 23x - 16$ by $x + 5$. What is the quotient and remainder?

$$\begin{array}{rl}
\begin{array}{r} 4x \\ x+5\,\overline{)\,4x^2+23x-16} \\ \underline{4x^2+20x} \\ 3x-16 \end{array} &
\begin{array}{l} \\ \text{Divide: } \frac{4x^2}{x} = 4x. \\ \text{Multiply: } 4x(x+5) = 4x^2 + 20x. \\ \text{Subtract to get } 3x\text{. Bring down } -16. \end{array}
\end{array}$$

Repeat the process of dividing, multiplying, and subtracting.

$$\begin{array}{rl}
\begin{array}{r} 4x\ +3 \\ x+5\,\overline{)\,4x^2+23x-16} \\ \underline{4x^2+20x} \\ 3x-16 \\ \underline{3x+15} \\ -31 \end{array} &
\begin{array}{l} \\ \\ \\ \text{Divide: } \frac{3x}{x} = 3 \\ \text{Multiply: } 3(x+5) = 3x + 15. \\ \text{Subtract to get } -31. \end{array}
\end{array}$$

The quotient is $4x + 3$ with remainder -31.

You can say, $4x + 3$, R -31.

Check

$$\begin{aligned} (x+5)(4x+3) - 31 &= (4x^2 + 3x + 20x + 15) - 31 && \text{Multiply } (x+5)(4x+3). \\ &= 4x^2 + 23x - 16 \ ✔ && \text{Simplify.} \end{aligned}$$

Think

How can you check your result?
Show that (divisor)(quotient) + remainder = dividend.

Got It? **1.** Use polynomial long division to divide $3x^2 - 29x + 56$ by $x - 7$. What is the quotient and remainder?

take note

Key Concept The Division Algorithm for Polynomials

You can divide polynomial $P(x)$ by polynomial $D(x)$ to get polynomial quotient $Q(x)$ and polynomial remainder $R(x)$. The result is $P(x) = D(x)Q(x) + R(x)$.

$$\begin{array}{r} Q(x) \\ D(x)\,\overline{)\,P(x)} \\ \vdots\ \\ \overline{R(x)} \end{array}$$

If $R(x) = 0$, then $P(x) = D(x)Q(x)$ and $D(x)$ and $Q(x)$ are factors of $P(x)$.

To use long division, $P(x)$ and $D(x)$ should be in standard form with zero coefficients where appropriate. The process stops when the degree of the remainder, $R(x)$, is less than the degree of the divisor, $D(x)$.

Problem 2 Checking Factors

A Is $x^2 + 1$ a factor of $3x^4 - 4x^3 + 12x^2 + 5$?

$$\begin{array}{rl}
 & 3x^2 - 4x + 9 \\
x^2 + 0x + 1 & \overline{)\,3x^4 - 4x^3 + 12x^2 + 0x + 5} \\
 & \underline{3x^4 + 0x^3 + 3x^2} \\
 & -4x^3 + 9x^2 + 0x \\
 & \underline{-4x^3 + 0x^2 - 4x} \\
 & 9x^2 + 4x + 5 \\
 & \underline{9x^2 + 0x + 9} \\
 & 4x - 4
\end{array}$$

Include 0x terms.

The degree of the remainder is less than the degree of the divisor. Stop!

The remainder is not zero. $x^2 + 1$ is not a factor of $3x^4 - 4x^3 + 12x^2 + 5$.

Plan

Can you use the Factor Theorem to help answer this question?
Yes; recall that if $P(a) = 0$, then $x - a$ is a factor of $P(x)$.

B Is $x - 2$ a factor of $P(x) = x^5 - 32$? If it is, write $P(x)$ as a product of two factors.

Step 1 Use the Factor Theorem to determine if $x - 2$ is a factor of $x^5 - 32$.

$$\begin{aligned} P(2) &= 2^5 - 32 \\ &= 32 - 32 \\ &= 0 \end{aligned}$$

Since $P(2) = 0$, $x - 2$ is a factor of $P(x)$.

Step 2 Use polynomial long division to find the other factor.

$$\begin{array}{rl}
 & x^4 + 2x^3 + 4x^2 + 8x + 16 \\
x - 2 & \overline{)\,x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 32} \\
 & \underline{x^5 - 2x^4} \\
 & 2x^4 + 0x^3 \\
 & \underline{2x^4 - 4x^3} \\
 & 4x^3 + 0x^2 \\
 & \underline{4x^3 - 8x^2} \\
 & 8x^2 + 0x \\
 & \underline{8x^2 - 16x} \\
 & 16x - 32 \\
 & \underline{16x - 32} \\
 & 0
\end{array}$$

$P(x) = (x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)$

Got It? **2. a.** Is $x^4 - 1$ a factor of $P(x) = x^5 + 5x^4 - x - 5$? If it is, write $P(x)$ as a product of two factors.

b. Reasoning Use the fact that $12 \cdot 31 = 372$ to write $3x^2 + 7x + 2$ as the product of two factors.

Synthetic division simplifies the long-division process for dividing by a linear expression $x - a$. To use synthetic division, write the coefficients (including zeros) of the polynomial in standard form. Omit all variables and exponents. For the divisor, reverse the sign (use a). This allows you to add instead of subtract throughout the process.

Problem 3 Using Synthetic Division

Use synthetic division to divide $x^3 - 14x^2 + 51x - 54$ by $x + 2$. What is the quotient and remainder?

Think

To divide by $x + 2$ what number do you use for the synthetic divisor?

$x + 2 = x - (-2)$ so use -2.

Step 1 Reverse the sign of +2. Write the coefficients of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & -14 & 51 & -54 \end{array}$$

Step 2 Bring down the first coefficient.

$$\begin{array}{r|rrrr} -2 & 1 & -14 & 51 & -54 \\ & & & & \\ \hline & 1 & & & \end{array}$$

Step 3 Multiply the coefficient by the divisor. Add to the next coefficient.

$$\begin{array}{r|rrrr} -2 & 1 & -14 & 51 & -54 \\ & & -2 & & \\ \hline & 1 & -16 & & \end{array}$$

Step 4 Continue multiplying and adding through the last coefficient.

$$\begin{array}{r|rrrr} -2 & 1 & -14 & 51 & -54 \\ & & -2 & 32 & -166 \\ \hline & 1 & -16 & 83 & -220 \end{array}$$

The quotient is $x^2 - 16x + 83$, R -220.

Got It? **3.** Use synthetic division to divide $x^3 - 57x + 56$ by $x - 7$. What is the quotient and remainder?

Problem 4 Using Synthetic Division to Solve a Problem

Plan

How can you use the picture to help solve the problem?

The picture gives the width of box. Remember for a rectangular prism, $V = \ell \times w \times h$.

Crafts **The polynomial $x^3 + 7x^2 - 38x - 240$ expresses the volume, in cubic inches, of the shadow box shown.**

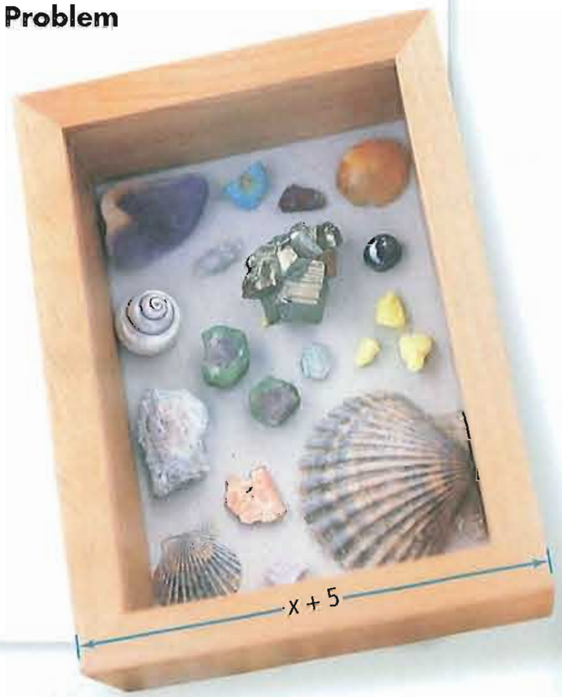
A **What are the dimensions of the box? (*Hint:* The length is greater than the height (or depth).)**

$$\begin{array}{r|rrrr} -5 & 1 & 7 & -38 & -240 \\ & & -5 & -10 & 240 \\ \hline & 1 & 2 & -48 & 0 \end{array}$$

$x^2 + 2x - 48 = (x - 6)(x + 8)$

So, $x^3 + 7x^2 - 38x - 240 = (x + 5)(x^2 + 2x - 48)$
$= (x + 5)(x - 6)(x + 8)$

The length, width, and height (or depth) of the box are $(x + 8)$ in., $(x + 5)$ in., and $(x - 6)$ in., respectively.



B **If the width of the box is 15 in., what are the other two dimensions?**

The width of the box is $x + 5$. So if $x + 5 = 15$, then $x = 10$.

Substitute for x to find the length and height (or depth).

Length: $x + 8 = 10 + 8 = 18$ in.
Height: $x - 6 = 10 - 6 = 4$ in.

Got It? **4.** If the polynomial $x^3 + 6x^2 + 11x + 6$ expresses the volume, in cubic inches, of the box, and the width is $(x + 1)$ in., what are the dimensions of the box?

The **Remainder Theorem** provides a quick way to find the remainder of a polynomial long-division problem.

take note

Theorem The Remainder Theorem

If you divide a polynomial $P(x)$ of degree $n \geq 1$ by $x - a$, then the remainder is $P(a)$.

Here's Why It Works When you divide polynomial $P(x)$ by $D(x)$, you find $P(x) = D(x)Q(x) + R(x)$.

$P(x) = (x - a)Q(x) + R(x)$ Substitute $(x - a)$ for $D(x)$.

$P(a) = (a - a)Q(a) + R(a)$ Evaluate $P(a)$. Substitute a for x.

$= R(a)$ Simplify.

Problem 5 Evaluating a Polynomial

GRIDDED RESPONSE

Think

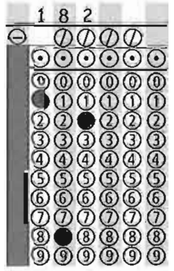
Is there a way to find $P(3)$ without substituting?
Use synthetic division. $P(3)$ is the remainder.

Given that $P(x) = x^5 - 2x^3 - x^2 + 2$, what is $P(3)$?

By the Remainder Theorem, $P(3)$ is the remainder when you divide $P(x)$ by $x - 3$.

$$\begin{array}{r|rrrrrr} 3 & 1 & 0 & -2 & -1 & 0 & 2 \\ & & 3 & 9 & 21 & 60 & 180 \\ \hline & 1 & 3 & 7 & 20 & 60 & 182 \end{array}$$

$P(3) = 182.$



Got It? **5.** Given that $P(x) = x^5 - 3x^4 - 28x^3 + 5x + 20$, what is $P(-4)$?

Lesson Check

Do you know HOW?

Divide using any method.

1. $(2x^2 + 7x + 11) \div (x + 2)$

2. $(x^3 + 5x^2 + 11x + 15) \div (x + 3)$

3. $(x^3 - x^2 - 4x + 4) \div (x - 2)$

4. $(4x^3 + 21x^2 - x - 24) \div (x + 5)$

5. $(9x^3 - 15x^2 + 4x) \div (x - 3)$

Do you UNDERSTAND?

6. **Reasoning** A polynomial $P(x)$ is divided by a binomial $x - a$. The remainder is 0. What conclusion can you draw? Explain.

7. **Writing** Explain why it is important to have the terms of both polynomials written in descending order of degree before dividing.

8. **Open-Ended** Write a polynomial division that has a quotient of $x + 3$ and a remainder of 2.

Practice and Problem-Solving Exercises

A Practice

Divide using long division. Check your answers.

See Problem 1.

9. $(x^2 - 3x - 40) \div (x + 5)$

10. $(3x^2 + 7x - 20) \div (x + 4)$

11. $(x^3 + 3x^2 - x + 2) \div (x - 1)$

12. $(2x^3 - 3x^2 - 18x - 8) \div (x - 4)$

13. $(3x^3 + 9x^2 + 8x + 4) \div (x + 2)$

14. $(9x^2 - 21x - 20) \div (x - 1)$

15. $(x^2 - 7x + 10) \div (x + 3)$

16. $(x^3 - 13x - 12) \div (x - 4)$

Determine whether each binomial is a factor of $x^3 + 4x^2 + x - 6$.

See Problem 2.

17. $x + 1$

18. $x + 2$

19. $x + 3$

20. $x - 3$

Divide using synthetic division.

See Problem 3.

21. $(x^3 + 3x^2 - x - 3) \div (x - 1)$

22. $(x^3 - 4x^2 + 6x - 4) \div (x - 2)$

23. $(x^3 - 7x^2 - 7x + 20) \div (x + 4)$

24. $(x^3 - 3x^2 - 5x - 25) \div (x - 5)$

25. $(x^2 + 3) \div (x - 1)$

26. $(3x^3 + 17x^2 + 21x - 9) \div (x + 3)$

27. $(x^3 + 27) \div (x + 3)$

28. $(6x^2 - 8x - 2) \div (x - 1)$

Use synthetic division and the given factor to completely factor each polynomial function.

See Problem 4.

29. $y = x^3 + 2x^2 - 5x - 6; (x + 1)$

30. $y = x^3 - 4x^2 - 9x + 36; (x + 3)$

31. **Geometry** The volume, in cubic inches, of the decorative box shown can be expressed as the product of the lengths of its sides as $V(x) = x^3 + x^2 - 6x$. What linear expressions with integer coefficients represent the length and height of the box?

Use synthetic division and the Remainder Theorem to find $P(a)$.

See Problem 5.

32. $P(x) = x^3 + 4x^2 - 8x - 6; a = -2$

33. $P(x) = x^3 + 4x^2 + 4x; a = -2$

34. $P(x) = x^3 - 7x^2 + 15x - 9; a = 3$

35. $P(x) = x^3 + 7x^2 + 4x; a = -2$

36. $P(x) = 6x^3 - x^2 + 4x + 3; a = 3$

37. $P(x) = 2x^3 - x^2 + 10x + 5; a = \frac{1}{2}$

38. $P(x) = 2x^3 + 4x^2 - 10x - 9; a = 3$

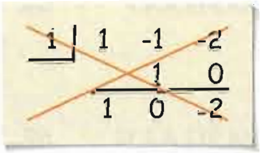
39. $P(x) = 2x^4 + 6x^3 + 5x^2 - 45; a = -3$

B Apply

40. Think About a Plan Your friend multiplies $x + 4$ by a quadratic polynomial and gets the result $x^3 - 3x^2 - 24x + 30$. The teacher says that everything is correct except for the constant term. Find the quadratic polynomial that your friend used. What is the correct result of multiplication?

- What does the fact that all the terms except for the constant are correct tell you?
- How can polynomial division help you solve this problem?
- What is the connection between the remainder of the division and your friend's error?

41. Error Analysis A student used synthetic division to divide $x^3 - x^2 - 2x$ by $x + 1$. Describe and correct the error shown.



42. Reasoning When a polynomial is divided by $(x - 5)$, the quotient is $5x^2 + 3x + 12$ with remainder 7. Find the polynomial.

43. Geometry The expression $\frac{1}{3}(x^3 + 5x^2 + 8x + 4)$ represents the volume of a square pyramid. The expression $x + 1$ represents the height of the pyramid. What expression represents the side length of the base? (*Hint:* The formula for the volume of a pyramid is $V = \frac{1}{3}Bh$.)

Divide.

44. $(2x^3 + 9x^2 + 14x + 5) \div (2x + 1)$

45. $(x^4 + 3x^2 + x + 4) \div (x + 3)$

46. $(x^5 + 1) \div (x + 1)$

47. $(x^4 + 4x^3 - x - 4) \div (x^3 - 1)$

48. $(3x^4 - 5x^3 + 2x^2 + 3x - 2) \div (3x - 2)$

Determine whether each binomial is a factor of $x^3 + x^2 - 16x - 16$.

49. $x + 2$

50. $x - 4$

51. $x + 1$

52. $x - 1$

Use synthetic division to determine whether each binomial is a factor of $3x^3 + 10x^2 - x - 12$.

53. $x + 3$

54. $x - 1$

55. $x + 2$

56. $x - 4$

Divide using synthetic division.

57. $(x^4 - 2x^3 + x^2 + x - 1) \div (x - 1)$

58. $(x^4 + 3x^3 + 3x^2 + 4x + 3) \div (x + 1)$

59. $(x^4 + 3x^3 + 7x^2 + 26x + 15) \div (x + 3)$

60. $(x^4 - 6x^2 - 27) \div (x + 2)$

61. $(x^4 - 5x^2 + 4x + 12) \div (x + 2)$

62. $\left(x^4 - \frac{9}{2}x^3 + 3x^2 - \frac{1}{2}x\right) \div \left(x - \frac{1}{2}\right)$

Challenge

63. **Reasoning** Divide. Look for patterns in your answers.

a. $(x^2 - 1) \div (x - 1)$ **b.** $(x^3 - 1) \div (x - 1)$ **c.** $(x^4 - 1) \div (x - 1)$

d. Using the patterns, factor $x^5 - 1$.

64. **Reasoning** The remainder from the division of the polynomial $x^3 + ax^2 + 2ax + 5$ by $x + 1$ is 3. Find a.

65. Use synthetic division to find $(x^2 + 4) \div (x - 2i)$.

66. **Writing** Suppose 3, −1, and 5 are zeros of a cubic polynomial function $f(x)$. What is the sign of $f(1) \cdot f(4)$? (*Hint:* Sketch the graph; consider all possibilities.)

Sunshine State Standards Practice

MA.912.A.4.4

67. What is the remainder when $x^2 - 5x + 7$ is divided by $x + 1$?

Ⓐ 1 Ⓑ 3 Ⓒ 11 Ⓓ 13

MA.912.A.4.7

68. What is the least degree of a polynomial that has a zero of multiplicity 3 at 1, a zero of multiplicity 1 at 0, and a zero of multiplicity 2 at 2?

Ⓕ 3 Ⓖ 4 Ⓗ 5 Ⓘ 6

MA.912.A.2.12

69. The equation $y = 0.17x$ represents your weight, in pounds, on the Moon y in relation to your weight on Earth x. If Al weighs 130 lb on Earth, what would he weigh on the Moon?

Ⓐ 22.1 lb Ⓑ 92.3 lb Ⓒ 130 lb Ⓓ 764.7 lb

MA.912.A.3.3

70. **Extended Response** The formula for the area of a circle is $A = \pi r^2$. Solve the equation for r. If the area of a circle is 78.5 cm^2, what is the radius? Use 3.14 for π.

Mixed Review

Find the real solutions of each equation by factoring. See Lesson 5-3.

71. $x^3 + 2x^2 + x = 0$ **72.** $2x^4 - 2x^3 + 2x^2 = 2x$ **73.** $5x^5 = 125x^3$

Solve each equation using the Quadratic Formula. See Lesson 4-7.

74. $x^2 + 3x - 2 = 0$ **75.** $2x^2 + 4x - 4 = 0$ **76.** $7x^2 - 2x - 5 = 0$

77. $x^2 - 5x = -5$ **78.** $x^2 - 6x = -7$ **79.** $x^2 + 7x + 11 = 0$

Find the solution of each system by graphing. See Lesson 3-3.

80. $\begin{cases} y < 2x + 3 \\ y > -x \end{cases}$ **81.** $\begin{cases} y > x - 4 \\ y > 4 - \frac{1}{3}x \end{cases}$ **82.** $\begin{cases} y < -|x| + 3 \\ y > x + 1 \end{cases}$

Get Ready! **To prepare for Lesson 5-5, do Exercises 83–85.**

Simplify each expression. See Lesson 4-8.

83. $(-4i)(6i)$ **84.** $(2 + i)(2 - i)$ **85.** $(4 - 3i)(5 + i)$

5 Mid-Chapter Quiz

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

For each polynomial function, describe the end behavior of its graph.

1. $f(x) = x^8 - 8x^4 + 6x^2$

2. $f(x) = -x^4 - x^3 + 1$

3. $f(x) = x^7 - 3x^5 - 5x^3$

4. What is the degree of the function that generates the data shown?

x	y
−3	159
−2	29
−1	−1
0	−3
1	−1
2	29
3	159

Find all the solutions of each equation by factoring.

5. $x^3 - 5x^2 = 36x$

6. $27x^3 = 8$

7. $x^4 - 20x^2 + 64 = 0$

8. $x^3 + 125 = 0$

9. Use the Remainder Theorem and synthetic division to find $P(4)$ for $P(x) = 2x^4 - 3x^2 + 4x - 1$.

You have several boxes with the same dimensions. They have a combined volume of $2x^4 + 4x^3 - 18x^2 - 4x + 16$. Determine whether each binomial below could represent the number of boxes you have.

10. $x - 1$

11. $x + 2$

12. $2x + 8$

Write each polynomial in standard form. Then classify it by degree and number of terms.

13. $-2x^3 + 6 - x^3 + 5x$

14. $3(x - 1)(x + 4)$

Describe the shape of the graph of each cubic function by determining the end behavior and number of turning points.

15. $y = -5x^3$

16. $y = 3x^3 + 4x^2 + 2x - 1$

Do you UNDERSTAND?

17. You buy one container each of strawberries, blueberries, and cherries. Cherries are \$1 more per container than blueberries, which are \$1 more per container than strawberries. The product of the 3 individual prices is 5 times the total cost of one container of each fruit.
  a. Write a polynomial function to model the cost of your purchase.
  b. Graph to find the price of each container.
  c. **Writing** Explain how you used the graph to find the prices.

18. A cylinder has a radius of $3x - 2$ and a height of $3 - 2x$.
  a. Use 3.14 as π and graph the equation for the volume.
  b. Find the relative maximum.
  c. **Reasoning** What kind of limitation on the radius would make your answer in part (b) the maximum possible volume?

19. **Open-Ended** Write a polynomial function in factored form with at least three zeros that are negative, one of which has multiplicity 2.

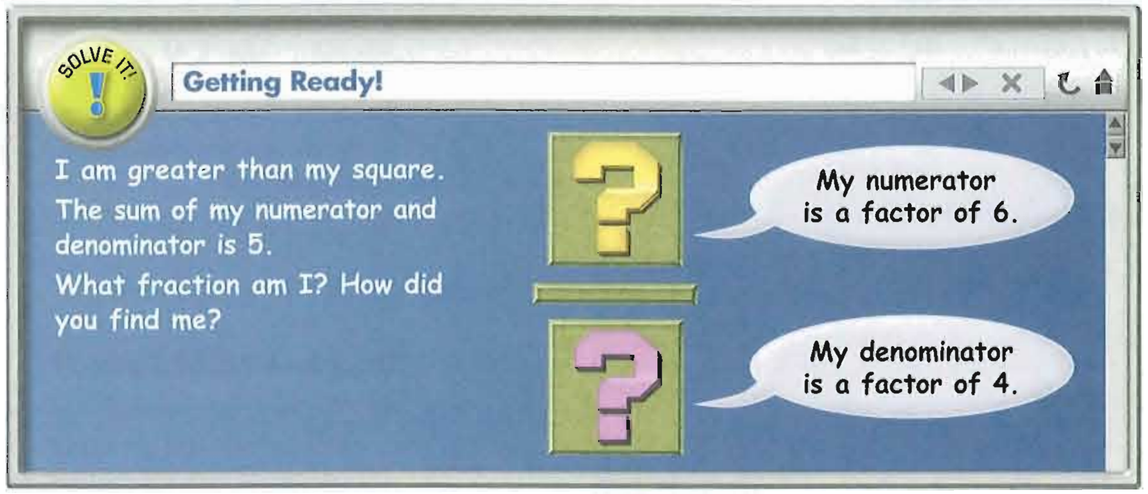
5-5 Theorems About Roots of Polynomial Equations

Sunshine State Standards

MA.912.A.4.6 Use the theorems of polynomial behavior including the Rational Root Theorem, Descartes' Rule of Signs, and the Conjugate Root Theorem.

MA.912.A.4.7 Write a polynomial equation for a given set of real and/or complex roots.

Objectives To solve equations using the Rational Root Theorem
To use the Conjugate Root Theorem



I am greater than my square. The sum of my numerator and denominator is 5.
What fraction am I? How did you find me?

Lesson Vocabulary
- Rational Root Theorem
- Conjugate Root Theorem
- Descartes' Rule of Signs

Factoring the polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ can be challenging, especially when both a_n and a_0 have many factors.

Essential Understanding The factors of the numbers a_n and a_0 in $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ can help you factor $P(x)$ and solve the equation $P(x) = 0$.

One way to find a root of the polynomial equation $P(x) = 0$ is to guess and check. This is inefficient unless there is a way to minimize the number of guesses, or possible roots. The **Rational Root Theorem** does just that.

take note

Theorem Rational Root Theorem

Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. There are a limited number of possible roots of $P(x) = 0$:
- Integer roots must be factors of a_0.
- Rational roots must have reduced form $\frac{p}{q}$ where p is an integer factor of a_0 and q is an integer factor of a_n.

Factors of the leading coefficient: $\pm1, \pm3, \pm7,$ and ±21.

$$21x^2 + 29x + 10 = 0$$

Factors of the constant term: $\pm1, \pm2, \pm5,$ and ±10.

$$x^2 + \frac{29}{21}x + \frac{10}{21} = 0$$

$$\left(x + \frac{2}{3}\right)\left(x + \frac{5}{7}\right) = 0$$

The roots are $-\frac{2}{3}$ and $-\frac{5}{7}$.

Plan

What information can you get from the equation?
The equation gives you the leading coefficient and the constant term.

Problem 1 Finding a Rational Root

What are the rational roots of $2x^3 - x^2 + 2x + 5 = 0$?

The only possible rational roots have the form $\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}$.

The constant factors are ±1, ±5. The leading coefficient factors are ±1, ±2.
The only possible rational roots are ±1, ±5, $\pm\frac{1}{2}$, $\pm\frac{5}{2}$.

The table shows the values of the function $y = P(x)$ for the possible roots.

x	1	−1	5	−5	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$
$P(x)$	8	0	240	−280	6	$\frac{7}{2}$	35	$-\frac{75}{2}$

The only rational root of $2x^3 - x^2 + 2x + 5 = 0$ is -1.

Got It? **1.** What are the rational roots of $3x^3 + 7x^2 + 6x - 8 = 0$?

Once you find one root, use synthetic division to factor the polynomial. Continue finding roots and dividing until you have a second-degree polynomial. Use the Quadratic Formula to find the remaining roots.

Problem 2 Using the Rational Root Theorem

What are the rational roots of $15x^3 - 32x^2 + 3x + 2 = 0$?

Know	Need	Plan
Coefficients and the constant term of the polynomial	The roots of the polynomial equation	• Find one root. • Factor until you get a quadratic. • Use the Quadratic Formula to find the other roots.

Step 1 The constant term factors are ±1 and ±2. The leading coefficient factors are ±1, ±3, ±5, and ±15.

Step 2 The possible rational roots are: $\pm1, \pm2, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{1}{15}$, and $\pm\frac{2}{15}$.

Step 3 Test each possible rational root in $15x^3 - 32x^2 + 3x + 2$ until you find a root.

Test 1: $15(1)^3 - 32(1)^2 + 3(1) + 2 = -12 \neq 0$

Test 2: $15(2)^3 - 32(2)^2 + 3(2) + 2 = 0$ So 2 is a root.

Step 4 Factor the polynomial by using synthetic division:
$P(x) = (x - 2)(15x^2 - 2x - 1)$.

$$\begin{array}{r|rrrr} 2 & 15 & -32 & 3 & 2 \\ & & 30 & -4 & -2 \\ \hline & 15 & -2 & -1 & 0 \end{array}$$

Step 5 Since $15x^2 - 2x - 1 = (5x + 1)(3x - 1)$, the other roots are $-\frac{1}{5}$ and $\frac{1}{3}$.

The rational roots of $15x^3 - 32x^2 + 3x + 2 = 0$ are 2, $-\frac{1}{5}$, and $\frac{1}{3}$.

Got It? **2.** What are the rational roots of $2x^3 + x^2 - 7x - 6 = 0$?

Recall from Lesson 4-8 that the complex numbers $a + bi$ and $a - bi$ are conjugates. Similarly, the irrational numbers $a + \sqrt{b}$ and $a - \sqrt{b}$ are conjugates. If a complex number or an irrational number is a root of a polynomial equation with rational coefficients, so is its conjugate.

take note

Theorem Conjugate Root Theorem

If $P(x)$ is a polynomial with *rational* coefficients, then irrational roots of $P(x) = 0$ that have the form $a + \sqrt{b}$ occur in conjugate pairs. That is, if $a + \sqrt{b}$ is an irrational root with a and b rational, then $a - \sqrt{b}$ is also a root.

If $P(x)$ is a polynomial with *real* coefficients, then the complex roots of $P(x) = 0$ occur in conjugate pairs. That is, if $a + bi$ is a complex root with a and b real, then $a - bi$ is also a root.

Problem 3 Using the Conjugate Root Theorem to Identify Roots

A quartic polynomial $P(x)$ has rational coefficients. If $\sqrt{2}$ and $1 + i$ are roots of $P(x) = 0$, what are the two other roots?

Think

Do you have real coefficients?
All rational numbers are real numbers. Therefore the rational coefficients are real coefficients.

Since $P(x)$ has rational coefficients and $0 + \sqrt{2}$ is a root of $P(x) = 0$, it follows from the Conjugate Root Theorem that $0 - \sqrt{2}$ is also a root.

Since $P(x)$ has real coefficients and $1 + i$ is a root of $P(x) = 0$, it follows that $1 - i$ is also a root.

The two other roots are $-\sqrt{2}$ and $1 - i$.

Got It? 3. A cubic polynomial $P(x)$ has real coefficients. If $3 - 2i$ and $\frac{5}{2}$ are two roots of $P(x) = 0$, what is one additional root?

Problem 4 Using Conjugates to Construct a Polynomial

Multiple Choice **What is a third-degree polynomial function $y = P(x)$ with rational coefficients so that $P(x) = 0$ has roots -4 and $2i$?**

Ⓐ $P(x) = x^3 - 2x^2 - 16x + 32$

Ⓑ $P(x) = x^3 - 4x^2 + 4x - 16$

Ⓒ $P(x) = x^3 + 4x^2 + 4x + 16$

Ⓓ $P(x) = x^3 + 4x^2 - 4x - 16$

Think

Does the Conjugate Root Theorem apply to -4?
No; the theorem does not apply because -4 is neither irrational nor imaginary.

Since $2i$ is a root, then $-2i$ is also a root.

$P(x) = (x + 2i)(x - 2i)(x + 4)$ Write the polynomial function.

$= (x^2 + 4)(x + 4)$ Multiply the complex conjugates.

$= x^3 + 4x^2 + 4x + 16$ Write the polynomial function in standard form.

The equation $x^3 + 4x^2 + 4x + 16 = 0$ has rational coefficients and has roots -4 and $2i$. The correct answer is C.

Got It? 4. What quartic polynomial equation has roots $2 - 3i$, 8, 2?

The French mathematician René Descartes (1596–1650) recognized a connection between the roots of a polynomial equation and the + and − signs of the standard form.

take note

Theorem Descartes' Rule of Signs

Let $P(x)$ be a polynomial with real coefficients written in standard form.

- The number of positive real roots of $P(x) = 0$ is either equal to the number of sign changes between consecutive coefficients of $P(x)$ or is less than that by an even number.
- The number of negative real roots of $P(x) = 0$ is either equal to the number of sign changes between consecutive coefficients of $P(-x)$ or is less than that by an even number.

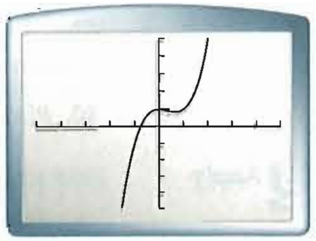
In both cases, count multiple roots according to their multiplicity.

Problem 5 Using Descartes' Rule of Signs

What does Descartes' Rule of Signs tell you about the real roots of $x^3 - x^2 + 1 = 0$?

There are two sign changes, + to − and − to +.
Therefore, there are either 0 or 2 positive real roots.

$P(-x) = (-x)^3 - (-x)^2 + 1 = -x^3 - x^2 + 1 = 0$ has only one sign change − to +. There is one negative real root.



Recall that graphs of cubic functions have zero or two turning points. Because the graph already shows two turning points, it will not change direction again. So there are no positive real roots.

Think

Why can't there be zero negative real roots?
The number of negative roots is equal to 1 or is less than 1 by an even number. Zero is less than 1 by an odd number.

Got It? **5. a.** What does Descartes' Rule of Signs tell you about the real roots of $2x^4 - x^3 + 3x^2 - 1 = 0$?

b. Reasoning Can you confirm real and complex roots graphically? Explain.

Lesson Check

Do you know HOW?

Use the Rational Root Theorem to list all possible rational roots for each equation.

1. $x^2 + x - 2 = 0$

2. $2x^3 - x^2 - 6 = 0$

3. $3x^4 + 2x^2 - 12 = 0$

Write a polynomial function with rational coefficients so that $P(x) = 0$ has the given roots.

4. 5 and 9

5. −4 and $2i$

Do you UNDERSTAND?

6. Vocabulary Give an example of a conjugate pair.

7. Reasoning In the statements below, r and s represent integers. Is each statement *always*, *sometimes*, or *never* true? Explain.

a. A root of the equation $3x^3 + rx^2 + sx + 8 = 0$ could be 5.

b. A root of the equation $3x^3 + rx^2 + sx + 8 = 0$ could be −2.

8. Error Analysis A student claims that $-4i$ is the only imaginary root of a polynomial equation that has real coefficients. What is the student's mistake?

Practice and Problem-Solving Exercises

A Practice

Use the Rational Root Theorem to list all possible rational roots for each equation. Then find any actual rational roots.

See Problems 1 and 2.

9. $x^3 - 4x + 1 = 0$

10. $x^3 + 2x - 9 = 0$

11. $2x^3 - 5x + 4 = 0$

12. $3x^3 + 9x - 6 = 0$

13. $4x^3 + 2x - 12 = 0$

14. $6x^3 + 2x - 18 = 0$

15. $7x^3 - x^2 + 4x + 10 = 0$

16. $8x^3 + 2x^2 - 5x + 1 = 0$

17. $10x^3 - 7x^2 + x - 10 = 0$

A polynomial function $P(x)$ with rational coefficients has the given roots. Find two additional roots of $P(x) = 0$.

See Problem 3.

18. $-2i$ and $\sqrt{10}$

19. $14 - \sqrt{2}$ and $-6i$

20. i and $7 + 8i$

21. $-\sqrt{3}$ and $5 - \sqrt{11}$

Write a polynomial function with rational coefficients so that $P(x) = 0$ has the given roots.

See Problem 4.

22. 7 and 12

23. -9 and -15

24. $-10i$

25. $3i + 9$

26. 4, 16, and $1 + 19i$

27. $13i$ and $5 + 10i$

28. $11 - 2i$ and $8 + 13i$

29. $17 - 4i$ and $12 + 5i$

What does Descartes' Rule of Signs say about the number of positive real roots and negative real roots for each polynomial function?

See Problem 5.

30. $P(x) = x^2 + 5x + 6$

31. $P(x) = 9x^3 - 4x^2 + 10$

32. $P(x) = 8x^3 + 2x^2 - 14x + 5$

B Apply

Find all rational roots for $P(x) = 0$.

33. $P(x) = 2x^3 - 5x^2 + x - 1$

34. $P(x) = 6x^4 - 13x^3 + 13x^2 - 39x - 15$

35. $P(x) = 7x^3 - x^2 - 5x + 14$

36. $P(x) = 3x^4 - 7x^3 + 10x^2 - x + 12$

37. $P(x) = 6x^4 - 7x^2 - 3$

38. $P(x) = 2x^3 - 3x^2 - 8x + 12$

Write a polynomial function $P(x)$ with rational coefficients so that $P(x) = 0$ has the given roots.

39. -6, 3, and $-15i$

40. $4 + \sqrt{5}$ and $8i$

41. $-5 - 7i$ and $2 - \sqrt{11}$

42. **Think About a Plan** You are building a square pyramid out of clay and want the height to be 0.5 cm shorter than twice the length of each side of the base. If you have 18 cm^3 of clay, what is the greatest height you could use for your pyramid?

- How can drawing a diagram help you solve this problem?
- What is the formula for the volume of a pyramid?
- What equation can you solve to find the height of the pyramid?

43. **Error Analysis** Your friend is using Descartes' Rule of Signs to find the number of negative real roots of $x^3 + x^2 + x + 1 = 0$. Describe and correct the error.

~~$P(-x) = (-x)^3 + (-x)^2 + (-x) + 1$~~
~~$= -x^3 - x^2 - x + 1$~~

~~Because there is only one sign change in P(-x), there must be one negative real root.~~

44. **Reasoning** A quartic equation with integer coefficients has two real roots and one imaginary root. Explain why the fourth root must be imaginary.

45. **Gardening** A gardener is designing a new garden in the shape of a trapezoid. She wants the shorter base to be twice the height and the longer base to be 4 feet longer than the shorter base. If she has enough topsoil to create a 60 ft^2 garden, what dimensions should she use for the garden?

46. **Open-Ended** Write a fourth-degree polynomial equation with integer coefficients that has two irrational roots and two imaginary roots.

Challenge

47. a. Find a polynomial equation in which $1 + \sqrt{2}$ is the only root.
b. Find a polynomial equation with root $1 + \sqrt{2}$ of multiplicity 2.
c. Find c such that $1 + \sqrt{2}$ is a root of $x^2 - 2x + c = 0$.

48. a. Using *real* and *imaginary* as types of roots, list all possible combinations of root type for a fourth-degree polynomial equation.
b. Repeat the process for a fifth-degree polynomial equation.
c. **Make a Conjecture** Make a conjecture about the number of real roots of an odd-degree polynomial equation.

49. **Writing** A student states that $2 + \sqrt{3}$ is a root of $x^2 - 2x - (3 + 2\sqrt{3}) = 0$. The student claims that $2 - \sqrt{3}$ is another root of the equation by the Conjugate Root Theorem. Explain how you would respond to the student.

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.4.6 **50.** What is a positive root of $-5x^3 - 2x^2 + 9x + 30 = 0$?

MA.912.A.4.4 **51.** What is the remainder when you divide $x^3 + 2x^2 - x - 6$ by $x - 1$?

MA.912.A.4.7 **52.** A polynomial with rational coefficients has roots $-3i$ and $8 + \sqrt{7}$. What is the minimum degree of the polynomial?

MA.912.A.3.14 **53.** What is the value of y in the solution of the system of equations?
$$\begin{cases} 10x + 24y = 9 \\ 8x + 60y = 14 \end{cases}$$

MA.912.A.7.3 **54.** What is the value of the greater solution of the equation $6x^2 - 17x + 5 = 0$?

Mixed Review

Divide.

See Lesson 5-4.

55. $(x^3 + 5x^2 - 3) \div (x - 1)$

56. $(8x^3 + 12x^2 + 7) \div (x + 6)$

57. $(7x^2 + 11x - 4) \div (x + 2)$

Solve.

See Lesson 4-8.

58. $7x^2 + 63 = 0$

59. $x^2 + 81 = 0$

60. $2x^2 + 288 = 0$

Get Ready! **To prepare for Lesson 5-6, do Exercises 61 and 62.**

Write each polynomial in standard form. Then classify it by degree and by number of terms.

See Lesson 5-1.

61. $6x^2 + 11 - 5x^4 + 9x$

62. $13x - 4x^5 + 7x^3 + 2$

Concept Byte

Use With Lesson 5-5

EXTENSION

Solving Polynomial Inequalities

Sunshine State Standard

MA.912.A.4.11 Solve a polynomial inequality by examining the graph with and without the use of technology.

You can use what you know about factoring and graphing polynomials to help you solve polynomial inequalities.

Example 1

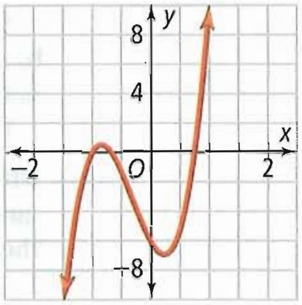
Solve $12x^3 + 11x^2 - 7x - 6 < 0$.

By the Rational Root Theorem and the Factor Theorem, $(x + 1)$ is one factor.

So, $12x^3 + 11x^2 - 7x - 6 = (x + 1)(12x^2 - x - 6)$

$$= (x + 1)(3x + 2)(4x - 3)$$

The zeros of the polynomial are -1, $-\frac{2}{3}$, and $\frac{3}{4}$.



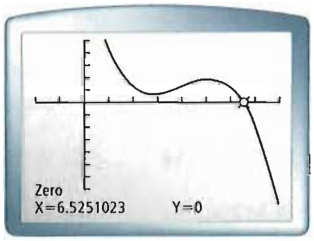
You know the graph of a cubic polynomial with a positive leading coefficient has the shape shown at the right. The polynomial is less than 0 where the graph is below the x-axis.

Thus, $x < -1$ and $-\frac{2}{3} < x < \frac{3}{4}$ is the solution of the inequality.

Example 2

Solve $-x^3 + 12x^2 - 44x + 54 > 0$.

You know the shape of the graph. Use a graphing calculator to find the only zero at $x = 6.53$.



Zero
X=6.5251023 Y=0

The graph is above the x-axis when $x < 6.53$. Therefore, the solution of the inequality is $x < 6.53$.

Exercises

Solve each inequality.

1. $8x^2 - 2x - 21 > 0$

2. $-16x^2 + 104x - 169 < 0$

3. $x^3 - 20x^2 - 100x + 2000 > 0$

4. $-10x^3 + x^2 - 10x + 1 > 0$

5. $x(x^2 + 1) > 2(2x^2 - 3)$

6. $7x^3 \leq x^2 + 66x$

7. $x^3 - 11x^2 + 32x - 28 \geq 0$

8. $11x^3 - 76x^2 + 80x + 192 > 0$

9. **Reasoning** If $Q(x)$ is a quartic polynomial, one possible solution of $Q(x) < 0$ is an interval of numbers, not including endpoints. Describe other possible solutions.

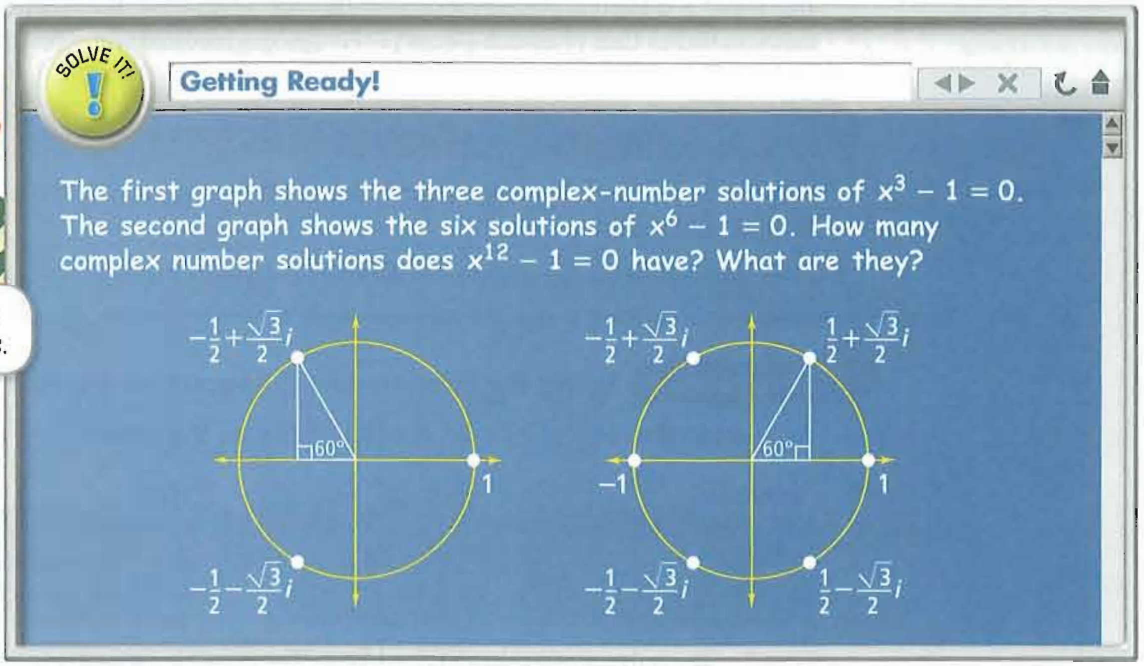
5-6 The Fundamental Theorem of Algebra

Sunshine State Standard

MA.912.A.4.6 Use the Fundamental Theorem of Algebra.

Objective To use the Fundamental Theorem of Algebra to solve polynomial equations with complex solutions


These graphs are on complex planes.



The first graph shows the three complex-number solutions of $x^3 - 1 = 0$. The second graph shows the six solutions of $x^6 - 1 = 0$. How many complex number solutions does $x^{12} - 1 = 0$ have? What are they?

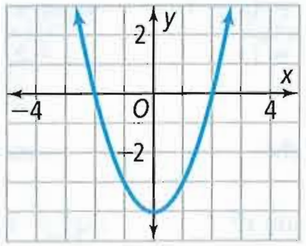
Lesson Vocabulary

- Fundamental Theorem of Algebra

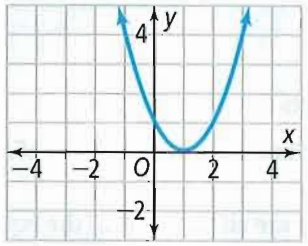
You can factor any polynomial of degree n into n linear factors, but sometimes the factors will involve imaginary numbers.

Essential Understanding The degree of a polynomial equation tells you how many roots the equation has.

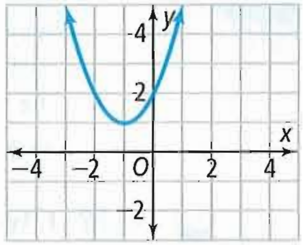
It is easy to see graphically that every polynomial function of degree 1 has a single zero, the x-intercept. However, there appear to be three possibilities for polynomials of degree 2. They correspond to these three graphs:



$y = x^2 - 4$
Two real zeros



$y = x^2 - 2x + 1$
One real zero



$y = x^2 + 2x + 2$
No real zeros

However, by factoring, you can see that each related equation has two roots.

$x^2 - 4 = (x - 2)(x + 2) = 0$	*two* real roots, 2 and −2
$x^2 - 2x + 1 = (x - 1)(x - 1) = 0$	a root of multiplicity *two* at 1
$x^2 + 2x + 2 = (x - (-1 + i))(x - (-1 - i)) = 0$	*two* complex roots, $-1 + i$ and $-1 - i$

Every quadratic polynomial equation has two roots, every cubic polynomial equation has three roots, and so on.

This result is related to the *Fundamental Theorem of Algebra.* The German mathematician Carl Friedrich Gauss (1777–1855) is credited with proving this theorem.

take note

Theorem The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly n roots, including multiple and complex roots.

ONLINE PROBLEMS

Problem 1 Using the Fundamental Theorem of Algebra

What are all the roots of $x^5 - x^4 - 3x^3 + 3x^2 - 4x + 4 = 0$?

Know

The polynomial equation has degree 5. There are 5 roots.

Need

The zeros of the function

Plan

Use the Rational Root and Factor Theorems, synthetic division, and factoring.

Think

How many linear factors will there be?

If there are five roots, there must be five linear factors.

Step 1 The polynomial is in standard form. The possible rational roots are ±1, ±2, ±4.

Step 2 Evaluate the related polynomial function for $x = 1$. Since $P(1) = 0$, 1 is a root and $x - 1$ is a factor. Use synthetic division to factor out $x - 1$:

$$\begin{array}{r|rrrrrr} 1 & 1 & -1 & -3 & 3 & -4 & 4 \\ & & 1 & 0 & -3 & 0 & -4 \\ \hline & 1 & 0 & -3 & 0 & -4 & 0 \end{array}$$

Step 3 Continue factoring until you have five linear factors.

$$\begin{aligned} x^5 - x^4 - 3x^3 + 3x^2 - 4x + 4 &= (x - 1)(x^4 - 3x^2 - 4) \\ &= (x - 1)(x^2 - 4)(x^2 + 1) \\ &= (x - 1)(x - 2)(x + 2)(x - i)(x + i) \end{aligned}$$

Step 4 The roots are 1, 2, −2, i, and $-i$.

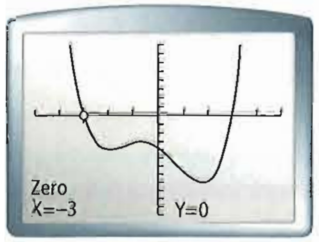
By the Fundamental Theorem of Algebra, these are the only roots.

Got It? **1.** What are all the roots of the equation $x^4 + 2x^3 = 13x^2 - 10x$?

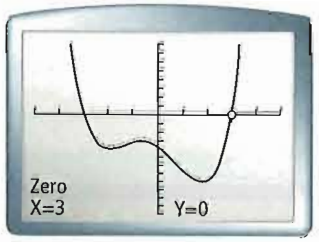
Problem 2 Finding All the Zeros of a Polynomial Function

What are the zeros of $f(x) = x^4 + x^3 - 7x^2 - 9x - 18$?

Step 1 Use a graphing calculator to find any real roots. The graph of $y = x^4 + x^3 - 7x^2 - 9x - 18$ shows real zeros at $x = -3$ and $x = 3$.



Zero
X=-3
Y=0



Zero
X=3
Y=0

Think

Does the graph show all of the real roots?
Yes; the graphs of quartic functions have one or three turning points. Since the graph shows three turning points, it will not turn again to cross the *x*-axis a third time.

Step 2 Factor out the linear factors $x + 3$ and $x - 3$. Use synthetic division twice.

$$\begin{array}{r|rrrrr} -3 & 1 & 1 & -7 & -9 & -18 \\ & & -3 & 6 & 3 & 18 \\ \hline & 1 & -2 & -1 & -6 & 0 \end{array} \qquad \begin{array}{r|rrrr} 3 & 1 & -2 & -1 & -6 \\ & & 3 & 3 & 6 \\ \hline & 1 & 1 & 2 & 0 \end{array}$$

$$\begin{aligned} x^4 + x^3 - 7x^2 - 9x - 18 &= (x+3)(x^3 - 2x^2 - x - 6) \\ &= (x+3)(x-3)(x^2 + x + 2) \end{aligned}$$

Step 3 Use the Quadratic Formula. Find the complex roots of $x^2 + x + 2 = 0$.

$a = 1, b = 1, c = 2$ Identify the values of *a*, *b*, and *c*.

$\frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$ Substitute.

$\frac{-1 \pm \sqrt{-7}}{2}$ Simplify.

The complex roots are $\frac{-1 + i\sqrt{7}}{2}$ and $\frac{-1 - i\sqrt{7}}{2}$.

Step 4 The four zeros of the function are -3, 3, $\frac{-1 + i\sqrt{7}}{2}$, and $\frac{-1 - i\sqrt{7}}{2}$.

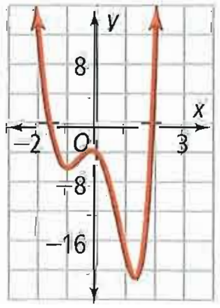
By the Fundamental Theorem of Algebra, there can be no other zeros.

Got It? **2. a.** What are all the zeros of the function $g(x) = 2x^4 - 3x^3 - x - 6$?

b. Reasoning The graph of $f(x) = x^5 + 4x^4 - 3x^3 - 12x^2 - 4x - 4$ is shown at the right.

i. Use the turning points to explain why the graph does NOT show all of the real zeros of the function.

ii. The graph of $g(x) = f(x) + 4$ is a translation of the graph of *f* up 4 units. How many real zeros of *g* will the graph of *g* show? Explain.



take note

Concept Summary The Fundamental Theorem of Algebra

Here are equivalent ways to state the Fundamental Theorem of Algebra. You can use any one of these statements to prove the others.

- Every polynomial equation of degree $n \geq 1$ has exactly n roots, including multiple and complex roots.
- Every polynomial of degree $n \geq 1$ has n linear factors.
- Every polynomial function of degree $n \geq 1$ has at least one complex zero.

Lesson Check

Do you know HOW?

Find the number of roots for each equation.

1. $5x^4 + 12x^3 - x^2 + 3x + 5 = 0$

2. $-x^{14} - x^8 - x + 7 = 0$

Find all the zeros for each function.

3. $y = x^3 - 5x^2 + 16x - 80$

4. $y = x^4 - 2x^3 + x^2 - 2x$

Do you UNDERSTAND?

5. Vocabulary Given a polynomial equation of degree n, explain how you determine the number of roots of the equation.

6. Open-Ended Write a polynomial function of degree 4 with rational coefficients and two complex zeros of multiplicity 2.

7. Writing Describe when to use synthetic division and when to use the Quadratic Formula to determine the linear factors of a polynomial.

Practice and Problem-Solving Exercises

A Practice

Without using a calculator, find all the roots of each equation.

See Problem 1.

8. $x^3 - 3x^2 + x - 3 = 0$

9. $x^3 + x^2 + 4x + 4 = 0$

10. $x^3 + 4x^2 + x - 6 = 0$

11. $x^3 - 5x^2 + 2x + 8 = 0$

12. $x^4 + 4x^3 + 7x^2 + 16x + 12 = 0$

13. $x^4 - 4x^3 + x^2 + 12x - 12 = 0$

14. $x^5 + 3x^3 - 4x = 0$

15. $x^5 - 8x^3 - 9x = 0$

Find all the zeros of each function.

See Problem 2.

16. $y = 2x^3 + x^2 + 1$

17. $f(x) = x^3 - 3x^2 + x - 3$

18. $g(x) = x^3 - 5x^2 + 5x - 4$

19. $y = x^3 - 2x^2 - 3x + 6$

20. $y = x^4 - 6x^2 + 8$

21. $f(x) = x^4 - 3x^2 - 4$

22. $y = x^3 - 3x^2 - 9x$

23. $y = x^3 + 6x^2 + x + 6$

24. $y = x^4 + 3x^3 + x^2 - 12x - 20$

25. $y = x^4 + x^3 - 15x^2 - 16x - 16$

B Apply

For each equation, state the number of complex roots, the possible number of real roots, and the possible rational roots.

26. $2x^4 - x^3 + 2x^2 + 5x - 26 = 0$

27. $x^5 - x^3 - 11x^2 + 9x + 18 = 0$

28. $-12 + x + 10x^2 + 3x^3 = 0$

29. $4x^6 - x^5 - 24 = 0$

Find all the zeros of each function.

30. $y = x^3 - 4x^2 + 9x - 36$

31. $f(x) = x^3 + 2x^2 - 5x - 10$

32. $y = 2x^3 - 3x^2 - 18x - 8$

33. $y = 3x^3 - 7x^2 - 14x + 24$

34. $g(x) = x^3 - 4x^2 - x + 22$

35. $y = x^3 - x^2 - 3x - 9$

36. $y = x^4 - x^3 - 5x^2 - x - 6$

37. $y = 2x^4 + 3x^3 - 17x^2 - 27x - 9$

38. **Think About a Plan** A polynomial function, $f(x) = x^4 - 5x^3 - 28x^2 + 188x - 240$, is used to model a new roller coaster section. The loading zone will be placed at one of the zeros. The function has a zero at 5. What are the possible locations for the loading zone?

- Can you determine how many zeros you need to find?
- How can you use polynomial division?
- What other methods can be helpful?

39. **Bridges** A twist in a river can be modeled by the function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - x$, $-3 \leq x \leq 2$. A city wants to build a road that goes directly along the x-axis. How many bridges would it have to build?

40. **Error Analysis** Maurice says: "Every linear function has exactly one zero. It follows from the Fundamental Theorem of Algebra." Cheryl disagrees. "What about the linear function $y = 2$?" she asks. "Its graph is a line, but it has no x-intercept." Whose reasoning is incorrect? Where is the flaw?

Determine whether each of the following statement is *always*, *sometimes*, or *never* true.

41. A polynomial function with real coefficients has real zeros.

42. Polynomial functions with complex coefficients have one complex zero.

43. A polynomial function that does not intercept the x-axis has complex roots only.

44. **Reasoning** A 4th degree polynomial function has zeros at 3 and $5 - i$. Can $4 + i$ also be a zero of the function? Explain your reasoning.

45. **Open-Ended** Write a polynomial function that has four possible rational zeros but no actual rational zeros.

46. Three roots of a polynomial equation with real coefficients are 3, $5 - 3i$, and $-3i$. Which number MUST also be a root of the equation?

I. -3 II. $5 + 3i$ III. $3i$

Ⓐ II only Ⓑ I and II only Ⓒ II and III only Ⓓ I, II, and III

Challenge

47. Use the Fundamental Theorem of Algebra and the Conjugate Root Theorem to show that any odd degree polynomial equation with real coefficients has at least one real root.

48. **Reasoning** What is the maximum number of points of intersection between the graphs of a quartic and a quintic polynomial function?

49. **Reasoning** What is the minimal degree of a polynomial with rational coefficients which has leading coefficient 1, constant term 5, and zeros at $\sqrt{2}$ and $\sqrt{3}$? Show that such minimal degree polynomial has a rational zero and indicate this zero.

Sunshine State Standards Practice

MA.912.A.4.6

50. How many roots does $f(x) = x^4 + 5x^3 + 3x^2 + 2x + 6$ have?

Ⓐ 5 Ⓑ 4 Ⓒ 3 Ⓓ 2

MA.912.A.2.10

51. Which translation takes $y = |x + 2| - 1$ to $y = |x| + 2$?

Ⓕ 2 units right, 3 units down

Ⓖ 2 units right, 3 units up

Ⓗ 2 units left, 3 units up

Ⓘ 2 units left, 3 units down

MA.912.A.4.3

52. What is the factored form of the expression $x^4 - 3x^3 + 2x^2$?

Ⓐ $x^2(x - 1)(x + 2)$

Ⓑ $x^2(x + 1)(x + 2)$

Ⓒ $x^2(x + 1)(x - 2)$

Ⓓ $x^2(x - 1)(x - 2)$

MA.912.A.3.14

53. **Short Response** How would you test whether (2, −2) is a solution of the system? $\begin{cases} y < -2x + 3 \\ y \geq x - 4 \end{cases}$

Mixed Review

54. Find a fourth-degree polynomial equation with real coefficients that has $2i$ and $-3 + i$ as roots.

See Lesson 5-5.

Solve each equation using the Quadratic Formula.

See Lesson 4-7.

55. $x^2 - 6x + 1 = 0$

56. $2x^2 + 5x = -9$

57. $2(x^2 + 2) = 3x$

Determine whether a quadratic model exists for each set of values. If so, write the model.

See Lesson 4-3.

58. $f(-1) = 0, f(2) = 3, f(1) = 4$

59. $f(-4) = 11, f(-5) = 5, f(-6) = 3$

Get Ready! **To prepare for Lesson 5-7, do Exercises 60–65.**

See Lesson 4-2.

Write each polynomial in standard form.

60. $(x + 1)^3$

61. $(x - 3)^3$

62. $(x - 2)^4$

63. $(x - 1)^2$

64. $(x + 5)^3$

65. $(4 - x)^3$

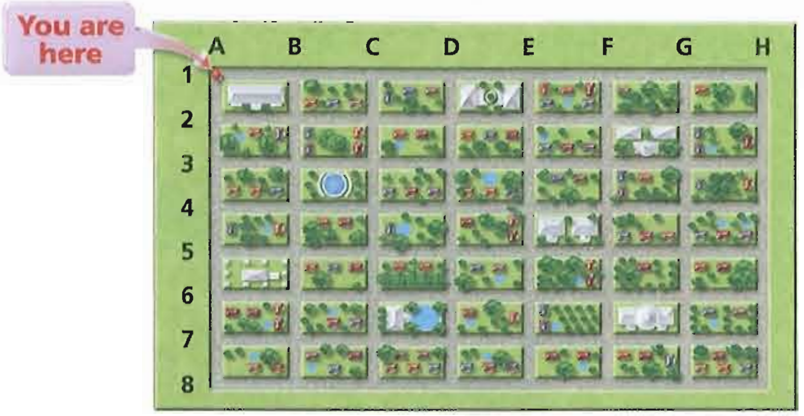
Concept Byte

For Use With Lesson 5-7

Pascal's Triangle

Sunshine State Standard
Prepares for MA.912.A.4.12 Apply the Binomial Thereom.

You are standing at the corner of the grid shown at the right (point A1). You are only permitted to travel down or to the right. The only way to get to point A2 is by traveling down one unit. To get to point B1, travel to the right one unit. There is one way to get to point B1. There is also one way to get to point A2, however there are 2 ways to get to point B2.



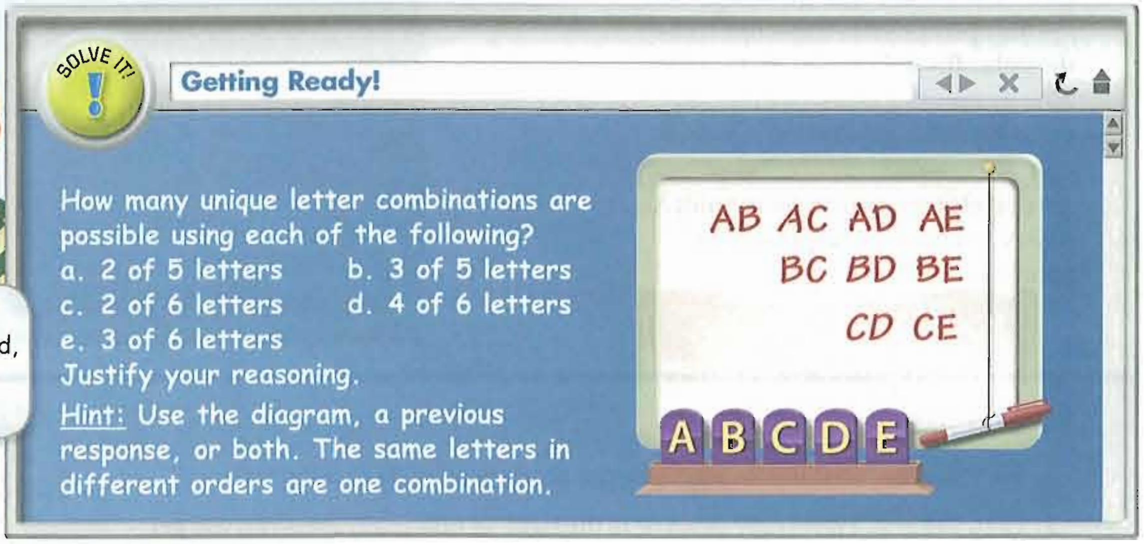
Exercises

1. What are the two different ways you can get from point A1 to point B2?
2. Copy the grid. Travel only down or to the right. In how many ways can you get from point A1 to point A3?
3. Use your copy of the grid from Exercise 2. In how many ways can you get from point A1 to point C2?
4. In how many ways can you get to point B5 from your starting point, A1?
5. Mark the number of ways you can get to each point on the grid from point A1.
6. **Reasoning** Describe any patterns you see in the numbers on the grid.
7. Describe how the numbers increase in each row. How can you tell the next number in each row without counting the number of paths from A1?
8. **a.** Make a copy of your completed grid. Color a shaded square around every number that is a multiple of 2. (You may need to extend the grid to see a pattern.)
   b. The pattern you see in part (a) resembles the Sierpinski triangle. Look at page 10, exercise 42 to see a Sierpinski triangle. Find another way to describe how to obtain this pattern.
9. **Writing** The completed grid is called Pascal's Triangle. Rotate your copy of the grid 45° clockwise so that point A1 is at the top. Explain why the grid is called a triangle.
10. Find the sum of the rows of the rotated grid. (These would be diagonals in the original grid.) What pattern do you find?

5-7 The Binomial Theorem

Sunshine State Standard
MA.912.A.4.12 Apply the Binomial Theorem.

Objectives To expand a binomial using Pascal's Triangle
To use the Binomial Theorem



How many unique letter combinations are possible using each of the following?
a. 2 of 5 letters b. 3 of 5 letters
c. 2 of 6 letters d. 4 of 6 letters
e. 3 of 6 letters
Justify your reasoning.
Hint: Use the diagram, a previous response, or both. The same letters in different orders are one combination.

Lesson Vocabulary
- expand
- Pascal's Triangle
- Binomial Theorem

There is a connection between the triangular pattern of numbers in the Solve It and the expansion of $(a + b)^n$.

Essential Understanding You can use a pattern of coefficients and the pattern

$$a^n, a^{n-1}b, a^{n-2}b^2, \dots, a^2b^{n-2}, ab^{n-1}, b^n \text{ to write the expansion of } (a + b)^n.$$

You can *expand* $(a + b)^3$ using the Distributive Property.

$$(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$$

To **expand** the power of a binomial in general, first multiply as needed. Then write the polynomial in standard form.

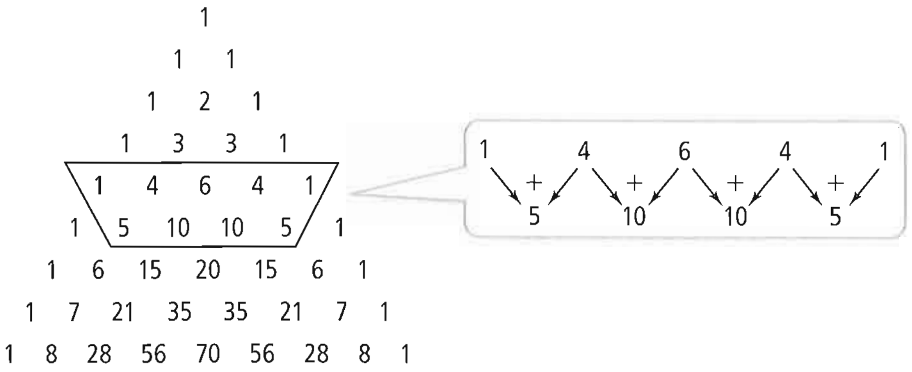
Consider the expansions of $(a + b)^n$ for the first few values of n:

Row	Power	Expanded Form	Coefficients Only
0	$(a + b)^0$	1	1
1	$(a + b)^1$	$1a^1 + 1b^1$	1 1
2	$(a + b)^2$	$1a^2 + 2a^1b^1 + 1b^2$	1 2 1
3	$(a + b)^3$	$1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$	1 3 3 1
4	$(a + b)^4$	$1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$	1 4 6 4 1

The "coefficients only" column matches the numbers in *Pascal's Triangle.* **Pascal's Triangle**, named for the French mathematician Blaise Pascal (1623–1662), is a triangular array of numbers in which the first and last number of each row is 1. Each of the other numbers in the row is the sum of the two numbers above it.

For example, to generate row 5, use the sums of the adjacent elements in the row above it.

Pascal's Triangle



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Problem 1 Using Pascal's Triangle

What is the expansion of $(a + b)^6$? Use Pascal's Triangle.

Plan

What row of Pascal's Triangle should you use for this expansion?
The expression is raised to the 6th power so use the 6th row.

The exponents for *a* begin with 6 and decrease to 0.

$$1a^6b^0 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + 1a^0b^6$$

The exponents for *b* begin with 0 and increase to 6.

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

Got It? **1.** What is the expansion of $(a + b)^8$? Use Pascal's Triangle.

The **Binomial Theorem** gives a general formula for expanding a binomial.

take note

Theorem Binomial Theorem

For every positive integer *n*,

$$(a + b)^n = P_0a^n + P_1a^{n-1}b + P_2a^{n-2}b^2 + \cdots + P_{n-1}ab^{n-1} + P_nb^n$$

where $P_0, P_1, \ldots, P_n$ are the numbers in the *n*th row of Pascal's Triangle.

When you use the Binomial Theorem to expand $(x - 2)^4$, $a = x$ and $b = -2$. To expand a binomial such as $(3x - 2)^5$, $a = 3x$ so remember that $a^4 = (3x)^4$ not $3x^4$.

Problem 2 Expanding a Binomial

What is the expansion of $(3x - 2)^5$? Use the Binomial Theorem.

Think	Write
For $(3x - 2)^5$, use the 5th row of Pascal's Triangle.	**Pascal's Triangle** $\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ \boxed{1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1} \end{array}$
The Binomial Theorem uses a binomial sum.	$(3x - 2)^5 = (3x + (-2))^5$
Apply the Binomial Theorem.	$= (3x)^5 + 5(3x)^4(-2)^1 + 10(3x)^3(-2)^2$ $+ 10(3x)^2(-2)^3 + 5(3x)^1(-2)^4 + 1(-2)^5$
Simplify.	$= 243x^5 - 810x^4 + 1080x^3 - 720x^2 + 240x - 32$

Got It? **2. a.** What is the expansion of $(2x - 3)^4$? Use the Binomial Theorem.

b. Reasoning Consider the following:

$11^0 = 1 \qquad 11^1 = 11 \qquad 11^2 = 121 \qquad 11^3 = 1331 \qquad 11^4 = 14641$

Why do these powers of 11 have digits that mirror Pascal's Triangle?

Lesson Check

Do you know HOW?

Use Pascal's Triangle to expand each binomial.

1. $(x + a)^3$

2. $(x - 2)^5$

3. $(2x + 4)^2$

4. $(3a - 2)^3$

Do you UNDERSTAND?

5. Vocabulary Tell whether each expression can be expanded using the Binomial Theorem.

a. $(2a - 6)^4$ **b.** $(5x^2 + 1)^5$ **c.** $(x^2 - 3x - 4)^3$

6. Writing Describe the relationship between Pascal's Triangle and the Binomial Theorem.

7. Reasoning Using Pascal's Triangle, determine the number of terms in the expansion of $(x + a)^{12}$. How many terms are there in the expansion of $(x + a)^n$?

Practice and Problem-Solving Exercises

A Practice

Expand each binomial.

See Problems 1 and 2.

8. $(x - y)^3$
9. $(a + 2)^4$
10. $(6 + a)^6$
11. $(x - 5)^3$
12. $(y + 1)^8$
13. $(x + 2)^{10}$
14. $(b - 4)^7$
15. $(b + 3)^9$
16. $(2x - y)^7$
17. $(a + 3b)^4$
18. $(4x + 2)^6$
19. $(4 - x)^8$
20. $(4x + 5)^2$
21. $(3a - 7)^3$
22. $(2a + 16)^6$
23. $(3y - 11)^4$

B Apply

24. **Think About a Plan** The side length of a cube is $\left(x^2 - \frac{1}{2}\right)$. Determine the volume of the cube.

- Rewrite the binomial as a sum.
- Consider $(a + b)^n$. Identify a and b in the given binomial.
- Which row of Pascal's Triangle can be used to expand the binomial?

25. In the expansion of $(2m - 3n)^9$, one of the terms contains m^3.
  a. What is the exponent of n in this term?
  b. What is the coefficient of this term?

Find the specified term of each binomial expansion.

26. Fourth term of $(x + 2)^5$
27. Third term of $(x - 3)^6$
28. Third term of $(3x - 1)^5$
29. Fifth term of $(a + 5b^2)^4$

30. **Reasoning** Explain why the coefficients in the expansion of $(x + 2y)^3$ do not match the numbers in the 3rd row of Pascal's Triangle.

31. **Compare and Contrast** What are the benefits and challenges of using the Binomial Theorem when expanding $(2x + 3)^2$? Using FOIL? Which method would you choose when expanding $(2x + 3)^6$? Why?

Expand each binomial.

32. $(2x - 2y)^6$
33. $(x^2 + 4)^{10}$
34. $(x^2 - y^2)^3$
35. $(a - b^2)^5$
36. $(3x + 8y)^3$
37. $(4x - 7y)^4$
38. $(7a + 2y)^{10}$
39. $(4x^3 + 2y^2)^6$
40. $(3b - 36)^7$
41. $(5a + 2b)^3$
42. $(b^2 - 2)^8$
43. $(-2y^2 + x)^5$

44. **Geometry** The side length of a cube is given by the expression $(2x + 8)$. Write a binomial power for the area of a face of the cube and for the volume of the cube. Then use the Binomial Theorem to expand and rewrite the powers in standard form.

45. **Writing** Explain why the terms of $(x - y)^n$ have alternating positive and negative signs.

46. **Error Analysis** A student expands $(3x - 8)^4$ as shown below. Describe and correct the student's error.

~~$(3x - 8)^4 = (3x)^4 + 4(3x)^3(-8) + 6(3x)^2(-8)^2 + 4(3x)(-8)^3 + (-8)^4$~~
~~$= 3x^4 - 96x^3 + 1152x^2 - 6144x + 4096$~~

Challenge **Use the Binomial Theorem to expand each complex expression.**

47. $(7 + \sqrt{-16})^5$ **48.** $(\sqrt{-81} - 3)^3$ **49.** $(x^2 - i)^7$

50. The first term in the expansion of a binomial $(ax + by)^n$ is $1024x^{10}$. Find a and n.

51. Determine the coefficient of x^7y in the expansion of $\left(\frac{1}{2}x + \frac{1}{4}y\right)^8$.

52. a. Expand $(1 + i)^4$.
b. Verify that $1 - i$ is a fourth root of -4 by repeating the process in part (a) for $(1 - i)^4$.

53. Verify that $-1 + \sqrt{3}i$ is a cube root of 8 by expanding $(-1 + \sqrt{3}i)^3$.

Sunshine State Standards Practice

MA.912.A.4.12

54. What is the fourth term in the expansion of $(2a + 4b)^5$?

Ⓐ $256a^4b$ Ⓑ $768a^3b^2$ Ⓒ $2560a^2b^3$ Ⓓ $2048ab^4$

MA.912.A.2.12

55. Suppose y varies directly with x. If x is 30 when y is 10, what is x when y is 9?

Ⓕ 3 Ⓖ 27 Ⓗ 29 Ⓘ $\frac{300}{9}$

MA.912.A.7.5

56. Which of following is a root of $9x^2 - 30x + 25 = 0$?

Ⓐ $x = \frac{3}{5}$ Ⓑ $x = \frac{5}{3}$ Ⓒ $x = -\frac{5}{3}$ Ⓓ $x = -\frac{3}{5}$

MA.912.A.3.15

57. Extended Response One company charges a monthly fee of \$7.95 and \$2.25 per hour for Internet access. Another company does not charge a monthly fee, but charges \$2.75 per hour for Internet access. Write a system of equations to represent the cost c for t hours of access in one month for each company. Then find how many hours of use it will take for the costs to be equal.

Mixed Review

Find all the roots of each equation. **See Lesson 5-6.**

58. $x^4 + 7x^3 + 20x^2 + 29x + 15 = 0$ **59.** $x^5 - x^4 + 10x^3 - 10x^2 + 9x - 9 = 0$

60. $2x^3 + 11x^2 + 14x + 8 = 0$ **61.** $x^4 - x^3 + 6x^2 - 13x + 7 = 0$

Simplify each expression. **See Lesson 4-8.**

62. $(5i - 4)(-2i + 7)$ **63.** $(-3i)(20i)(10i)$

64. $\frac{-6 - 2i}{3 + i}$ **65.** $\frac{11i + 9}{2 - i}$

Get Ready! **To prepare for Lesson 5-8, do Exercises 66–68.**

Write each polynomial in standard form. Then classify it by degree and by number of terms. **See Lesson 5-1.**

66. $5x^2 - x + 2x^3 + 9$ **67.** $1 + 4x - 7x^2$ **68.** $-9x^2 + x - 3x^3 - 8 + 12x^4$

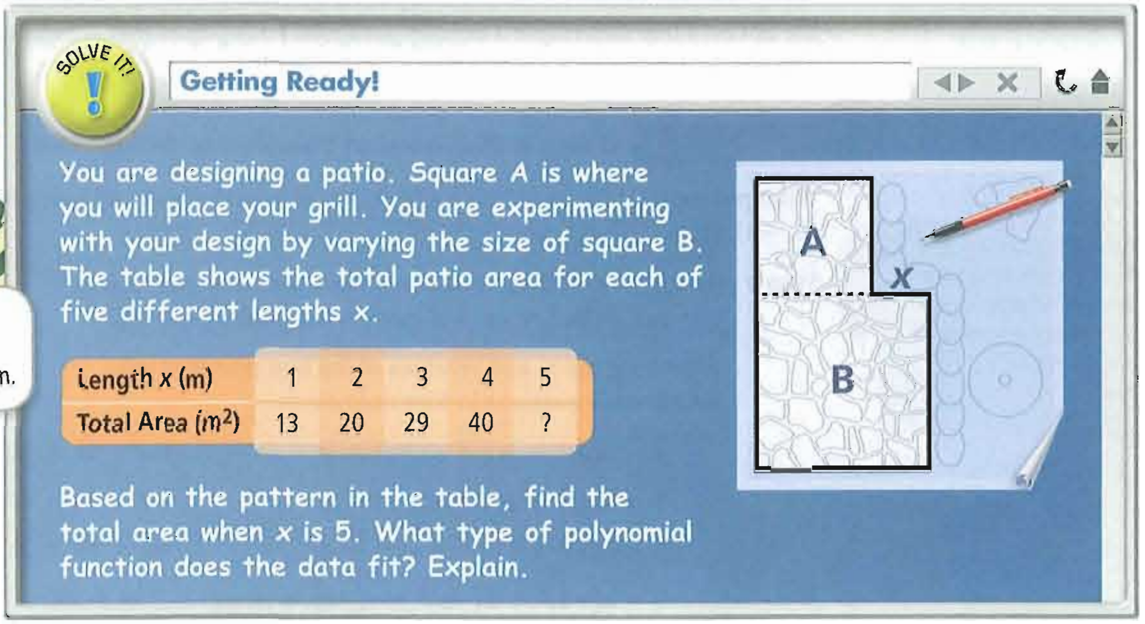
5-8 Polynomial Models in the Real World

Sunshine State Standards

MA.912.A.4.10 Use polynomial equations to solve real-world problems.

MA.912.A.4.5 Graph polynomial functions with and without technology and describe end behavior.

Objective To fit data to linear, quadratic, cubic, or quartic models



Getting Ready!

You are designing a patio. Square A is where you will place your grill. You are experimenting with your design by varying the size of square B. The table shows the total patio area for each of five different lengths x.

Length x (m)	1	2	3	4	5
Total Area (m^2)	13	20	29	40	?

Based on the pattern in the table, find the total area when x is 5. What type of polynomial function does the data fit? Explain.

Polynomial functions can be degree 0 (constant), degree 1 (linear), degree 2 (quadratic), degree 3 (cubic), and so on.

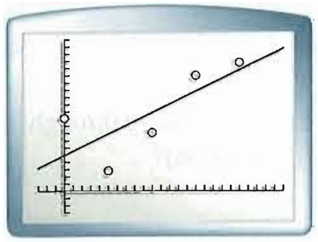
Essential Understanding You can use polynomial functions to model many real-world situations. The behavior of the graphs of polynomial functions of different degrees can suggest what type of polynomial will best fit a particular data set.

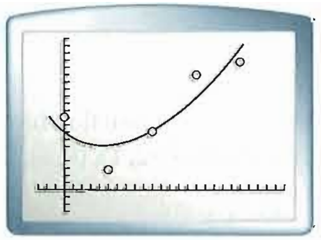
You can use a graphing calculator to help you find functions that model the data in the table shown here.

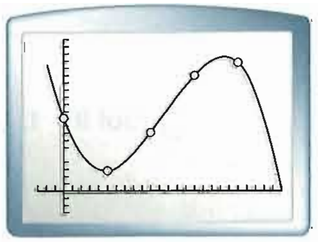
Enter the data into calculator lists **L1** and **L2**. Use three different regressions: **LINREG**, **QUADREG**, and **CUBICREG**.

Graph each regression function and the scatter plot of the data in the same window. For this data, the cubic function appears to be a perfect fit.

x	y
0	10.1
5	2.8
10	8.1
15	16.0
20	17.8







take note

Key Concept The $(n + 1)$ Point Principle

For any set of $n + 1$ points in the coordinate plane that pass the vertical line test, there is a unique polynomial of degree at most n that fits the points perfectly.

This principle confirms that any two points determine a unique line. Three points that are not on a line determine a unique parabola. Four points that are not on a line or a parabola determine a unique cubic, and so forth.

Problem 1 Using A Polynomial Function to Model Data

What polynomial function has a graph that passes through the four points $(0, -3)$, $(1, -1)$, $(2, 5)$, and $(-1, -7)$?

Plan

How can you use the four points to find a system of equations? Substitute each point into a polynomial function.

Step 1 By the $(n + 1)$ Point Principle, there is a cubic polynomial $y = ax^3 + bx^2 + cx + d$ that fits the points perfectly.

Substitute the x- and y-values of the four points to get four linear equations in four unknowns.

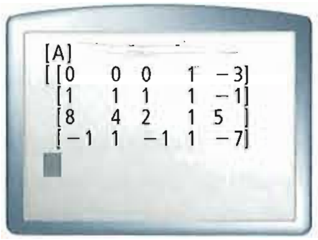
$$-3 = a(0)^3 + b(0)^2 + c(0) + d \qquad 0a + 0b + 0c + 1d = -3$$

$$-1 = a(1)^3 + b(1)^2 + c(1) + d \qquad 1a + 1b + 1c + 1d = -1$$

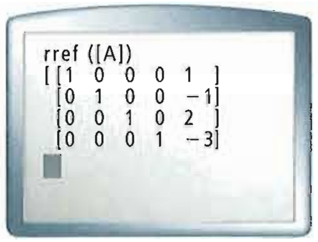
$$5 = a(2)^3 + b(2)^2 + c(2) + d \qquad 8a + 4b + 2c + 1d = 5$$

$$-7 = a(-1)^3 + b(-1)^2 + c(-1) + d \qquad -1a + 1b - 1c + 1d = -7$$

Step 2 Write the system in augmented matrix form. Use the **RREF()** function to find the coefficient values a, b, c, and d.



```
[A]
[[0   0  0   1  -3]
 [1   1  1   1  -1]
 [8   4  2   1  5 ]
 [-1  1  -1  1  -7]
```

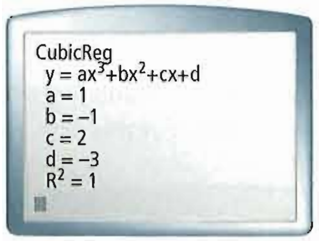


```
rref ([A])
[[1  0  0  0  1 ]
 [0  1  0  0  -1]
 [0  0  1  0  2 ]
 [0  0  0  1  -3]
```

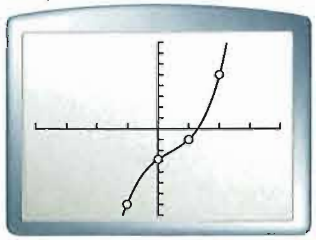
Step 3 $a = 1$, $b = -1$, $c = 2$, and $d = -3$. The polynomial function is $y = x^3 - x^2 + 2x - 3$.

Check Use **CUBICREG** with the four given points.

The solution checks.



```
CubicReg
 y = ax³+bx²+cx+d
 a = 1
 b = -1
 c = 2
 d = -3
 R² = 1
```



Got It? **1.** What polynomial function has a graph that passes through the four points $(-2, 1)$, $(0, 5)$, $(2, 9)$, and $(3, 36)$?

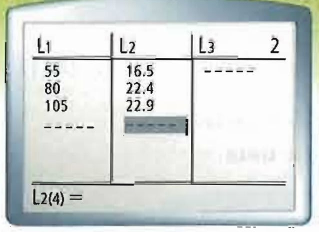
Problem 2 Modeling Data

Food Production **The chart shows how much milk Wisconsin dairy farms produced in 1955, 1980, and 2005. What linear model best fits this data? Use the model to estimate milk production in 2000.**

Milk Production (in billions of lbs)

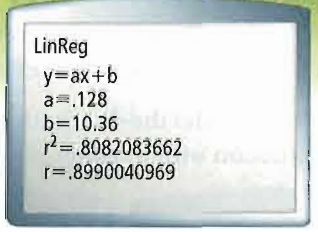
Think

What data should you enter?
Enter the years, after 1900, and billions of pounds of milk produced.



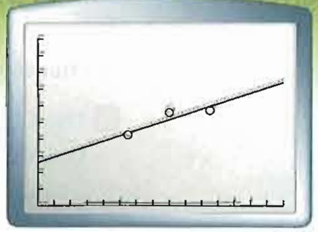
```
L1      L2      L3     2
55      16.5    -----
80      22.4
105     22.9
-----   -----

L2(4) =
```

Enter the data. Let x represent years after 1900. Let y represent billions of pounds of milk.



```
LinReg
 y=ax+b
 a=.128
 b=10.36
 r²=.8082083662
 r=.8990040969
```

Use **LINREG** to find a linear model: $f(x) = 0.128x + 10.36$. Notice the value of r^2.



Show the scatter plot of the data and a graph of $f(x)$.

The closer r^2 is to 1, the better the fit.

Use the model to estimate milk production in 2000.

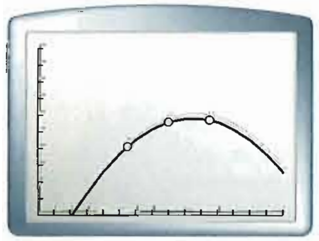
$$f(100) = 0.128(100) + 10.36 \approx 23.2$$

In 2000, Wisconsin produced about 23.2 billion pounds of milk.

Got It? **2.** Use the linear model. Estimate Wisconsin milk production in 1995.

Problem 3 Comparing Models

Food Production **The graph shows the quadratic model for the milk production data in Problem 2. The quadratic model fits the data points exactly because of the $(n + 1)$ Point Principle. Given that both models are good fits, which seems more likely to represent milk production over time?**



Think

Is the quadratic model reasonable?
No; the quadratic model will show milk production eventually decreasing to below zero.

Linear Model: This model continually rises.

Quadratic Model: This model has down-and-down end behavior. It shows slowing growth, a turning point, and then a decline, eventually to 0 and negative values.

Despite the R^2 value of 1 for the quadratic model, the linear model is more likely to represent milk production over time since it shows a continuing increase.

Got It? **3.** If four data points were given, would a cubic function be the best model for the data? Explain your answer.

When deciding whether a model is reliable, it is important to consider the source of the data. Data generated by a law of physics or a geometric formula will have a mathematical model that fits the data and yields accurate predictions. With other data, such as sales records, you can approximate the data only within, or close to, the domain over which it was generated.

Using a model to predict a y-value "outside" the domain of a data set is *extrapolation*. Estimating within the domain is *interpolation*. Interpolation usually yields reliable estimates. Extrapolation becomes less reliable as you move farther away from the data.

Problem 4 Using Interpolation and Extrapolation

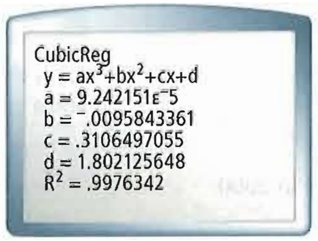
Cheese Consumption **The table shows average annual consumption of cheese per person in the U. S. for selected years from 1910 to 2001.**

Cheese Consumption

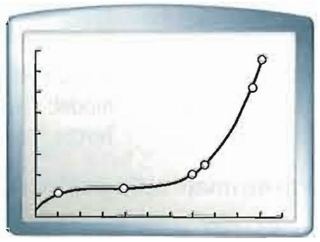
Year	Pounds Consumed
1910	4
1940	5
1970	8
1975	10
1995	25
2001	30

Source: U.S. Department of Agriculture

A **Use CUBICREG. Model the data with a cubic function. Graph the function with a scatter plot of the data.**



```
CubicReg
 y = ax³+bx²+cx+d
 a = 9.242151ᴇ⁻5
 b = ⁻.0095843361
 c = .3106497055
 d = 1.802125648
 R² = .9976342
```



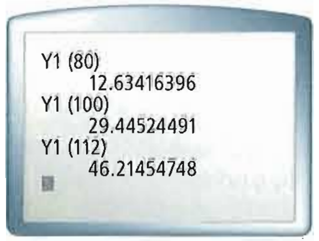
Since R^2 is close to 1, the fit is good. The cubic model is approximately $y = 0.0000924x^3 - 0.00958x^2 + 0.311x + 1.802$.

B **Use the model to estimate cheese consumption for 1980, 2000, and 2012. In which estimate do you have the most confidence? The least confidence? Explain.**

Use the cubic model from part A to estimate the cheese consumption for each year:

1980: 12.6 lb of cheese per person
2000: 29.4 lb of cheese per person
2012: 46.2 lb of cheese per person

You can be confident in interpolating the estimates for 1980 (**Y1(80)**) and 2000 (**Y1(100)**) because the cheese consumption fits the pattern of increase shown in the table. You should have the least confidence in the extrapolated 2012 (**Y1(112)**) estimate because the cubic model increases so quickly beyond 2001.



```
Y1 (80)
        12.63416396
Y1 (100)
        29.44524491
Y1 (112)
        46.21454748
▮
```

Think

What affects your confidence in drawing conclusions from the model?
Your confidence can waver where model behavior is extreme or where there are large gaps in the data.

Got It? **4. a.** Use **LINREG** to find a linear model for cheese consumption. Graph it with a scatter plot.

b. **Reasoning** Use the model to estimate consumption for 1980, 2000, and 2012. In which of these estimates do you have the most confidence? The least confidence? Explain.

Lesson Check

Do you know HOW?

Determine which type of model best fits each set of points.

1. $(-2, -1)$, $(0, 3)$, and $(2, 7)$

2. $(0, 3)$, $(3, 4)$, and $(5, 6)$

3. $(2, 3)$, $(4, 2)$, $(6, 4)$, and $(8, 5)$

4. $(-5, 6)$, $(-4, 3)$, $(0, 2)$, $(2, 4)$, and $(5, 10)$

Do you UNDERSTAND?

5. **Vocabulary** Explain which form of estimation, interpolation or extrapolation, is more reliable.

6. **Reasoning** Is it possible to create a cubic function that passes through $(0, 0)$, $(-1, 1)$, $(-2, 2)$, and $(-3, 9)$? Explain.

7. **Writing** The R^2 value for a quartic model is 0.94561. The R^2 value for a cubic model of the same data is 0.99817. Which model seems to show a better fit? Explain.

Practice and Problem-Solving Exercises

A Practice

Find a polynomial function whose graph passes through each set of points. See Problem 1.

8. $(0, 5)$ and $(2, -13)$

9. $(-2, -4)$ and $(8, 1)$

10. $(-5, 14)$ and $(1, -16)$

11. $(7, 13)$, $(10, -11)$, and $(0, 4)$

12. $(-2, -16)$, $(3, 11)$, and $(0, 2)$

13. $(-1, 8)$, $(5, -4)$, and $(7, 8)$

14. $(-1, -15)$, $(1, -7)$, and $(6, -22)$

15. $(-1, 9)$, $(0, 6)$, $(1, 5)$, and $(2, 18)$

For each set of data, compare two models and determine which one best fits the data. Which model seems more likely to represent each set of data over time? See Problems 2 and 3.

16. U.S. Federal Spending

Year	Total (billions $)
1965	630
1980	1,300
1995	1,950
2005	2,650

17. World Population

Year	Average Growth Rate (%)
1972	1.96
1982	1.73
1992	1.5
2002	1.22

18. U.S. Homes

Year	Average Sale Price (thousands $)
1990	149
1995	158
2000	207

19. U.S. Crude Oil and Petroleum

Month (2008)	Products Supplied (millions of barrels/day)
2	19.782
4	19.768
6	19.553

Use your models from Exercises 16-19 to make predictions.

See Problem 4.

20. Estimate total U.S. federal spending for 1990 and 2010.

21. Estimate the average annual growth rate of the world population for 1950, 1988, and 2010.

22. Estimate the average sale price of homes sold in the United States for 1985, 1999, and 2020.

23. Estimate the number of barrels of crude oil and petroleum supplied per day for January, March, and October of 2008.

B Apply

Find a cubic and a quartic model for each set of values. Explain why one models the data better.

24.

x	−2	−1	0	2	3
y	−25	−4	3	23	40

25.

x	−2	−1	0	1	2
y	−65	−14	−4	2	90

Find a polynomial function whose graph passes through the points.

26. $(-14, 14)$, $(-10, 0)$, $(0, -1)$, $(8, 0)$, and $(12,4)$

27. $(-3, -50)$, $(-2, -4)$, $(-1, 10)$, $(0, 7)$, and $(2, -23)$

28. **Think About a Plan** The table at the right shows the amount of carbon dioxide in the Earth's atmosphere for selected years. Predict the amount of carbon dioxide in the Earth's atmosphere in 2022. How confident are you in your prediction?

- How can you plot the data? (*Hint:* Let x equal the years after 1900.)
- What polynomial model should you use?

Year	CO$_2$ in atmosphere (ppm)
1968	324.14
1983	343.91
1998	367.68
2003	376.68
2008	385.60

Source: The Weather Channel

Find a cubic model for each set of values. Then use the regression coefficient of each model to determine whether the model is a good fit.

29. $(-5, -60)$, $(-1, -5)$, $(0, 0.5)$, $(1, 8)$, $(5, 17)$, $(10, 32)$

30. $(8, -101)$, $(-1, 10)$, $(-8, 47)$, $(-10, 59)$

31. **Air Travel** The table shows the percent of on-time flights for selected years. Find a polynomial function to model the data. Use 1998 as Year 0.

Year	1998	2000	2002	2004	2006
On-time Flights (%)	77.20	72.59	82.14	78.08	75.45

Source: U.S. Bureau of Transportation Statistics

32. **Writing** Explain two ways to find a polynomial function to model a given set of data.

33. **Error Analysis** The table at the right shows the number of students enrolled in a high school personal finance course. A student says that a cubic model would best fit the data based on the $(n + 1)$ Point Principle. Explain why a quadratic model might be more appropriate.

Year	Number of Students Enrolled
2000	50
2004	65
2008	94
2010	110

34. **Compare and Contrast** The table shows the United States gross domestic product for selected years. Construct curves using cubic regression and quartic regression to model the data. Which curve seems most likely to model gross domestic product over the years?

Year	1960	1970	1980	1990	2000
GDP (billions $)	526.4	1038.5	2789.5	5803.1	9817.0

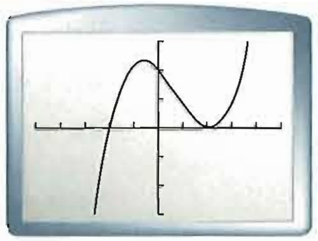
35. The table below shows the percentage of the U.S. labor force in unions for selected years between 1955 and 2005.

Year	1955	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
%	33.2	31.4	28.4	27.3	25.5	21.9	18.0	16.1	14.9	13.5	12.5

a. Make a scatter plot of the data. Which kind of polynomial model seems to be most appropriate?
b. Use a graphing calculator to find the type of model from part (a).
c. Use the model you found in part (b) to predict the percent of the labor force in unions in the year 2020.
d. **Reasoning** Do you have much confidence in this prediction? Explain.

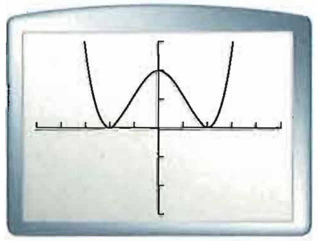
Challenge

36. Your friend's teacher showed the class a graph of a cubic polynomial in the ZDecimal window, which is $[-4.7, 4.7]$ by $[-3.1, 3.1]$. She then challenged the class to find the polynomial *without using cubic regression on their calculators*, and your friend succeeded. Follow your friend's steps and see if you can find the polynomial.



a. The graph resembles a parabola with vertex $(2, 0)$ near $x = 2$. Find the equation in standard form for that parabola.
b. Find the equation of a line in slope-intercept form through $(-2, 0)$ with slope 1. Multiply the linear expression by the quadratic expression from part (a) to get a cubic. (Leave it in factored form.) Graph the cubic function. What do you notice about the zeros and the y-intercept of the cubic function?
c. Multiply the cubic by a constant to change the y-intercept to 2. Graph the function to see if you've found the right polynomial. What is the function?

37. The graph at the right is that of a certain quartic polynomial in the ZDecimal window, which is $[-4.7, 4.7]$ by $[-3.1, 3.1]$. Find the equation of the quartic *without using quartic regression on your calculator*. You may leave it in factored form.



Sunshine State Standards Practice

GRIDDED RESPONSE

Files	Time(s)
100	0.5
200	0.9
300	3.5
400	8.2
500	14.8

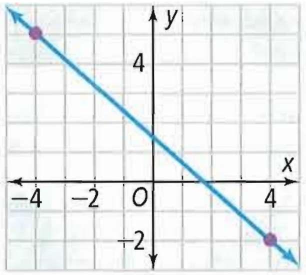
MA.912.A.4.10 **38.** The table shows the time it takes a computer program to run, given the number of files used as input. Using a cubic model, what do you predict the run time will be if the input consists of 1000 files?

MA.912.A.4.9 **39.** Suppose you hit a ball and its flight follows the graph of $f(x) = -16x^2 + 20x + 3$. How many seconds will it take for the ball to hit the ground? Round your answer to the nearest second.

MA.912. A.4.8 **40.** What is the multiplicity of the zeros of $y = 16x^2 - 8x + 1$?

MA.912.A.3.10 **41.** What is the slope of the line shown?

MA.912. A.2.6 **42.** What is the degree of the polynomial function $y = -9x^3 - 5x^2 - 2x^5 + 4x + x^4 + 1$?



Mixed Review

Expand each binomial. **See Lesson 5-7.**

43. $(2x + 3)^5$

44. $(11x - 1)^3$

45. $(8 - 3x)^4$

46. $(4 + 9x)^3$

Write each compound inequality as an absolute value inequality. **See Lesson 1-6.**

47. $7 < x < 9$

48. $\frac{1}{4} \leq x \leq \frac{1}{2}$

49. $1.7 < y < 3.9$

50. $500 < t < 1000$

Solve each formula for the indicated variable. **See Lesson 1-4.**

51. $A = s^2$, for s

52. $P = 2(l + w)$, for l

53. $C = 2\pi r$, for r

54. $A = bh$, for b

Get Ready! **To prepare for Lesson 5-9, do Exercises 55–58.**

Graph each function. **See Lesson 4-1.**

55. $y = x^2$

56. $y = -4x^2$

57. $y = x^2 + 3$

58. $y = -7x^2 - 1$

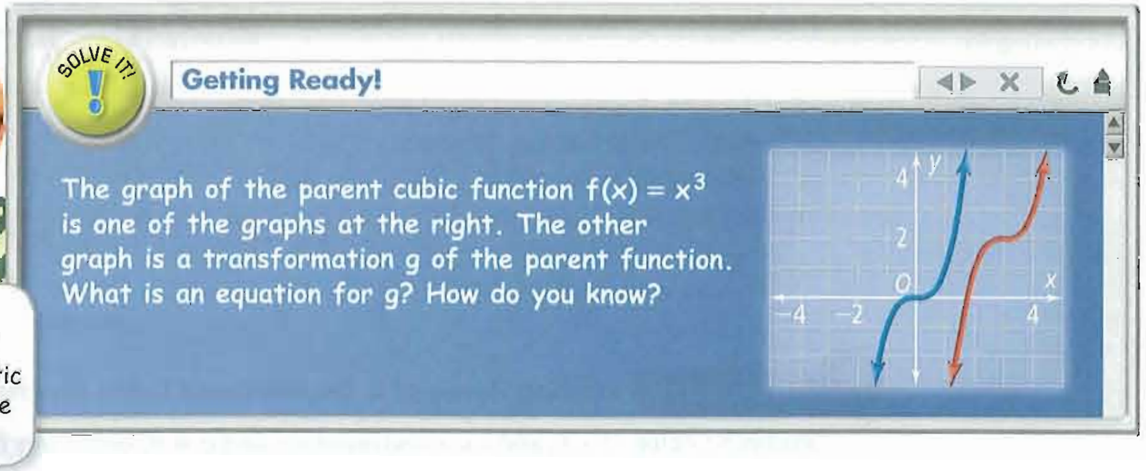
5-9 Transforming Polynomial Functions

Sunshine State Standards

MA.912.A.2.10 Describe and graph transformations of functions.

MA.912.A.4.7 Write a polynomial equation for a given set of real and/or complex roots.

Objective To apply transformations to graphs of polynomials



The graph of the parent cubic function $f(x) = x^3$ is one of the graphs at the right. The other graph is a transformation g of the parent function. What is an equation for g? How do you know?

Lesson Vocabulary

- power function
- constant of proportionality

Recall that you can obtain the graph of any quadratic function from the graph of the parent quadratic function, $y = x^2$, using one or more basic transformations. You will find that this is not true of cubic functions.

Essential Understanding The graph of the function $y = af(x - h) + k$ is a vertical stretch or compression by the factor $|a|$, a horizontal shift of h units, and a vertical shift of k units of the graph of $y = f(x)$.

Problem 1 Transforming $y = x^3$

What is an equation of the graph of $y = x^3$ under a vertical compression by the factor $\frac{1}{2}$ followed by a reflection across the x-axis, a horizontal translation 3 units to the right, and then a vertical translation 2 units up?

Step 1 Multiply by $\frac{1}{2}$ to compress.

$$y = x^3 \longrightarrow y = \frac{1}{2}x^3$$

Step 2 Multiply by -1 to reflect.

$$y = \frac{1}{2}x^3 \longrightarrow y = -\frac{1}{2}x^3$$

Step 3 Replace x with $x - 3$ to translate horizontally.

$$y = -\frac{1}{2}x^3 \longrightarrow y = -\frac{1}{2}(x - 3)^3$$

Step 4 Add 2 to translate vertically.

$$y = -\frac{1}{2}(x - 3)^3 \longrightarrow y = -\frac{1}{2}(x - 3)^3 + 2$$

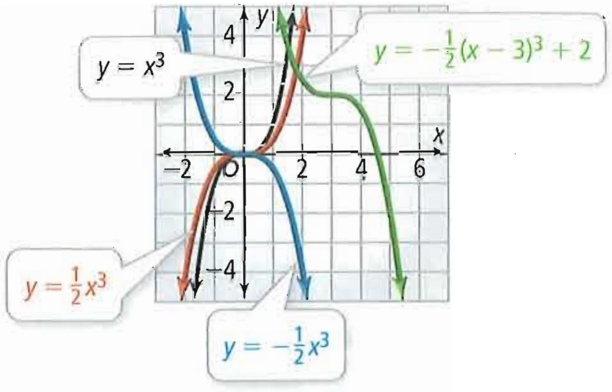
Think

How is translating this cubic function like translating a quadratic function?
In each case you replace x with $x - h$ to translate h units to the right.

Got It? **1.** What is an equation of the graph of $y = x^3$ under a vertical stretch by the factor 2 followed by a horizontal translation 3 units to the left and then a vertical translation 4 units down?

The graph shows $y = x^3$ and the graphs that result from the transformations in Problem 1.

In general, $y = a(x - h)^3 + k$ represents all of the cubic functions you can obtain by stretching, compressing, reflecting, or translating the cubic parent function $y = x^3$.



Problem 2 Finding Zeros of a Transformed Cubic Function

Plan

How can you find the zeros of a function? Set the function equal to 0 and solve for x.

Multiple Choice **If a, h, and k are real numbers and $a \neq 0$, how many distinct real zeros does $y = -a(x - h)^3 + k$ have?**

Ⓐ 0 Ⓑ 1 Ⓒ 2 Ⓓ 3

$-a(x - h)^3 + k = 0$	x is a zero means it is an x-intercept, so $y = 0$.
$-a(x - h)^3 = -k$	Subtract k from each side.
$(x - h)^3 = \frac{k}{a}$	Divide each side by $-a$.
$x - h = \sqrt[3]{\frac{k}{a}}$	Take the cube root of each side.
$x = \sqrt[3]{\frac{k}{a}} + h$	Solve for x.

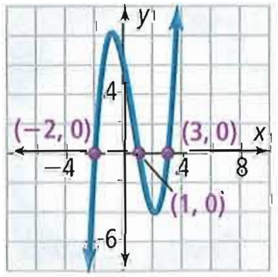
Disregarding multiplicities, the function has a single real zero. The correct answer is B.

Got It? **2.** What are all the real zeros of the function $y = 3(x - 1)^3 + 6$?

Problems 1 and 2 together illustrate that the graph of an "offspring" function of the parent cubic function $y = x^3$ has only one x-intercept.

The graph of the cubic function $y = x^3 - 2x^2 - 5x + 6$ has three x-intercepts. You cannot obtain this function or others like it by transforming the parent cubic function $y = x^3$ using stretches, reflections, and translations.

Similarly, some quartic functions are simple transformations of $y = x^4$ and some are not.



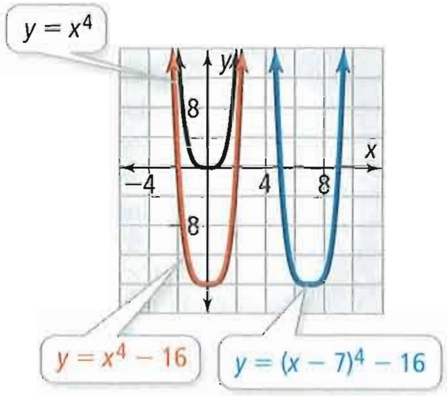
Problem 3 Constructing a Quartic Function with Two Real Zeros

What is a quartic function with only two real zeros, $x = 5$ and $x = 9$?

Method 1 Use transformations.

First, find a quartic with zeros at ±2.
Translate the basic quartic 16 units down:
$y = x^4 \rightarrow y = x^4 - 16$

9 is 7 units to the right of 2.
Translate 7 units to the right.
$y = x^4 - 16 \rightarrow y = (x - 7)^4 - 16$



A quartic function with its only real zeros at 5 and 9 is $y = (x - 7)^4 - 16$.

Think

What should you use for $Q(x)$?
Choose $Q(x)$ to be a quadratic with no real zeros. Keep it simple, such as $x^2 + 1$.

Method 2 Use algebraic methods.

$$\begin{aligned} y &= (x - 5)(x - 9) \cdot Q(x) \\ &= (x - 5)(x - 9)(x^2 + 1) \\ &= (x^2 - 14x + 45)(x^2 + 1) \\ &= x^4 - 14x^3 + 46x^2 - 14x + 45 \end{aligned}$$

Make $Q(x)$ a quadratic with no real zeros.

Another quartic function with its only real zeros at 5 and 9 is $y = x^4 - 14x^3 + 46x^2 - 14x + 45$.

Got It? **3.** **a.** What is a quartic function $f(x)$ with only two real zeros, $x = 0$ and $x = 6$?
b. **Reasoning** Does the quartic function $-f(x)$ have the same zeros? Explain.

The "offspring" of the parent function $y = x^4$ is a subfamily of all quartic polynomials. This subfamily consists of quartics of the form $y = a(x - h)^4 + k$. These functions also belong to another category of polynomials, and in this category you can generate families as usual.

take note Key Concept Power Functions

Definition	**Examples**	
A **power function** is a function of the form $y = a \cdot x^b$, where a and b are nonzero real numbers.	$y = 0.5x^6$	$y = \frac{1}{2}x^2$
	$y = -4x^{\frac{2}{3}}$	$y = x^{0.25}$

If the exponent b in $y = ax^b$ is a positive integer, the function is also a *monomial function.*

If $y = ax^b$ describes y as a power function of x, then y *varies directly with*, or *is proportional to*, the bth power of x. The constant a is the **constant of proportionality**. Power functions arise in many real-world contexts related to the concept of direct variation, which you studied in Chapter 2.

Problem 4 Modeling With a Power Function

Think

How is this problem like the direct variation you studied in Chapter 2?
With the direct variation, $y = kx$, you use a first power. Here you use a third power.

Wind-Generated Power Wind farms are a source of renewable energy found around the world. The power P (in kilowatts) generated by a wind turbine varies directly as the cube of the wind speed v (in meters per second). The picture shows the power output of one turbine at one wind speed. To the nearest kilowatt, how much power does this turbine generate in a 10 m/s wind?

The formula for P as a power function of v is $P = a \cdot v^3$. From the picture, $P = 600$ when $v = 8$, or $600 = a \cdot 8^3$. Solve for a.

$$600 = a \cdot 8^3$$ Use values of P and v to find a.

$$600 = 512a$$

$$a \approx 1.1719$$

$$P \approx 1.1719v^3.$$ Use the value of a in the original formula.

$$P \approx 1.1719 \cdot 10^3 = 1171.9.$$ Substitute 10 for v and simplify.

This turbine generates about 1172 kW of power in a 10 m/s wind.

Got It? **4.** Another turbine generates 210 kW of power in a 12 mi/h wind. How much power does this turbine generate in a 20 mi/h wind?

Lesson Check

Do you know HOW?

Find all the real zeros of each function.

1. $y = -(x + 3)^3 + 1$

2. $y = -8(x - 5)^3 - 64$

3. $y = \frac{9}{2}(x - 1)^3 + \frac{4}{3}$

Do you UNDERSTAND?

4. **Vocabulary** Is the function $y = 4x^3 + 5$ an example of a power function? Explain.

5. **Error Analysis** Your friend says that he has found a way to transform the graph of $y = x^3$ to obtain three real roots. Using the graph of the function, explain why this is impossible.

6. **Compare and Contrast** How are the graphs of $y = x^3$ and $y = 4x^3$ alike? How are they different? What transformation was used to get the second equation?

Practice and Problem-Solving Exercises

A Practice

Determine the cubic function that is obtained from the parent function $y = x^3$ after each sequence of transformations.

See Problem 1.

7. a vertical stretch by a factor of 3;
a reflection across the x-axis;
a vertical translation 2 units up;
and a horizontal translation 1 unit right

8. a vertical stretch by a factor of 2;
a vertical translation 4 units up;
and a horizontal translation 3 units left

9. a reflection across the y-axis;
a vertical translation 1 unit down;
and a horizontal translation 5 units left

10. a vertical translation 3 units down;
and a horizontal translation 2 units right

11. a vertical stretch by a factor of 3;
a reflection across the y-axis;
a vertical translation $\frac{3}{4}$ unit up;
and a horizontal translation $\frac{1}{2}$ unit left

12. a vertical stretch by a factor of $\frac{5}{3}$;
a reflection across the x-axis;
a vertical translation 4 units down;
and a horizontal translation 3 units right

Find all the real zeros of each function.

See Problem 2.

13. $y = -27(x - 2)^3 + 8$

14. $y = -\frac{1}{8}(x - 7)^3 - 8$

15. $y = -3\left(x + \frac{4}{5}\right)^3 + \frac{8}{9}$

16. $y = -16(x + 3)^3 + 9$

17. $y = 4(x - 1)^3 + 10$

18. $y = 2(x + 5)^3 + 10$

Find a quartic function with the given x-values as its only real zeros.

See Problem 3.

19. $x = 2$ and $x = -1$

20. $x = -3$ and $x = -4$

21. $x = -1$ and $x = 3$

22. $x = 4$ and $x = 2$

23. $x = -4$ and $x = -1$

24. $x = -3$ and $x = 2$

See Problem 4.

25. Cooking The number of pepperoni slices that Kim puts on a pizza varies directly as the square of the diameter of the pizza. If she puts 15 slices on a 10" diameter pizza, how many slices should she put on a 16" diameter pizza?

26. Volume The amount of water that a spherical tank can hold varies directly as the cube of its radius. If a tank with radius 7.5 ft holds 1767 ft^3 of water, how much water can a tank with radius 16 ft hold?

B Apply

27. Think About a Plan The kinetic energy generated by a 5 lb ball is represented by the formula $K = \frac{1}{2}(5)v^2$. If the ball is thrown with a velocity of 6 ft/sec, how much kinetic energy is generated?

- What does 5 represent in the function?
- What number should you substitute for v?

Determine whether each function can be obtained from the parent function, $y = x^n$, using basic transformations. If so, describe the sequence of transformations.

28. $y = 3x^3$

29. $y = 2(x - 3)^2 + 5$

30. $y = x^3 - x$

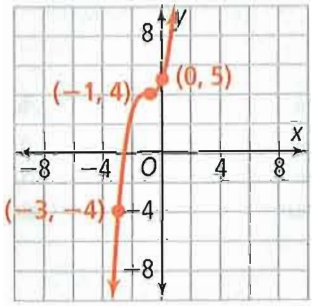
31. $y = x^2 - 8x + 7$

32. $y = (x + 2)^4$

33. $y = -4x^3$

Determine the transformations that were used to change the graph of the parent function $y = x^3$ to each of the following graphs.

34.



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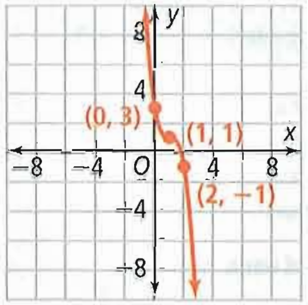

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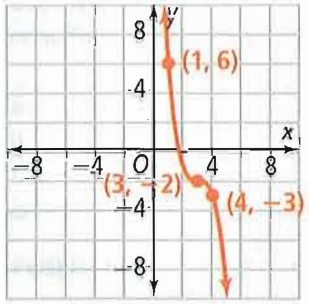

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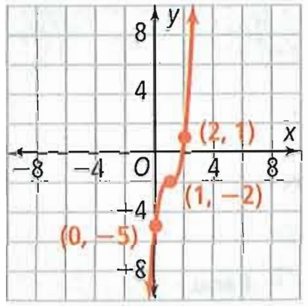

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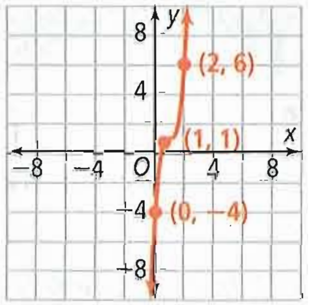

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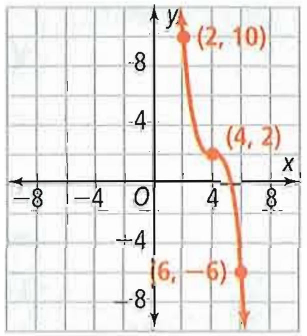
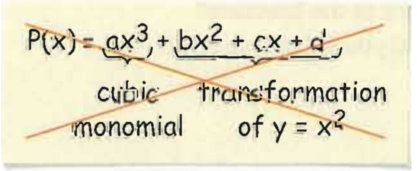


40. **Physics** The formula $K = \frac{1}{2}mv^2$ represents the kinetic energy of an object. If the kinetic energy of a ball is 10 lb-ft^2/s^2 when it is thrown with a velocity of 4 ft/s, how much kinetic energy is generated if the ball is thrown with a velocity of 8 ft/s?

41. **Reasoning** Explain why the basic transformations of the parent function $y = x^5$ will only generate functions that can be written in the form $y = a(x - h)^5 + k$.

42. **Reasoning** Explain why some quartic polynomials cannot be written in the form $y = a(x - h)^4 + k$. Give two examples.

43. **Error Analysis** Your friend claims he can write any cubic polynomial as the sum of two functions: (1) a cubic monomial and (2) a transformation of $y = x^2$. Explain why your friend's claim is incorrect.



$$\cancel{P(x) = \underbrace{ax^3}_{\text{cubic monomial}} + \underbrace{bx^2 + cx + d}_{\text{transformation of } y = x^2}}$$

Challenge

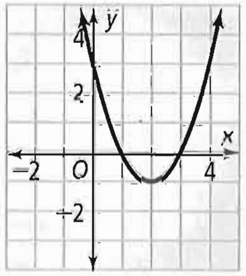
44. Reasoning Find a sequence of basic transformations by which the polynomial function $y = 2x^3 - 6x^2 + 6x + 5$ can be derived from the cubic function $y = x^3$.

45. Physics For a constant resistance R (in ohms), the power P (in watts) dissipated across two terminals of a battery varies directly as the square of the current I (in amps). If a battery connected in a circuit dissipates 24 watts of power for 2 amps of current flow, how much power would be dissipated when the current flow is 5 amps?

46. Writing Give an argument that shows that *every* polynomial family of degree $n > 2$ contains polynomials that cannot be generated from the basic function $y = x^n$ by using stretches, compressions, reflections, and translations.

Sunshine State Standards Practice

Use the graph to answer questions 47–49.



MA.912.A.2.6 **47.** Which equation does the graph represent?

Ⓐ $y = (x + 2)^2 - 1$

Ⓑ $y = (x - 2)^2 - 1$

Ⓒ $y = (x - 2)^2 + 1$

Ⓓ $y = (x - 2)^4 - 1$

MA.912.A.4.8 **48.** If $y = f(x)$ is an equation for the graph, what are factors of $f(x)$?

Ⓕ $(x - 1)$ and $(x + 3)$

Ⓖ $(x - 1)$ and $(x - 3)$

Ⓗ $(x + 1)$ and $(x - 3)$

Ⓘ $(x + 1)$ and $(x + 3)$

MA.912.A.7.4 **49. Short Response** If $y = ax^2 + bx + c$ is an equation for the graph, what type of number is its discriminant?

Mixed Review

Find a polynomial function whose graph passes through the given points. **See Lesson 5-8.**

50. $(-1, 4), (0, -2), (1, -2), (2, -8)$

51. $(-2, -17), (0, -3), (1, -5), (3, 63)$

Write an equation of each line. **See Lesson 2-4.**

52. slope $= -\frac{4}{5}$; through $(-1, 4)$

53. slope $= -3$; through $(2, -1)$

Determine whether each relation is a function. **See Lesson 2-1.**

54. $\{(0, -1), (-1, 3), (2, 3), (-3, 3)\}$

55. $\{(-4, 0), (-7, 0), (-4, 1), (-7, 1)\}$

Get Ready! **To prepare for Lesson 6-1, do Exercises 56–58.**

Factor each expression. **See Lesson 5-2.**

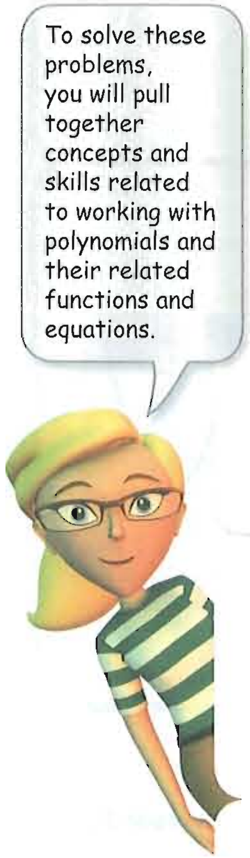
56. $x^{10} + x^2$

57. $x^4 - y^4$

58. $169x^6y^{12} - 13x^3y^6$

5

Pull It All Together



BIG idea Function

You can represent quantities using variables and algebraic expressions. You can represent some relationships between quantities using equations.

Task 1

The polynomial $2x^3 + 9x^2 + 4x - 15$ represents the volume in cubic feet of a rectangular holding tank at a fish hatchery. The depth of the tank is $(x - 1)$ feet. The length is 13 feet.

a. Use synthetic division to help you factor the volume polynomial. How many linear factors should you look for? What are they?

b. Assume the length is the greatest dimension. Which linear factor represents the 13-ft length? What are the dimensions of the tank? What is its volume? What is the value of x? Do you get the same volume if you substitute the value of x into $2x^3 + 9x^2 + 4x - 15$?

BIG idea Equivalence

You can use the Binomial Theorem and properties of algebra to rewrite some powers.

Task 2

Show that the following equation is true for all values of a and b.

$$[(a - b) + 1]^5 = a^5 - 5a^4(b - 1) + 10a^3(b - 1)^2 - 10a^2(b - 1)^3 + 5a(b - 1)^4 - (b - 1)^5$$

BIG idea Solving Equations and Inequalities

A polynomial $P(x)$ of degree n, $n \geq 1$, and its related polynomial function $y = P(x)$ have n complex zeros. The zeros are identical to the n complex roots of the related polynomial equation $P(x) = 0$.

Task 3

Four of these five polynomial functions have identical zeros. The fifth has exactly two zeros in common with each of the other functions. Write this fifth function as a product of its linear factors.

$P_1(x) = x^3 - 6x^2 + 11x - 6$
$P_2(x) = x^4 - 5x^3 + 7x^2 - 5x + 6$
$P_3(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$
$P_4(x) = x^4 - 8x^3 + 23x^2 - 28x + 12$
$P_5(x) = x^4 - 9x^3 + 29x^2 - 39x + 18$

5 Chapter Review

Connecting BIG ideas and Answering the Essential Questions

1 Function

A polynomial of degree n has n linear factors. The graph of the related function crosses the x-axis an even or odd number of times depending on whether n is even or odd.

2 Equivalence

$(x - a)$ is a linear factor if and only if a is a zero, and if and only if $(a, 0)$ is an x-intercept when a is a real number.

3 Solving Equations and Inequalities

$(x - a)$ is a linear factor if and only if a is a root of the related polynomial equation.

Polynomial Functions, Zeros, and Linear Factors (Lessons 5-1 and 5-2)

$y = 2x^3 + 7x^2 - 9$ has 3 linear factors $(x + 3)$, $(2x + 3)$, $(x - 1)$, it crosses the x-axis 3 times—at $(-3, 0)$, $(-\frac{3}{2}, 0)$, and $(1, 0)$. Its end behavior is down and up.

Theorems About Roots of Polynomial Equations (Lesson 5-5)

$P(x) = 2x^3 + 7x^2 - 9$ and $Q(x) = 2x^3 + 5x^2 + 9$ each have degree 3 so $P(x) = 0$ and $Q(x) = 0$ each have 3 complex roots.

Each has $\pm 1, \pm 3, \pm 9, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}$ as its possible rational roots.

$P(-3) = 0$, so $x + 3$ is a factor of $2x^3 + 7x^2 - 9$.

$Q(-3) = 0$, so $x + 3$ is a factor of $2x^3 + 5x^2 + 9$.

Solving Polynomial Equations (Lesson 5-3)

$2x^3 + 7x^2 - 9 = 0$ has factored form $(x + 3)(2x + 3)(x - 1) = 0$. It has 3 roots or solutions, $x = -3$, $x = -\frac{3}{2}$, and $x = 1$.

The Fundamental Theorem of Algebra (Lesson 5-6)

$2x^3 + 5x^2 + 9 = 0$ has factored form $(x + 3)(2x^2 - x + 3) = 0$. It has 3 roots or solutions, $x = -3$, $x = \frac{1}{4} - \frac{\sqrt{23}}{4}i$, and $x = \frac{1}{4} + \frac{\sqrt{23}}{4}i$.

Chapter Vocabulary

- Binomial Theorem (p. 327)
- Conjugate Root Theorem (p. 314)
- constant of proportionality (p. 341)
- degree of a monomial (p. 280)
- degree of a polynomial (p. 280)
- Descartes' Rule of Signs (p. 315)
- difference of cubes (p. 297)
- end behavior (p. 282)
- expand a binomial (p. 326)
- Factor Theorem (p. 290)
- Fundamental Theorem of Algebra (p. 320)
- monomial (p. 280)
- multiple zero (p. 291)
- multiplicity (p. 291)
- Pascal's Triangle (p. 327)
- polynomial (p. 280)
- polynomial function (p. 280)
- power function (p. 341)
- Rational Root Theorem (p. 312)
- relative maximum (p. 291)
- relative minimum (p. 291)
- Remainder Theorem (p. 307)
- standard form of a polynomial function (p. 281)
- sum of cubes (p. 297)
- synthetic division (p. 306)
- turning point (p. 282)

Match each vocabulary term with the description that best fits it.

1. Conjugate Root Theorem
2. Fundamental Theorem of Algebra
3. Rational Root Theorem
4. Remainder Theorem

A. determines $P(a)$ by dividing the polynomial by $x - a$
B. the degree equals the number of roots
C. minimizes guessing fraction and integer solutions
D. complex numbers as roots come in pairs

5-1 Polynomial Functions

Quick Review

The **standard form of a polynomial function** is $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is a nonnegative integer and the coefficients are real numbers. A **polynomial function** is classified by degree. Its degree is the highest degree among its monomial term(s). The degree determines the possible number of **turning points** in the graph and the **end behavior** of the graph.

Example

Write the polynomial function in standard form and classify it by degree. How many terms does it have? What are the possible numbers of turning points of the graph of $P(x)$ given the degree of the polynomial?

$$P(x) = -4x^2 + x^4$$

Standard form arranges the terms by decreasing exponents, or $P(x) = x^4 - 4x^2$. Its degree is 4, so $x^4 - 4x^2$ is a quartic binomial. It has two terms. The graph of a quartic polynomial function can have either one or three turning points.

Exercises

Write each polynomial function in standard form, classify it by degree, and determine the end behavior of its graph.

5. $y = 12 - x^4$

6. $y = x^2 + 7 - x$

7. $y = 2x^3 - 6x + 3x^2 - x^4 + 12$

8. $y = 2x^2 + 8 - 4x + x^3$

9. $y = 10 - 3x^3 + 3x^2 + x^4$

10. If the volume of a cube can be represented by a polynomial of degree 9, what is the degree of the polynomial that represents each side length?

11. A polynomial function $P(x)$ has degree n. If n is even, is the number of turning points of the graph of $P(x)$ even or odd? What can you say about the number of turning points if n is odd?

5-2 Polynomials, Linear Factors, and Zeros

Quick Review

For any real number a and polynomial $P(x)$, if $x - a$ is a **factor** of $P(x)$, then a is:

- a **zero** of $y = P(x)$
- a **root** (or **solution**) of $P(x) = 0$, and
- an ***x*-intercept** of the graph of $y = P(x)$.

If a is a **multiple zero**, its **multiplicity** is the same as the number of times $x - a$ appears as a factor.

A turning point is a **relative maximum** or **relative minimum** of a polynomial function.

Example

Find the zeros for $y = 3x^3 - 6x^2 + 3x$ and state the multiplicity of any multiple zeros.

$y = 3x(x^2 - 2x + 1)$ Factor out the GCF, 3x.

$y = 3x(x - 1)(x - 1)$ Factor the quadratic.

The zeros are 0, and 1 with multiplicity 2.

Exercises

Write a polynomial function with the given zeros.

12. $x = -1, -1, 6$

13. $x = -1, 0, 2$

14. $x = 1, 2, 3$

15. $x = -2, 1, 4$

Find the zeros of each function. State the multiplicity of any multiple zeros.

16. $y = 3x(x + 2)^3$

17. $y = x^4 - 8x^2 + 16$

18. $y = 4x^3 - 2x^2 - 2x$

19. $y = (x - 5)(x + 2)^2$

Use a graphing calculator to find the relative maximum, relative minimum, and zeros of each function.

20. $y = x^4 - 5x^3 + 5x^2 - 3$

21. $y = 5x^3 + x^2 - 9x + 4$

22. $y = x^4 - 4x - 1$

23. $y = x^3 - 3x^2 - 3x - 4$

5-3 Solving Polynomial Equations

Quick Review

One way to solve a polynomial equation is by factoring. First write the equation in the form $P(x) = 0$, where $P(x)$ is the polynomial. Then factor the polynomial. Last, use the Zero-Product Property to find the solutions, or roots. The solutions may be real or imaginary. Real solutions and approximations of irrational solutions can also be found by using a graphing calculator.

Example

Solve $x^3 + 4x^2 = 12x$ by factoring.

$x^3 + 4x^2 - 12x = 0$ Set equal to 0.

$x(x - 2)(x + 6) = 0$ Factor the left side.

$x = 0, x - 2 = 0, x + 6 = 0$ Zero Product Property.

$x = 0, x = 2, x = -6$ Solve each equation.

The solutions are 0, 2, and -6.

Exercises

Find the real or imaginary solutions of each equation by factoring.

24. $x^2 - 11x = -24$

25. $4x^2 = -4x - 1$

26. $3x^3 + 3x^2 = 27x$

27. $2x^2 + 3 = 4x$

Find the real roots of each equation by graphing.

28. $x^4 + 3x^2 - 2x + 5 = 0$

29. $x^2 + 3 = x^3 - 5$

30. The height and width of a rectangular prism are each 2 inches shorter than the length of the prism. The volume of the prism is 40 cubic inches. Approximate the dimensions of the prism to the nearest hundredth.

5-4 Dividing Polynomials

Quick Review

You can divide a polynomial by one of its factors to find another factor. When you divide by a linear factor, you can simplify this division by writing only the coefficients of each term. This is called **synthetic division**. The **Remainder Theorem** says that $P(a)$ is the remainder when you divide $P(x)$ by $x - a$.

Example

Let $P(x) = 3x^2 - 13x + 15$. What is $P(3)$?

According to the Remainder Theorem, $P(3)$ is the remainder when you divide $P(x)$ by $x - 3$.

$$\begin{array}{r|rrr} 3 & 3 & -13 & 15 \\ & & 9 & -12 \\ \hline & 3 & -4 & 3 \end{array}$$

Put the opposite of the constant in the divisor at the top left.

The quotient is $3x - 4$ with remainder 3, so $P(3) = 3$.

Exercises

Divide using long division. Check your answers.

31. $(x^3 + 7x^2 + 15x + 9) \div (x + 1)$

32. $(2x^3 - 7x^2 - 7x + 13) \div (x - 4)$

Determine whether each binomial is a factor of $x^3 + x^2 - 10x + 8$.

33. $x - 2$

34. $x - 4$

Divide using synthetic division.

35. $(x^3 + 5x^2 - x - 5) \div (x + 5)$

36. $(2x^3 + 14x^2 - 58x) \div (x + 10)$

37. $(5x^3 + 8x^2 - 60) \div (x - 2)$

Use the Remainder Theorem to determine the value of $P(a)$.

38. $P(x) = 2x^3 + 5x^2 + 7x - 4, a = -2$

39. $P(x) = x^3 - 4x^2 + 2x + 3, a = 1$

5-5 Theorems About Roots of Polynomial Equations

Quick Review

The **Rational Root Theorem** gives a way to determine the possible roots of a polynomial equation $P(x) = 0$. If the coefficients of $P(x)$ are all integers, then every root of the equation can be written in the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient.

The **Conjugate Root Theorem** states that if $P(x)$ is a polynomial with rational coefficients, then irrational roots that have the form $a + \sqrt{b}$ and imaginary roots of $P(x) = 0$ come in conjugate pairs. Therefore, if $a + \sqrt{b}$ is an irrational root, where a and b are rational, then $a - \sqrt{b}$ is also a root. Likewise, if $a + bi$ is a root, where a and b are real and i is the imaginary unit, then $a - bi$ is also a root.

Descartes' Rule of Signs gives a way to determine the possible number of positive and negative real roots by analyzing the signs of the coefficients. The number of positive real roots is equal to the number of sign changes in consecutive coefficients of $P(x)$, or is less than that by an even number. The number of negative real roots is equal to the number of sign changes in consecutive coefficients of $P(-x)$, or is less than that by an even number.

Example

Find the rational roots of $P(x) = 0$ if $P(x) = 2x^3 - 4x^2 - 10x + 12$.

List the possible roots: $\pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm2, \pm3, \pm4, \pm6, \pm12$.
Use synthetic division to test roots.

$$\begin{array}{r|rrrr} 3 & 2 & -4 & -10 & 12 \\ & & 6 & 6 & -12 \\ \hline & 2 & 2 & -4 & 0 \end{array}$$

So $x - 3$ and $(2x^2 + 2x - 4)$ are factors of $P(x)$.

$P(x) = (x - 3)(2x^2 + 2x - 4)$

Factor the quadratic.

$P(x) = 2(x - 3)(x + 2)(x - 1)$

Solve $2(x - 3)(x + 2)(x - 1) = 0$.

$x = 3;\ x = -2,\text{ or } x = 1$

The rational roots are 3, −2, and 1.

Exercises

List the possible rational roots of $P(x)$ given by the Rational Root Theorem.

40. $P(x) = x^3 + 4x^2 - 10x + 6$

41. $P(x) = 3x^3 - x^2 - 7x + 2$

42. $P(x) = 4x^4 - 2x^3 + x^2 - 12$

43. $P(x) = 3x^4 - 4x^3 - x^2 - 7$

Find any rational roots of $P(x)$.

44. $P(x) = x^3 + 2x^2 + 4x + 21$

45. $P(x) = x^3 + 5x^2 + x + 5$

46. $P(x) = 2x^3 + 7x^2 - 5x - 4$

47. $P(x) = 3x^4 + 2x^3 - 9x^2 + 4$

A polynomial $P(x)$ has rational coefficients. Name additional roots of $P(x)$ given the following roots.

48. $1 - i$ and 5

49. $5 + \sqrt{3}$ and $-\sqrt{2}$

50. $-3i$ and $7i$

51. $-2 + \sqrt{11}$ and $-4 - 6i$

Write a polynomial function with the given roots.

52. 7 and 10

53. −3 and $5i$

54. $6 - i$

55. $3 + i$, 2, and −4

Determine the possible number of positive real zeros and negative real zeros for each polynomial function given by Descartes' Rule of Signs.

56. $P(x) = 5x^3 + 7x^2 - 2x - 1$

57. $P(x) = -3x^3 + 11x^2 + 12x - 8$

58. $P(x) = 6x^4 - x^3 + 5x^2 - x + 9$

59. $P(x) = -x^4 - 3x^3 + 8x^2 + 2x - 14$

5-6 The Fundamental Theorem of Algebra

Quick Review

The **Fundamental Theorem of Algebra** states that if $P(x)$ is a polynomial of degree n, where $n \geq 1$, then $P(x) = 0$ has exactly n roots. This includes multiple and complex roots.

Example

Use the Fundamental Theorem of Algebra to determine the number of roots for $x^4 + 2x^2 - 3 = 0$.

Because the polynomial is of degree 4, it has 4 roots.

Exercises

Find the number of roots for each equation.

60. $x^3 - 2x + 5 = 0$

61. $2 - x^4 + x^2 = 0$

62. $-x^5 - 6 = 0$

63. $5x^4 - 7x^6 + 2x^3 + 8x^2 + 4x - 11 = 0$

Find all the zeros for each function.

64. $P(x) = x^3 + 5x^2 - 4x - 2$

65. $P(x) = x^4 - 4x^3 - x^2 + 20x - 20$

66. $P(x) = 2x^3 - 3x^2 + 3x - 2$

67. $P(x) = x^4 - 4x^3 - 16x^2 + 21x + 18$

5-7 The Binomial Theorem

Quick Review

Rows 0–5 of Pascal's Triangle are shown below.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

The **Binomial Theorem** uses Pascal's Triangle to expand binomials. For a positive integer n, $(a + b)^n = P_0a^n + P_1a^{n-1}b + P_2a^{n-2}b^2 + \cdots + P_{n-1}ab^{n-1} + P_nb^n$, where $P_0, P_1, \ldots P_n$ are the coefficients of the nth row of Pascal's Triangle.

Example

Use the binomial theorem to expand $(2x + 3)^3$.

$$(2x + 3)^3$$
$$= 1(2x)^3 + 3(2x)^2(3) + 3(2x)(3)^2 + 1(3)^3$$
$$= 8x^3 + 36x^2 + 54x + 27$$

Exercises

68. How many numbers are in the eighth row of Pascal's Triangle?

69. List the numbers in the eighth row of Pascal's Triangle.

70. How many numbers are in the fifteenth row of Pascal's Triangle?

71. What is the third number in the fifteenth row of Pascal's Triangle?

Use the Binomial Theorem to expand each binomial.

72. $(x + 9)^3$

73. $(b + 2)^4$

74. $(3a + 1)^3$

75. $(x - 5)^3$

76. $(x - 2y)^3$

77. $(3a + 4b)^5$

78. $(x + 1)^6$

79. $(2x - 1)^6$

Find the coefficient of the x^2 term in each binomial expansion.

80. $(3x + 4)^3$

81. $(ax - c)^4$

5-8 Polynomial Models in the Real World

Quick Review

A data set can be modeled by a polynomial function. Methods of finding a model that fits the data include the $(n + 1)$ Point Principle and **regression**. Linear, quadratic, cubic, and quartic regressions can be performed on a graphing calculator. A higher R^2 value means a better fit. Once the equation that models the data is known, it can be used to make predictions.

Example

For the data set (8, 30), (10, 45), and (11, 65), predict y when $x = 15$.

Enter 8, 10, and 11 in **L1** and 30, 45, and 65 in **L2**. Choose **LINREG** to find the regression model $y \approx 11.071x - 60.357$. The r^2 value is about 0.928.

Now try **QUADREG**. The model is $y \approx 4.17x^2 - 67.5x + 303.3$ with an R^2 value of 1. Assuming the model makes sense in context, it fits the data better.

Using the quadratic model, when $x = 15$, $y \approx 228.3$.

Exercises

82. Write a polynomial function whose graph passes through (0, 5), (2, 10), and (1, 4). Use a regression to check your answer.

83. Find a linear, a quadratic, and a cubic model for the data. Which model best fits the data?

x	3	8	15	21
y	7	11	26	44

84. Use **CUBICREG** to model the data below. Then use the model to estimate the population in 2008. Let x be the number of years after 2000.

Year	2004	2007	2009	2010
Population	457	910	1244	1315

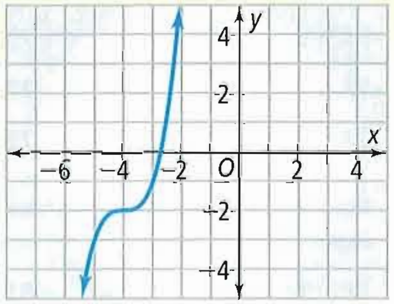
5-9 Transforming Polynomial Functions

Quick Review

A polynomial function can be transformed into other polynomial functions using stretches, reflections, and translations. The monomial function $y = ax^b$ is called a **power function**.

Example

This is the graph of a cubic function. Determine which sequence of transformations you can apply to the graph of the parent function $y = x^3$ to get this graph. Write an equation for the graph.



Translate the parent function 4 units left and 2 units down: $y = (x + 4)^3 - 2$

Exercises

Determine the cubic function obtained from the parent function $y = x^3$ after each sequence of transformations.

85. a reflection across the x-axis;
a translation 1 unit up;
and a translation 2 units right

86. a vertical stretch by a factor of 6;
and a translation 3 units left

87. Find a quartic function whose only real zeros are 4 and 6.

88. The parent power function $y = x^5$ is translated 3 units up and is compressed by the factor 0.3. Write the function.

5 Chapter Test

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Do you know HOW?

Write each polynomial function in standard form. Then classify it by degree and by number of terms and describe its end behavior.

1. $y = 3x^2 - 7x^4 + 9 - x^4$

2. $y = 2x(x^2 - 3)(x^2 + 2)$

3. $y = (t - 2)(t + 1)(t + 1)$

Write a polynomial function for each set of zeros.

4. $x = 1, 2, \frac{3}{5}$

5. $x = \sqrt{2}, -i$

6. $x = 3 + i, -1 - \sqrt{5}$

Find the quotient and remainder.

7. $(x^2 + 3x - 4) \div (x - 1)$

8. $(x^3 + 7x^2 - 5x - 6) \div (x + 2)$

9. $(2x^3 + 9x^2 + 11x + 3) \div (2x + 3)$

For each equation, state the number of complex roots, the possible number of real roots, and the possible rational roots.

10. $3x^4 + 5x^3 - 2x^2 + x - 9 = 0$

11. $x^7 - 2x^5 - 4x^3 - 2x - 1 = 0$

For each equation, find all the roots.

12. $3x^4 - 11x^3 + 15x^2 - 9x + 2 = 0$

13. $x^3 - x^2 - x - 2 = 0$

14. One x-intercept of the graph of the cubic function $f(x) = x^3 - 2x^2 - 111x - 108$ is -9. What are the other zeros?

Use synthetic division and the Remainder Theorem to find $P(a)$.

15. $P(x) = 6x^4 + 19x^3 - 2x^2 - 44x - 24; a = -\frac{2}{3}$

16. $P(x) = x^4 + 3x^3 - 7x^2 - 9x + 12; a = 3$

17. $P(x) = x^3 + 3x^2 - 5x - 4; a = -1$

Expand each binomial.

18. $(x + z)^5$

19. $(1 - 2t)^2$

20. Graph and write the equation of the cubic function that is obtained from the parent function $y = x^3$ after this sequence of transformations: vertical stretch by a factor of 2, reflection across the x-axis, translation 3 units down and 4 units right.

Do you UNDERSTAND?

21. **Physics** You take measurements of the distance traveled by an object that is increasing its speed at a constant rate. The distance traveled as a function of time can be modeled by a quadratic function.

a. Write a quadratic function that models distances of 10 ft at 1 sec, 30 ft at 2 sec, and 100 ft at 4 sec.

b. Find the zeros of the function.

c. **Reasoning** Describe what each zero represents for this real-world situation.

22. **Writing** For the polynomial $x^6 - 64$, could you apply the Difference of Cubes? Difference of Squares? Explain your answers.

23. The number of pairs of shoes Emily buys varies directly as the square of the area of the floor of her closet. If she can fit 12 pairs of shoes when her closet was 10 square feet, how many pairs of shoes will she fit when the area of her closet floor is 18 square feet?

5 Sunshine State Standards Practice

TIPS FOR SUCCESS

Some problems require you to simplify expressions that contain imaginary numbers.

TIP 1

When multiplying binomials, you should use the Distributive Property.

Which expression is equivalent to $(3 - 4i)(2 + i)$?

Ⓐ $2 - 5i$

Ⓑ $2 + 5i$

Ⓒ $10 - 5i$

Ⓓ $10 + 5i$

TIP 2

Recall that $i^2 = -1$.

Think It Through

$$\begin{aligned}&(3 - 4i)(2 + i)\\ &= 6 + 3i - 8i - 4i^2\\ &= 6 - 5i - 4(-1)\\ &= 10 - 5i\end{aligned}$$

The correct answer is C.

Vocabulary Review

As you solve test items, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. multiplicity

B. synthetic division

C. Conjugate Root Theorem

D. relative maximum

E. polynomial

I. the process of dividing a polynomial by a linear factor, omitting all variables and exponents

II. if $a + bi$ is an irrational root where a and b are real numbers, then $a - bi$ is also a root

III. a monomial or the sum of monomials

IV. the greatest y-value in a region of a graph

V. the number of times a zero is repeated in a polynomial function

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

1. Which statement is true about this system of linear equations?

$$\begin{cases} 3x - 4y = 12 \\ 6x - 8y = 12 \end{cases}$$

Ⓐ The solution is $(0, -1)$.

Ⓑ The solution is $(8, 4)$.

Ⓒ There is no solution because the lines are parallel.

Ⓓ There are infinitely many solutions because the lines are coinciding.

2. Which of the following statements is *never* true about a quartic function?

Ⓕ The end behavior of the function is up and up.

Ⓖ The function has 4 zeros.

Ⓗ The function has 4 turning points.

Ⓘ The function has complex roots.

3. Tia deposited x dollars in a bank account that paid 4% interest. She also deposited y dollars in a bank account that paid 8% interest. The system below represents one year's interest on Tia's deposits.

$$\begin{cases} 0.04x + 0.08y = 240 \\ 0.04x = 0.08y \end{cases}$$

Based on the solution of the system of equations, which of the following can you conclude?

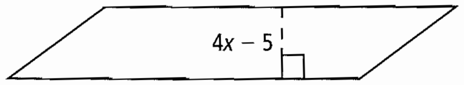
(A) Tia deposited \$3000 in each account, and the amounts of interest earned were \$240 and \$120.

(B) Tia deposited \$3000 in each account, and the amount of interest earned in each account was \$120.

(C) The deposit amounts were \$3000 and \$1500, and the amounts of interest earned in each account were \$240 and \$120.

(D) The deposit amounts were \$3000 and \$1500, and the amount of interest earned in each account was \$120.

4. The total area of the parallelogram below is $4x^4 + 3x^3 - 14x^2 + 33x - 35$. Which of the following expressions best represents the length of the base of the parallelogram? (*Hint:* $A = bh$)



(F) $x^3 + 2x^2 - x + 7$

(G) $x^3 - 2x^2 + x - 7$

(H) $4x^4 + 3x^3 - 14x^2 + 33x - 7$

(I) $x^3 + 5x^2 - x + 5$

5. Which point corresponds to a zero of the function $f(x) = x^2 + 2x - 15$?

(A) (0, −15)

(B) (5, 0)

(C) (−5, 0)

(D) (−3, 0)

6. Solve the equation $x^2 + 3w = P$ for x.

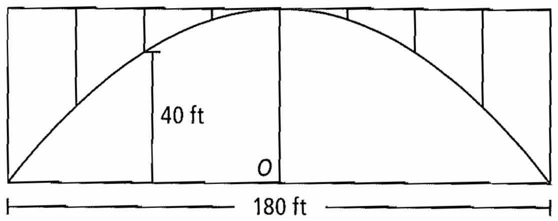
(F) $x = P - 3w$

(G) $x = \pm\sqrt{\frac{P}{3w}}$

(H) $x = \pm\sqrt{P - 3w}$

(I) $x = \pm\sqrt{P + 3w}$

7. A bridge supported by a parabolic arch spans a stream of water 180 feet wide. There must be a clearance of at least 40 feet over a 100-foot channel in the middle of the stream. The origin is placed at water level directly below the center of the arch. Which equation best represents the situation?



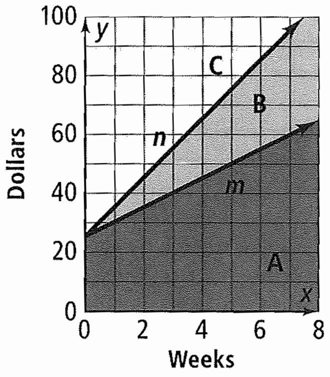
(A) $y = 140(x + 180)(x - 180)$

(B) $y = -\frac{1}{140}(x + 90)(x - 90)$

(C) $y = -\frac{1}{140}(x + 40)(x - 40)$

(D) $y = 140(x + 40)(x - 40)$

8. Sofia has \$25 in her savings account. She plans to deposit between \$5 and \$10 each week into her account. On the graph, line m represents a deposit of exactly \$5 per week and line n represents a deposit of exactly \$10 per week.



If Sofia deposits between \$5 and \$10 per week, which region on the graph represents all possible balances in her account?

(F) A

(G) B

(H) C

(I) A and C combined

9. Solve the absolute value inequality $-2|x - 3| \leq -16$.

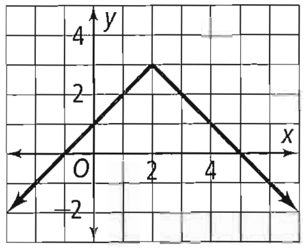
Ⓐ $x \leq -3$ or $x \geq 4$

Ⓑ $x \leq -5$ or $x \geq 11$

Ⓒ $-5 \leq x \leq 11$

Ⓓ $-11 \leq x \leq 5$

10. Which equation does the graph represent?



Ⓕ $y = -|x - 2| + 3$

Ⓖ $y = -|x - 3| + 2$

Ⓗ $y = |x - 2| + 3$

Ⓘ $y = |x + 2| + 3$

11. Which equation shows one of the steps in solving $x^2 - 4x = 8$ by completing the square?

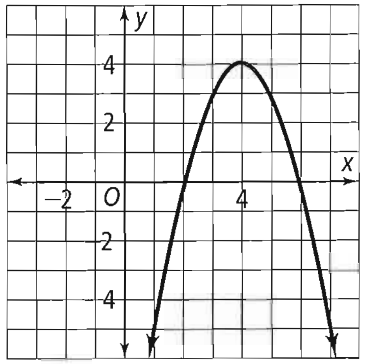
Ⓐ $x^2 - 4x + \left(\frac{-4}{2}\right)^2 = 8$

Ⓑ $x^2 - 4x + 4 = 4$

Ⓒ $x^2 - 4x - \left(\frac{4}{2}\right)^2 = 8 - \left(\frac{4}{2}\right)^2$

Ⓓ $x^2 - 4x + 4 = 8 + 4$

12. Describe the sequence of transformations that would take the graph of the parent function $y = x^2$ to the graph shown.



Ⓕ Stretch the graph of $y = x^2$ vertically by the factor 2, translate 4 units to the right and 4 units up. Then reflect in the x-axis.

Ⓖ Reflect the graph of $y = x^2$ in the x-axis, stretch the graph vertically by the factor 2, then translate 4 units to the right and 4 units up.

Ⓗ Reflect the graph of $y = x^2$ in the x-axis, then translate 4 units to the right and 4 units up.

Ⓘ Translate the graph of $y = x^2$ by 4 units to the right and 4 units up. Then reflect in the x-axis.

GRIDDED RESPONSE

13. The power created by a wind turbine varies directly as the cube of the wind speed in miles per hour. A turbine with 30% efficiency spinning in a 50 mile per hour wind can be expected to produce approximately 10,000 watts of electricity. How many watts would the same turbine produce in a 25 mile per hour wind? If necessary, round your answer to the nearest whole number.

14. One root of a cubic equation is $2i$. How many real roots does the equation have?

15. What is the value of the real part of the quotient of $(6 + 4i)$ and $(5 - i)$? Express your answer as a fraction.

16. If the solutions of an equation are -1, 2, and 5, what is the sum of the zeros of the related function?

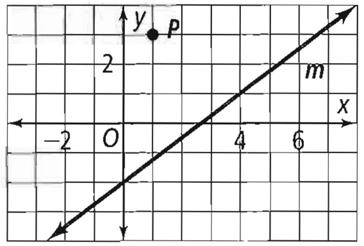
17. Assume y varies directly with x. If $y = -3$ when $x = -\frac{2}{5}$, what is x when y is 45?

18. What is the x-coordinate of the point where a relative maximum of $g(x) = -2x^3 + 6x^2 - 10$ occurs?

19. Using a graph, find the real zero of the function $y = 2x^3 - 10x^2 + 18x - 90$.

20. How many imaginary roots does $x^2 - 5x + 10 = 0$ have?

21. The graph shows line m and point P. What is the value of the y-intercept of the line n that goes through point P and is perpendicular to line m?



22. A cat, walking along the x-axis, avoids a telephone pole at (2, 0) by following the graph of $y = -x^2 + 4x - 1$ in the first quadrant. At the halfway point on the detour around the pole, what is the y-value of the cat's position?

CHAPTER 6

Get Ready!

Lessons 2-1 and 4-1

Finding the Domain and Range of Functions

Find the domain and range of each function.

1. $\{(1, 2), (2, 3), (3, 4), (4, 5)\}$

2. $\{(1, 2), (2, 2), (3, 2), (4, 2)\}$

3. $f(x) = (x - 4)^2 - 8$

4. $f(x) = 2x^2 + 3$

Lesson 4-1

Graphing Quadratic Functions

Graph each function.

5. $y = 2x^2 - 4$

6. $y = -3(x^2 + 1)$

7. $y = \frac{1}{2}(x - 3)^2 + 1$

Lesson 4-4

Multiplying Binomials

Multiply.

8. $(3y - 2)(y - 4)$

9. $(7a + 10)(7a - 10)$

10. $(x - 3)(x + 6)(x + 1)$

Lessons 4-5 and 5-3

Solving by Factoring

Solve each equation by factoring.

11. $x^2 - 5x - 14 = 0$

12. $2x^2 - 11x + 15 = 0$

13. $3x^2 + 10x - 8 = 0$

14. $12x^2 - 12x + 3 = 0$

15. $8x^2 - 98 = 0$

16. $x^4 - 14x^2 + 49 = 0$

Looking Ahead Vocabulary

17. Combining two or more elements forms composite chemical mixtures. In some cases, if you change the order in which you mix two chemicals, it can produce very different results. A *composite function* is made by combining two functions. If you are buying a $60 shirt and there is a 50% off sale and you have a $10 coupon, does it make a difference which discount is applied first?

18. One-to-one relationships describe situations where people are matched with unique identifiers, such as their social security numbers. A function is a relation that matches *x* values to *y* values. What do you suppose a *one-to-one function* is?

19. In an orchestra, the principal player is chosen among all the other musicians that play a certain instrument to sit in the first chair and lead his section. In math, what do you suppose a *principal root* is?

CHAPTER

6

Radical Functions and Rational Exponents

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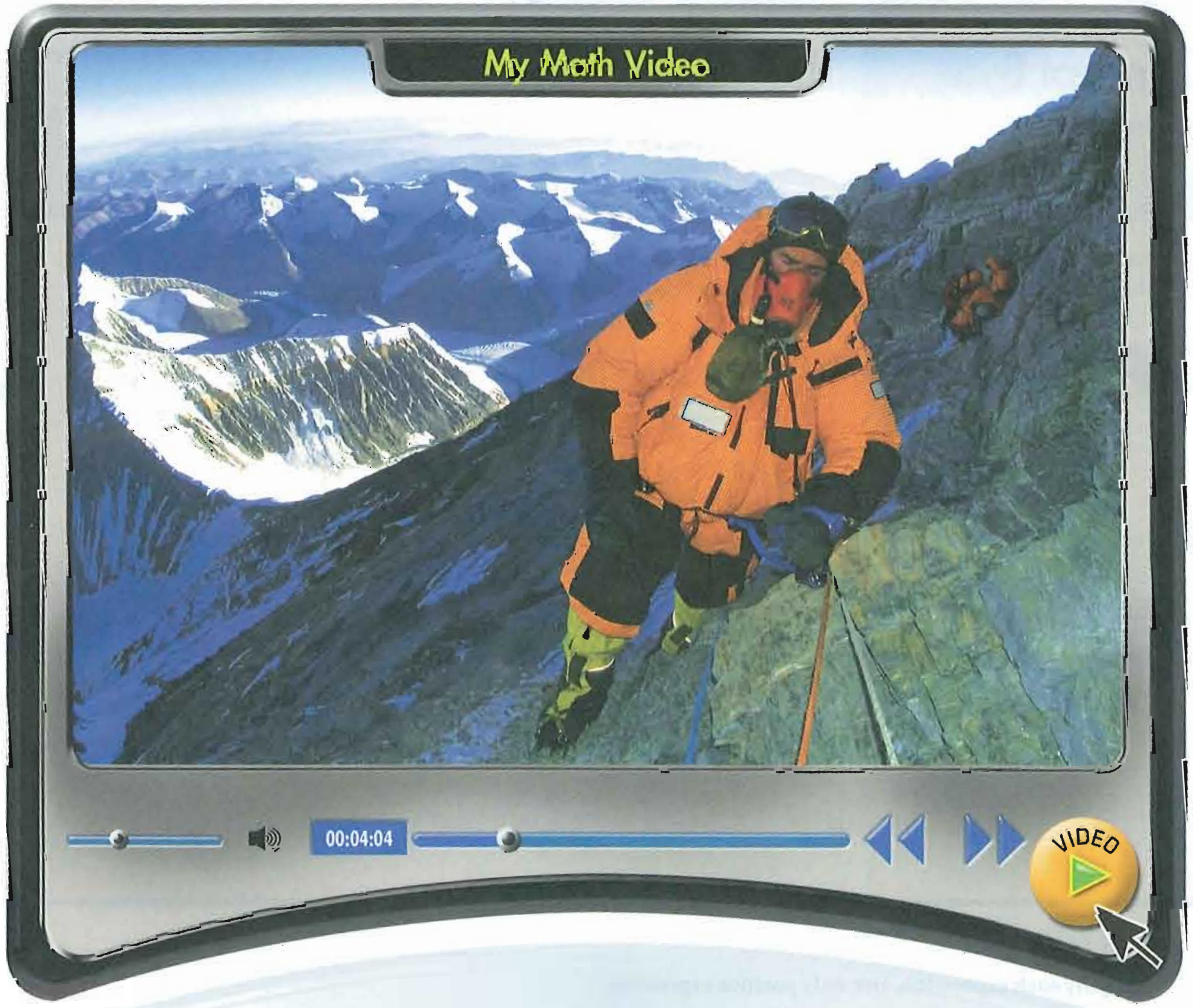
In this chapter, you will learn how to work with radicals, whether they occur by themselves or as parts of functions or equations; whether they appear with the symbol $\sqrt{}$ or as fractional exponents. In a sense, radicals are the inverses of powers. Mountain climbers need to be aware of air pressure, which can be calculated using fractional exponents.

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
composite function, *p. 399*	función compuesta
inverse function, *p. 405*	función inversa
*n*th root, *p. 361*	raíz *n*-ésima
principal root, *p. 361*	raíz principal
radical equation, *p. 390*	ecuación radical
radicand, *p. 362*	radicando
rational exponent, *p. 382*	exponente racional
rationalize the denominator, *p. 369*	racionalizar el denominador
square root equation, *p. 390*	ecuación de raíz cuadrada
square root function, *p. 415*	función de raíz cuadrada



BIG ideas

1 Equivalence

Essential Question To simplify the *n*th root of an expression, what must be true about the expression?

2 Solving Equations and Inequalities

Essential Question When you square each side of an equation, is the resulting equation equivalent to the original?

3 Function

Essential Question How are a function and its inverse function related?

Chapter Preview

Concept Byte

For Use With Lesson 6-1

REVIEW

Properties of Exponents

Sunshine State Standard
Prepares for MA.912.A.6.3 Simplify expressions using properties of rational exponents.

Exponents indicate powers. The table below lists the properties of exponents. Assume that no denominator is equal to zero and that *m* and *n* are integers.

take note

Properties **Properties of Exponents**

- $a^0 = 1, a \neq 0$
- $\frac{a^m}{a^n} = a^{m-n}$
- $a^{-n} = \frac{1}{a^n}$
- $(ab)^n = a^n b^n$
- $(a^m)^n = a^{mn}$
- $a^m \cdot a^n = a^{m+n}$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example

Simplify and rewrite each expression using only positive exponents.

a. $(5a^3)(-3a^{-4})$

$$\begin{aligned}(5a^3)(-3a^{-4}) &= 5(-3)a^{(3+(-4))}\\ &= -15a^{-1}\\ &= \frac{-15}{a}, \text{ or } -\frac{15}{a}\end{aligned}$$

b. $(-4x^{-3}y^5)^2$

$$\begin{aligned}(-4x^{-3}y^5)^2 &= (-4)^2(x^{-3})^2(y^5)^2\\ &= 16x^{-6}y^{10}\\ &= \frac{16y^{10}}{x^6}\end{aligned}$$

c. $\frac{4ab^6c^3}{a^5bc^3}$

$$\begin{aligned}\frac{4ab^6c^3}{a^5bc^3} &= 4a^{(1-5)}b^{(6-1)}c^{(3-3)}\\ &= 4a^{-4}b^5c^0\\ &= \frac{4b^5}{a^4}\end{aligned}$$

Exercises

Simplify each expression. Use only positive exponents.

1. $(2a^3)(5a^4)$

2. $(-3x^2)(-4x^{-2})$

3. $(3x^2y^3)^2$

4. $(3x^{-4}y^3)^2$

5. $\frac{4a^8}{2a^4}$

6. $\frac{12x^5y^3}{4x^{-1}}$

7. $\frac{(6x^3)^0}{3xy^2}$

8. $\left(\frac{2x^4}{3}\right)^3$

9. $(-4m^2n^3)(2mn)$

10. $(2x^3y^7)^{-2}$

11. $\frac{(3r^{-2}s^3t^0)^{-3}}{3rs}$

12. $(h^7k^3)^0$

13. $\frac{r^2s^4t^6}{r^3s^4t^{-6}}$

14. $\frac{x^2y}{4} \cdot \frac{16x}{y}$

15. $(s^4t)^2(st)$

16. $\left(\frac{1}{h^{-2}}\right)^{-1} \cdot h^3$

17. $\frac{1}{a^2b^{-3}}(a^2b^{-3})^{-1}$

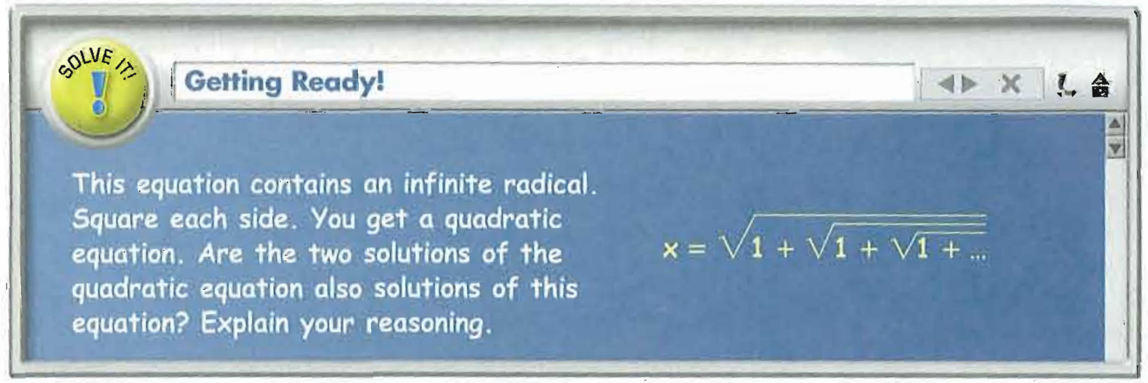
18. $\left(\frac{r^{-1}s^2t^{-3}}{r^{-2}s^0t^1}\right)^{-1}$

19. **Reasoning** Your friend tells you that $(k^2)^{-5} = -k^{10}$. Did she apply the properties of exponents correctly? Explain why or why not.

6-1 Roots and Radical Expressions

Sunshine State Standard
Prepares for MA.912.A.6.2 Add, subtract, multiply, and divide radical expressions.

Objective To find *n*th roots



SOLVE IT! **Getting Ready!**

This equation contains an infinite radical. Square each side. You get a quadratic equation. Are the two solutions of the quadratic equation also solutions of this equation? Explain your reasoning.

$$x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$$

Dynamic Activity
Simplifying Radical Expressions

Lesson Vocabulary
- *n*th root
- principal root
- radicand
- index

In Chapter 5, you used *root* to represent a solution of an equation. For example, 2 is a root of the equation $x^3 = 8$. For such a simple power equation, you can simply refer to 2 as a cube root of 8.

Essential Understanding Corresponding to every power, there is a root. For example, just as there are squares (second powers), there are square roots. Just as there are cubes (third powers), there are cube roots, and so on.

$5^2 = 25$	5 is a square root of 25.
$5^3 = 125$	5 is a cube root of 125.
$5^4 = 625$	5 is a fourth root of 625.
$5^5 = 3125$	5 is a fifth root of 3125.

This pattern suggests a definition of an *n*th root.

take note

Key Concept The *n*th Root

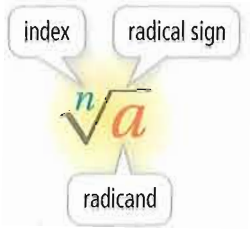
If $a^n = b$, with a and b real numbers and n a positive integer, then a is an ***n*th root** of b.

If *n* is odd...
there is one real *n*th root of b, denoted in radical form as $\sqrt[n]{b}$.

If *n* is even...
- and b is positive, there are two real *n*th roots of b. The positive root is the **principal root** (or principal *n*th root) and its symbol is $\sqrt[n]{b}$. The negative root is its opposite, or $-\sqrt[n]{b}$.
- and b is negative, there are no real *n*th roots of b.

The only *n*th root of 0 is 0.

You use a radical sign to indicate a root. The number under the radical sign is the **radicand**. The **index** gives the degree of the root.



Problem 1 Finding All Real Roots

Plan

How many real cube roots are there?
A cube root is the same as a third root, and 3 is odd. So there is only one real cube root of a number.

A What are the real cube roots of 0.008, −1000, and $\frac{1}{27}$?

$0.008 = (0.2)^3$ 0.2 is the only real cube root of 0.008.

$-1000 = (-10)^3$ −10 is the only real cube root of −1000.

$\frac{1}{27} = \left(\frac{1}{3}\right)^3$ $\frac{1}{3}$ is the only real cube root of $\frac{1}{27}$.

B What are the real fourth roots of 1, −0.0001, and $\frac{16}{81}$?

Since 1 is positive, there are two real fourth roots.

$1 = 1^4$ 1 is a real fourth root of 1.

$1 = (-1)^4$ −1 is the other real fourth root of 1.

Since −0.0001 is negative, there are no real fourth roots of −0.0001.

Since $\frac{16}{81}$ is positive, there are two real fourth roots.

$\frac{16}{81} = \left(\frac{2}{3}\right)^4$ $\frac{2}{3}$ is a real fourth root of $\frac{16}{81}$.

$\frac{16}{81} = \left(-\frac{2}{3}\right)^4$ $-\frac{2}{3}$ is the other real fourth root of $\frac{16}{81}$.

Got It? **1. a.** What are the real fifth roots of 0, −1, and 32?

b. What are the real square roots of 0.01, −1, and $\frac{36}{121}$?

c. Reasoning Explain why a negative real number b has no real nth roots if n is even.

According to the Fundamental Theorem of Algebra, $x^4 - 1 = 0$ has four roots, only two of which are real. In this chapter, the focus is on real roots only.

Problem 2 Finding Roots

Plan

How can you find a cube root?
Work backwards. Find a number whose cube is the radicand.

What is each real-number root?

A $\sqrt[3]{-8}$

$(-2)^3 = -8$

So, $\sqrt[3]{-8} = -2$.

B $\sqrt{0.04}$

$(0.2)^2 = 0.04$

So, $\sqrt{0.04} = 0.2$.

$(-0.2)^2 = 0.04$ also, but $\sqrt{0.04}$ represents the positive square root.

C $\sqrt[4]{-1}$

There is no real root because there is no real number whose fourth power is −1.

D $\sqrt{(-2)^2}$

$\sqrt{(-2)^2} = \sqrt{4} = 2.$

Got It? **2.** What is each real-number root?

a. $\sqrt[3]{-27}$ **b.** $\sqrt[4]{-81}$ **c.** $\sqrt{(-7)^2}$ **d.** $\sqrt{-49}$

It is tempting to conclude that $\sqrt[n]{a^n} = a$ for all real numbers a, but part (d) of Problem 2 shows that this is not the case. If n is even, then $\sqrt[n]{a^n}$ is positive even if a itself is negative.

take note

Property **nth Roots of nth Powers**

For any real number a, $\sqrt[n]{a^n} = \begin{cases} a \text{ if } n \text{ is odd} \\ |a| \text{ if } n \text{ is even} \end{cases}$

It is easy to overlook this rule for simplifying radicals. It is particularly important that you remember it when the radicand contains a variable expression. You must *include* the absolute value when n is even, and you must *omit* it when n is odd.

Problem 3 Simplifying Radical Expressions

Plan

How can you get started?
You're simplifying a square root, so use properties of exponents to write the *entire* radicand as a perfect square.

What is a simpler form of each radical expression?

A $\sqrt{16x^8}$

$\sqrt{16x^8} = \sqrt{4^2(x^4)^2} = \sqrt{(4x^4)^2} = |4x^4| = 4x^4$

You need to include absolute value symbols because the index of a square root is 2, which is even. However, $|4x^4| = 4x^4$ because x^4 is always nonnegative.

B $\sqrt[3]{a^6b^9}$

$\sqrt[3]{a^6b^9} = \sqrt[3]{(a^2)^3(b^3)^3} = \sqrt[3]{(a^2b^3)^3} = a^2b^3$

The index is odd, so you cannot include absolute value symbols here.

C $\sqrt[4]{x^8y^{12}}$

$\sqrt[4]{x^8y^{12}} = \sqrt[4]{(x^2)^4(y^3)^4} = \sqrt[4]{(x^2y^3)^4} = x^2|y^3|$

The index is even. The absolute value symbols ensure that the root is positive when y^3 is negative. Absolute value symbols are not needed for x^2 since x^2 is always nonnegative.

Got It? **3.** What is the simplified form of each radical expression?

a. $\sqrt{81x^4}$ **b.** $\sqrt[3]{a^{12}b^{15}}$ **c.** $\sqrt[4]{x^{12}y^{16}}$

Problem 4 Using a Radical Expression

Academics Some teachers adjust test scores when a test is difficult. One teacher's formula for adjusting scores is $A = 10\sqrt{R}$, where A is the adjusted score and R is the raw score. If the raw scores on one test range from 36 to 90, what is the range of the adjusted scores?

Think	Write
You have to adjust the lowest raw score and the highest raw score.	$10\sqrt{36} = 10(6) = 60$ $10\sqrt{90} \approx 10(9.487) = 94.87 \approx 95$
The other adjusted scores must be between the lowest and highest adjusted scores.	The adjusted scores range from 60 to 95.

Got It? **4.** In Problem 4, what are the adjusted scores for raw scores of 0 and 100?

Lesson Check

Do you know HOW?

Find all the real square roots of each number.

1. 25 **2.** 0.16 **3.** −64

Simplify each radical expression.

4. $\sqrt{9b^2}$ **5.** $\sqrt{a^8b^{18}}$ **6.** $\sqrt[3]{-125a^3}$

Do you UNDERSTAND?

7. **Error Analysis** A student said the only fourth root of 16 is 2. Describe and correct his error.

8. **Vocabulary** Explain the difference between a real root and the principal root.

9. **Reasoning** A number has only one real nth root. What can you conclude about the index n?

Practice and Problem-Solving Exercises

A Practice

Find all the real square roots of each number. **See Problem 1.**

10. 225 **11.** 0.0049 **12.** $-\frac{1}{121}$ **13.** $\frac{64}{169}$

Find all the real cube roots of each number.

14. −64 **15.** 0.125 **16.** $-\frac{27}{216}$ **17.** 0.000343

Find all the real fourth roots of each number.

18. 16 **19.** −16 **20.** 0.0081 **21.** $\frac{10,000}{81}$

Find each real root.

See Problem 2.

22. $\sqrt{36}$

23. $\sqrt{0.25}$

24. $-\sqrt[3]{64}$

25. $\sqrt[3]{-27}$

Simplify each radical expression. Use absolute value symbols when needed.

See Problem 3.

26. $\sqrt{16x^2}$

27. $\sqrt[3]{27y^6}$

28. $\sqrt[4]{x^{20}y^{28}}$

29. $\sqrt[5]{32y^{10}}$

See Problem 4.

30. **Grades** In many classes, a passing test grade is 70. Using the formula $A = 10\sqrt{R}$, what raw score would a student need to get a passing grade after her score is adjusted?

B Apply

Find the two real solutions of each equation.

31. $x^2 = 100$

32. $x^4 = 1$

33. $x^2 = 0.25$

34. $x^4 = \frac{16}{81}$

35. **Think About a Plan** The radius of a spherical balloon can be expressed as $r = \sqrt[3]{\frac{3V}{4\pi}}$ inches, where r is the radius and V is the volume of the balloon in cubic inches. If air is pumped to inflate the balloon from 500 cubic inches to 800 cubic inches, by how many inches has the radius of the balloon increased?

- What was the radius of the balloon originally?
- What was the radius after inflating the balloon to 800 cubic inches?
- How can you use the two radii to find the amount of increase?

36. **Electricity** The voltage V of an audio system's speaker can be represented by $V = 4\sqrt{P}$, where P is the power of the speaker. An engineer wants to design a speaker with 400 watts of power. What will the voltage be?

37. **Boat Building** Boat builders share an old rule of thumb for sailboats. The maximum speed K in knots is 1.35 times the square root of the length L in feet of the boat's waterline.

a. A customer is planning to order a sailboat with a maximum speed of 12 knots. How long should the waterline be?

b. How much longer would the waterline have to be to achieve a maximum speed of 15 knots?

Simplify each radical expression. Use absolute value symbols when needed.

38. $\sqrt[3]{0.125}$

39. $\sqrt[3]{\frac{8}{216}}$

40. $\sqrt[4]{0.0016}$

41. $\sqrt[4]{\frac{1}{256}}$

42. $\sqrt[4]{16c^4}$

43. **Open-Ended** Write three radical expressions that simplify to $-2x^2$.

44. **Reasoning** For what positive integers n is each of the statements true?

a. If $x^n = b$, then x is an nth root of b.

b. If $x^n = b$, then $x = \sqrt[n]{b}$.

Is each equation *always, sometimes,* or *never* true? Explain your answer.

45. $\sqrt{x^4} = x^2$

46. $\sqrt{x^6} = x^3$

47. $\sqrt[3]{x^8} = x^2$

48. $\sqrt[3]{x^3} = |x|$

Challenge **Simplify each radical expression if n is even, and then if n is odd.**

49. $\sqrt[n]{m^n}$ **50.** $\sqrt[n]{m^{2n}}$ **51.** $\sqrt[n]{m^{3n}}$ **52.** $\sqrt[n]{m^{4n}}$

53. **Reasoning** How many square roots of integers are in the interval between 24 and 25?

54. **Reasoning** The square root of a positive integer is either a positive integer or an irrational number. Is this a true statement or not? Explain your reasoning.

55. **Geometry** Without using a calculator, determine which is greater: the altitude of an equilateral triangle with side 8 or the diagonal of a square with side 5. Show your work.

Sunshine State Standards Practice

MA.912.A.6.2 **56.** Which equation has more than one real-number solution?

Ⓐ $x^2 = 0$ Ⓑ $x^2 = 1$ Ⓒ $x^2 = -1$ Ⓓ $x^3 = -1$

MA.912.A.4.6 **57.** According to the Rational Root Theorem, which of the following is NOT a possible root of the polynomial equation $7x^5 + 3x^2 - 4x + 21 = 0$?

Ⓕ $\frac{1}{7}$ Ⓖ $\frac{1}{3}$ Ⓗ 3 Ⓘ 7

MA.912.A.7.6 **58.** The fuse of a three-break firework rocket is programmed to ignite three times with 2-second intervals between the ignitions. When the rocket is shot vertically in the air, its height h in feet after t seconds is given by the formula $h(t) = -5t^2 + 70t$. At how many seconds after the shot should the firework technician set the timer of the first ignition to make the second ignition occur when the rocket is at its highest point?

Ⓐ 3 Ⓑ 9 Ⓒ 5 Ⓓ 7

MA.912.A.4.5 **59.** **Extended Response** Write a system of equations to find a cubic polynomial that goes through $(-3, -35)$, $(0, 1)$, $(2, 3)$, and $(4, 7)$.

Mixed Review

Determine the cubic function that is obtained from the parent function $y = x^3$ after each sequence of transformations. See Lesson 5-9.

60. translation up 3 units and to the left 2 units

61. vertical compression by a factor of $\frac{1}{2}$, translation down 2 units

Solve each equation by using the Quadratic Formula. See Lesson 4-7.

62. $-4x^2 + 7x - 3 = 0$ **63.** $3x^2 - 5x + 3 = 0$ **64.** $36x^2 - 132x + 121 = 0$

Get Ready! **To prepare for Lesson 6-2, do Exercises 65–67.**

Simplify each algebraic expression. See Lesson 1-3.

65. $\frac{14x^7y^9}{7x^4y^6}$ **66.** $\frac{3abc}{9b}$ **67.** $\frac{20x}{5x^3}$

6-2 Multiplying and Dividing Radical Expressions

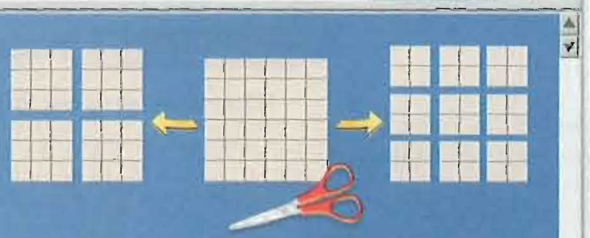
Sunshine State Standard
MA.912.A.6.2 Multiply and divide radical expressions.

Objective To multiply and divide radical expressions

Solve It! **Getting Ready!**

Cutting *n*-squares into 1-squares doesn't count.

You can cut the 36-square into four 9-squares or nine 4-squares. What other *n*-square can you cut into sets of smaller squares in two ways? Is there a square you can cut into smaller squares in three ways? Explain your reasoning.



Lesson Vocabulary
- simplest form of a radical
- rationalize the denominator

Knowing the perfect squares greater than 1 (namely, 4, 9, 16, and so on) will help you simplify some radical expressions.

Essential Understanding You can simplify a radical expression when the exponent of one factor of the radicand is a multiple of the radical's index.

You can simplify the product of powers that have the same exponent. Similarly, you can simplify the product of radicals that have the same index.

Same Exponent	Same Index
$2^2 \cdot 3^2 = (2 \cdot 3)^2$	$\sqrt{2} \cdot \sqrt{3} = \sqrt{2 \cdot 3}$
$4^3 \cdot 5^3 = (4 \cdot 5)^3$	$\sqrt[3]{4} \cdot \sqrt[3]{5} = \sqrt[3]{4 \cdot 5}$

take note **Property** **Combining Radical Expressions: Products**

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.

Problem 1 Multiplying Radical Expressions

Plan
What allows you to use the property for multiplying radicals?
The radicals must be real numbers. The indexes must be the same.

Can you simplify the product of the radical expressions? Explain.

A $\sqrt[3]{6} \cdot \sqrt{2}$

No. The indexes are different. The property above does not apply.

B $\sqrt[3]{-4} \cdot \sqrt[3]{2}$

Yes. $\sqrt[3]{-4} \cdot \sqrt[3]{2} = \sqrt[3]{-4(2)} = \sqrt[3]{-8} = -2$.

Got It? **1.** Can you simplify the product of the radical expressions? Explain.

a. $\sqrt[4]{7} \cdot \sqrt[5]{7}$ **b.** $\sqrt[5]{-5} \cdot \sqrt[5]{-2}$

If the radicand of $\sqrt[n]{a}$ has a perfect *n*th power among its factors, you can *reduce* the radical. If you reduce a radical as much as possible, the radical is in **simplest form**. For example, consider $\sqrt{24}$ and $\sqrt[3]{24}$.

$\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{2^2 \cdot 6} = \sqrt{2^2} \cdot \sqrt{6} = 2\sqrt{6}$ $2\sqrt{6}$ is in simplest form.

$\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{2^3 \cdot 3} = \sqrt[3]{2^3} \cdot \sqrt[3]{3} = 2\sqrt[3]{3}$ $2\sqrt[3]{3}$ is in simplest form.

Problem 2 Simplifying a Radical Expression

What is the simplest form of $\sqrt[3]{54x^5}$?

$\sqrt[3]{54x^5} = \sqrt[3]{3^3 \cdot 2 \cdot x^2 \cdot x^3}$ Find all perfect cube factors.

$= \sqrt[3]{3^3x^3} \cdot \sqrt[3]{2x^2}$ $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$= 3x\sqrt[3]{2x^2}$ Simplify.

Think

How do you know when you are done simplifying?
You are done when the radicand contains no perfect cube factors.

Got It? **2.** What is the simplest form of $\sqrt[3]{128x^7}$?

Problem 2 involves simplifying a cube root, so absolute value symbols are not needed. Remember that to combine $\sqrt[n]{a}$ and $\sqrt[n]{b}$ by multiplication, both radical expressions must be real numbers.

Problem 3 Simplifying a Product

What is the simplest form of $\sqrt{72x^3y^2} \cdot \sqrt{10xy^3}$?

Think

You need to multiply the radicands and find the perfect square factors.

Now find square roots. Since $\sqrt{72x^3y^2}$ and $\sqrt{10xy^3}$ must be real numbers, *x* and *y* are nonnegative, so no absolute value symbols are needed.

Write

$$\begin{aligned}\sqrt{72x^3y^2} \cdot \sqrt{10xy^3} &= \sqrt{(72x^3y^2)(10xy^3)} \\ &= \sqrt{720x^4y^5} \\ &= \sqrt{12^2(5)(x^2)^2(y^2)^2y} \\ &= \sqrt{12^2(x^2)^2(y^2)^2} \cdot \sqrt{5y} \\ &= 12|x^2y^2| \cdot \sqrt{5y} \\ &= 12x^2y^2\sqrt{5y}\end{aligned}$$

The simplest form is $12x^2y^2\sqrt{5y}$.

Got It? **3.** What is the simplest form of $\sqrt{45x^5y^3} \cdot \sqrt{35xy^4}$?

Since you define division in terms of multiplication, you can extend the property for multiplying radical expressions. If the indexes are the same, you can write a quotient of roots as a root of a quotient.

Multiplying	Dividing
$\sqrt{2} \cdot \sqrt{3} = \sqrt{2 \cdot 3}$	$\frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$
$\sqrt[3]{4} \cdot \sqrt[3]{5} = \sqrt[3]{4 \cdot 5}$	$\frac{\sqrt[3]{4}}{\sqrt[3]{5}} = \sqrt[3]{\frac{4}{5}}$

take note

Property Combining Radical Expressions: Quotients

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $b \neq 0$, then $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.

Problem 4 Dividing Radical Expressions

What is the simplest form of the quotient?

Think

Do you need to include absolute value symbols?
No. Both the divisor and dividend already require that x be nonnegative.

A $\frac{\sqrt{18x^5}}{\sqrt{2x^3}}$

$$\frac{\sqrt{18x^5}}{\sqrt{2x^3}} = \sqrt{\frac{18x^5}{2x^3}}$$
$$= \sqrt{9x^2}$$
$$= 3x$$

B $\frac{\sqrt[3]{162y^5}}{\sqrt[3]{3y^2}}$

$$\frac{\sqrt[3]{162y^5}}{\sqrt[3]{3y^2}} = \sqrt[3]{\frac{162y^5}{3y^2}}$$
$$= \sqrt[3]{54y^3}$$
$$= \sqrt[3]{27y^3} \cdot \sqrt[3]{2}$$
$$= \sqrt[3]{3^3y^3} \cdot \sqrt[3]{2}$$
$$= 3y\sqrt[3]{2}$$

Got It? **4. a.** What is the simplest form of $\frac{\sqrt{50x^6}}{\sqrt{2x^4}}$?

b. Reasoning Can you simplify the expression in Problem 4(a) by first simplifying $\sqrt{18x^5}$ and $\sqrt{2x^3}$? Explain.

Another way to simplify a radical expression is to **rationalize the denominator**. You rewrite the expression so that there are no radicals in any denominator and no denominator in any radical.

Multiply by 1.

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}$$

The product of $\sqrt{2}$ and itself is a rational number, 2.

Problem 5 Rationalizing the Denominator

Multiple Choice What is the simplest form of $\sqrt[3]{\frac{5x^2}{12y^2z}}$?

Ⓐ $\frac{\sqrt[3]{90x^2yz^2}}{6yz}$ Ⓑ $\frac{\sqrt[3]{5x^2}}{\sqrt[3]{12y^2z}}$ Ⓒ $\frac{5\sqrt[3]{x^2yz^2}}{yz}$ Ⓓ $5\sqrt[3]{x^2z}$

Think

How do you choose what to multiply by?
Choose a cube root with a radicand that will make each factor of the radicand in the denominator a perfect cube.

$$\sqrt[3]{\frac{5x^2}{12y^2z}} = \frac{\sqrt[3]{5x^2}}{\sqrt[3]{2^2 \cdot 3y^2z}}$$

The radicand in the denominator needs 2, 3^2, y, and z^2 to make the factors perfect cubes.

$$= \frac{\sqrt[3]{5x^2}}{\sqrt[3]{2^2 \cdot 3y^2z}} \cdot \frac{\sqrt[3]{2 \cdot 3^2yz^2}}{\sqrt[3]{2 \cdot 3^2yz^2}}$$

Multiply the numerator and denominator by $\sqrt[3]{2 \cdot 3^2yz^2}$.

$$= \frac{\sqrt[3]{90x^2yz^2}}{\sqrt[3]{2^3 \cdot 3^3y^3z^3}}$$

$$= \frac{\sqrt[3]{90x^2yz^2}}{2 \cdot 3yz}$$

Simplify.

$$= \frac{\sqrt[3]{90x^2yz^2}}{6yz}$$

The correct answer is A.

Got It? **5. a.** What is the simplest form of $\frac{\sqrt[3]{7x}}{\sqrt[3]{5y^2}}$?

b. Reasoning Which choice in Problem 5 could be eliminated immediately? Explain your reasoning.

Lesson Check

Do you know HOW?

Multiply, if possible. Then simplify.

1. $\sqrt{2} \cdot \sqrt{5}$

2. $\sqrt[3]{-27} \cdot \sqrt[3]{4}$

3. $\sqrt[3]{2} \cdot \sqrt[2]{7}$

4. $\sqrt{3} \cdot \sqrt{-4}$

Divide and simplify.

5. $\frac{\sqrt[3]{15x^2}}{\sqrt[3]{5x}}$

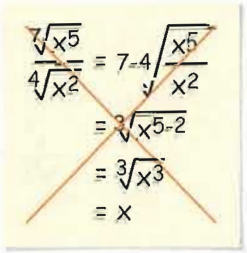
6. $\frac{\sqrt{21x^{10}}}{\sqrt{7x^5}}$

Do you UNDERSTAND?

7. Vocabulary Write the simplest form of $\sqrt[3]{32x^4}$.

8. Reasoning For what values of x is $\sqrt{-4x^3}$ real? Justify your reasoning.

9. Error Analysis Explain the error in this simplification of radical expressions.



$$\frac{\sqrt[7]{x5}}{\sqrt[4]{x2}} = \sqrt[7-4]{\frac{x5}{x2}}$$
$$= \sqrt[3]{x5\text{-}2}$$
$$= \sqrt[3]{x3}$$
$$= x$$

Practice and Problem-Solving Exercises

A Practice

Multiply, if possible. Then simplify. **See Problem 1.**

10. $\sqrt{8} \cdot \sqrt{32}$ **11.** $\sqrt[3]{4} \cdot \sqrt[3]{16}$ **12.** $\sqrt[3]{9} \cdot \sqrt[3]{-81}$

13. $\sqrt[4]{8} \cdot \sqrt[3]{32}$ **14.** $\sqrt{-5} \cdot \sqrt{5}$ **15.** $\sqrt[3]{-5} \cdot \sqrt[3]{-25}$

16. $\sqrt[3]{9} \cdot \sqrt[3]{-24}$ **17.** $\sqrt[3]{-12} \cdot \sqrt[3]{-18}$ **18.** $\sqrt{50} \cdot \sqrt{75}$

Simplify. **See Problem 2.**

19. $\sqrt{20x^3}$ **20.** $\sqrt[3]{81x^3}$ **21.** $\sqrt{50x^5}$

22. $\sqrt[3]{32a^5}$ **23.** $\sqrt[3]{54y^{10}}$ **24.** $\sqrt{200a^6b^7}$

25. $\sqrt[3]{-250x^6y^5}$ **26.** $\sqrt[4]{64x^3y^6}$ **27.** $\sqrt[5]{-32x^6y^7}$

Multiply and simplify. **See Problem 3.**

28. $\sqrt[3]{6} \cdot \sqrt[3]{16}$ **29.** $\sqrt{8y^5} \cdot \sqrt{40y^2}$ **30.** $\sqrt{8x^5} \cdot \sqrt{3x}$

31. $4\sqrt{2x} \cdot 5\sqrt{6xy^2}$ **32.** $3\sqrt[3]{5y^3} \cdot 2\sqrt[3]{50y^4}$ **33.** $-\sqrt[3]{2x^2y^2} \cdot 2\sqrt[3]{15x^5y}$

34. $\sqrt[4]{81x^5y^4} \cdot \sqrt[4]{32x^3y}$ **35.** $2\sqrt[3]{2xy^2} \cdot \sqrt[3]{4x^2y^5}$ **36.** $3\sqrt[4]{18a^9} \cdot \sqrt[4]{6ab^2}$

Divide and simplify. **See Problem 4.**

37. $\frac{\sqrt{500}}{\sqrt{5}}$ **38.** $\frac{\sqrt{48x^3}}{\sqrt{3xy^2}}$ **39.** $\frac{\sqrt{56x^5y^5}}{\sqrt{7xy}}$

40. $\frac{\sqrt[3]{250x^7y^3}}{\sqrt[3]{2x^2y}}$ **41.** $\frac{\sqrt[3]{48x^3y^2}}{\sqrt[3]{6x^4y}}$ **42.** $\frac{\sqrt{20ab}}{\sqrt{45a^2b^3}}$

Rationalize the denominator of each expression. **See Problem 5.**

43. $\frac{\sqrt{x}}{\sqrt{2}}$ **44.** $\frac{\sqrt{5}}{\sqrt{8x}}$ **45.** $\frac{\sqrt[3]{x}}{\sqrt[3]{2}}$

46. $\sqrt[3]{\frac{5}{3x}}$ **47.** $\frac{\sqrt[4]{2}}{\sqrt[4]{5}}$ **48.** $\frac{15\sqrt{60x^5}}{3\sqrt{12x}}$

49. $\frac{\sqrt{3xy^2}}{\sqrt{5xy^3}}$ **50.** $\frac{\sqrt{5x^4y}}{\sqrt{2x^2y^3}}$ **51.** $\frac{\sqrt[3]{12ab^3c^2}}{\sqrt[3]{10a^3bc}}$

B Apply

52. Think About a Plan The formula $t = \sqrt{\frac{2s}{a}}$ shows the time t that any vehicle takes to travel a distance s at a constant acceleration a, starting from rest. What is the difference in time between a car accelerating at 16 m/s^2 and one accelerating at 25 m/s^2 for a distance of 200 m?

- What is the time that a car accelerating at 16 m/s^2 takes to travel 200 m?
- What is the time that a car accelerating at 25 m/s^2 takes to travel 200 m?

53. Geometry The base of a triangle is $\sqrt{18}$ cm and its height is $\sqrt{8}$ cm. Find its area.

54. **Physics** The formula $F = \frac{mv^2}{r}$ gives the centripetal force F of an object of mass m moving along a circle of radius r, where v is the tangential velocity of the object. Solve the formula for v. Rationalize the denominator.

55. **Satellites** The circular velocity v in miles per hour of a satellite orbiting Earth is given by the formula $v = \sqrt{\frac{1.24 \times 10^{12}}{r}}$, where r is the distance in miles from the satellite to the center of the Earth. How much greater is the velocity of a satellite orbiting at an altitude of 100 mi than the velocity of a satellite orbiting at an altitude of 200 mi? (The radius of the Earth is 3950 mi.)

56. **a.** Simplify $\frac{\sqrt{2} + \sqrt{3}}{\sqrt{75}}$ by multiplying the numerator and denominator by $\sqrt{75}$.
    b. Simplify the expression in (a) by multiplying by $\sqrt{3}$ instead of $\sqrt{75}$.
    c. Explain how you would simplify $\frac{\sqrt{2} + \sqrt{3}}{\sqrt{98}}$.

Simplify each expression. Rationalize all denominators.

57. $\sqrt{5} \cdot \sqrt{50}$
58. $\sqrt[3]{4} \cdot \sqrt[3]{80}$
59. $\sqrt{x^5y^5} \cdot 3\sqrt{2x^7y^6}$
60. $5\sqrt{2xy^6} \cdot 2\sqrt{2x^3y}$
61. $\sqrt{2}(\sqrt{50} + 7)$
62. $\sqrt{5}(\sqrt{5} + \sqrt{15})$
63. $\frac{\sqrt{5x^4}}{\sqrt{2x^2y^3}}$
64. $\frac{5\sqrt{2}}{3\sqrt{7x}}$
65. $\frac{1}{\sqrt[3]{9x}}$
66. $\frac{10}{\sqrt[3]{5x^2}}$
67. $\frac{\sqrt[3]{14}}{\sqrt[3]{7x^2y}}$
68. $\frac{3\sqrt{11x^3y}}{-2\sqrt{12x^4y}}$

69. **Physics** The mass m of an object is $\sqrt{80}$ g and its volume V is $\sqrt{5}$ cm^3. Use the formula $D = \frac{m}{V}$ to find the density D of the object.

70. **Writing** Does $\sqrt{x^3} = \sqrt[3]{x^2}$ for *all*, *some*, or *no* values of x? Explain.

71. **Open-Ended** Of the equivalent expressions $\sqrt{\frac{2}{3}}$, $\frac{\sqrt{2}}{\sqrt{3}}$, and $\frac{\sqrt{6}}{3}$, which do you prefer to use for finding a decimal approximation with a calculator? Justify your reasoning.

72. **Error Analysis** Explain the error in this simplification of radical expressions.

$$\cancel{\sqrt{-2} \cdot \sqrt{-8} = \sqrt{-2(-8)} = \sqrt{16} = 4}$$

Determine whether each expression is *always*, *sometimes*, or *never* a real number. Assume that x can be any real number.

73. $\sqrt[3]{-x^2}$
74. $\sqrt{-x^2}$
75. $\sqrt{-x}$

Challenge

Simplify each expression. Rationalize all denominators.

76. $\sqrt{\sqrt{16x^4y^4}}$
77. $\sqrt{\sqrt[3]{8000}}$
78. $\sqrt[6]{\frac{y^{-3}}{x^{-4}}}$

79. **Reasoning** When $\sqrt{x^ay^b}$ is simplified, the result is $\frac{1}{x^cy^{3d}}$, where c and d are positive integers. Express a in terms of c, and b in terms of d.

Sunshine State Standards Practice

MA.912.A.6.2 **80.** What is the simplified form of the expression $\frac{3}{\sqrt{18xy^2}}$ if x and y are positive?

Ⓐ $\frac{\sqrt{2x}}{2xy}$ Ⓑ $\frac{\sqrt{2y}}{2xy}$ Ⓒ $\frac{\sqrt{54xy^2}}{2xy}$ Ⓓ $\frac{\sqrt{27xy^2}}{2xy}$

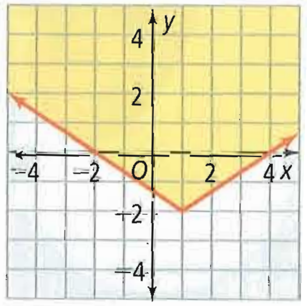
MA.912.A.7.5 **81.** What are the solutions, in simplest form, of the quadratic equation $3x^2 + 6x - 5 = 0$?

Ⓕ $\frac{-6 \pm \sqrt{96}}{6}$ Ⓖ $\frac{-6 \pm i\sqrt{24}}{6}$ Ⓗ $\frac{-3 \pm 2\sqrt{6}}{3}$ Ⓘ $\frac{-3 \pm i\sqrt{6}}{3}$

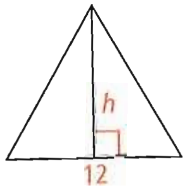
MA.912.A.2.5 **82.** Which inequality is shown by the graph at the right?

Ⓐ $y \geq \frac{2}{3}|x - 1| - 2$ Ⓒ $y \geq \frac{3}{2}|x - 1| - 2$

Ⓑ $y \geq \frac{2}{3}|x - 2| - 1$ Ⓓ $y \geq \left|\frac{2}{3}x - 1\right| - 2$



MA.912.A.2.12 **83.** A triangle has the dimensions shown below.



What is the height of a triangle with equal area but a base of 36?

Ⓕ $\frac{h}{3}$ Ⓖ $\frac{2h}{3}$ Ⓗ $2h$ Ⓘ $3h$

MA.912.A.7.6 **84. Short Response** Find the axis of symmetry of the graph of the function $y = -2x^2 - 5x + 4$. Show your work.

Mixed Review

Simplify each radical expression. Use absolute value symbols when needed. See Lesson 6-1.

85. $\sqrt{121a^{90}}$ **86.** $\sqrt{81c^{48}d^{64}}$ **87.** $\sqrt[3]{64a^{81}}$ **88.** $\sqrt[5]{32y^{25}}$

Divide using synthetic division. See Lesson 5-4.

89. $(y^3 - 64) \div (y + 4)$ **90.** $(6a^3 + a^2 - a + 4) \div (a + 1)$

Complete each square. See Lesson 4-6.

91. $x^2 + 10x + ■$ **92.** $x^2 - 10x + ■$ **93.** $x^2 + 11x + ■$ **94.** $x^2 - 11x + ■$

Get Ready! To prepare for Lesson 6-3, do Exercises 95–98.

Write each quotient as a complex number in the form $a \pm bi$. See Lesson 4-8.

95. $\frac{2}{3 - i}$ **96.** $\frac{5}{2 + 3i}$ **97.** $\frac{4}{4 + i}$ **98.** $\frac{-1}{7 - 5i}$

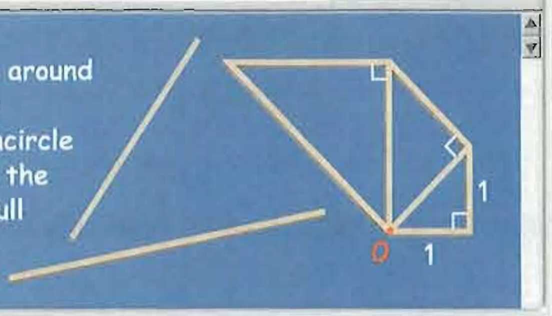
6-3 Binomial Radical Expressions

Sunshine State Standard
MA.912.A.6.2 Add, subtract, multiply, and divide radical expressions (square roots and higher).

Objective To add and subtract radical expressions

SOLVE IT! **Getting Ready!**

You are building right isosceles triangles around point O in the pattern shown. How many triangles must you build to completely encircle O? Explain your reasoning. What will be the area of the figure once you've made a full circle? What will be its perimeter?



Lesson Vocabulary
- like radicals

Like radicals are radical expressions that have the same index and radicand.

Essential Understanding You can combine like radicals using properties of real numbers.

Here is how you can combine like radicals using the Distributive Property.

Like Radicals With Numbers	Like Radicals With Variables
$\sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$	$\sqrt{5xy} + 8\sqrt{5xy} = 9\sqrt{5xy}$
$\sqrt[3]{7} - 5\sqrt[3]{7} = -4\sqrt[3]{7}$	$\sqrt[3]{9x^2y} - 8\sqrt[3]{9x^2y} = -7\sqrt[3]{9x^2y}$

take note

Property Combining Radical Expressions: Sums and Differences

Use the Distributive Property to add or subtract like radicals.

$$a\sqrt[n]{x} + b\sqrt[n]{x} = (a + b)\sqrt[n]{x} \qquad a\sqrt[n]{x} - b\sqrt[n]{x} = (a - b)\sqrt[n]{x}$$

Combining radical expressions is different from *adding* them. The sum of any two real numbers is a real number, so you can add $\sqrt{2}$ and $\sqrt{3}$ to get the real number $\sqrt{2} + \sqrt{3}$.

However, you cannot *combine* the result into a single radical, so $\sqrt{2} + \sqrt{3} \neq \sqrt{5}$.

$$\begin{array}{rl} \sqrt{2} \approx 1.414 & \quad \sqrt{5} \approx 2.236 \\ + \sqrt{3} \approx 1.732 & \\ \hline \sqrt{2} + \sqrt{3} \approx 3.146 & \quad \neq 2.236 \end{array}$$

Problem 1 Adding and Subtracting Radical Expressions

What is the simplified form of each expression?

A $3\sqrt{5x} - 2\sqrt{5x}$

$3\sqrt{5x} - 2\sqrt{5x} = (3 - 2)\sqrt{5x}$ Distributive Property

$= \sqrt{5x}$ Simplify.

B $6x^2\sqrt{7} + 4x\sqrt{5}$

The radicands are different. You cannot combine the expressions.

C $12\sqrt[3]{7xy} - 8\sqrt[5]{7xy}$

The indexes are different. You cannot combine the expressions.

Think

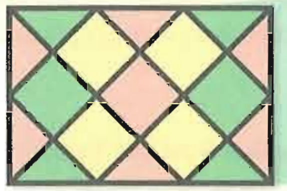
Can you always simplify a radical sum?
No. The radicands and the indexes must be the same.

Got It? **1.** What is the simplified form of each expression?

a. $7\sqrt[3]{5} - 4\sqrt{5}$ **b.** $3x\sqrt{xy} + 4x\sqrt{xy}$ **c.** $17\sqrt[5]{3x^2} - 15\sqrt[5]{3x^2}$

Problem 2 Using Radical Expressions

Architecture **In the stained-glass window design, the side of each small square is 5 in. Find the perimeter of the window to the nearest tenth of an inch.**



Length of the diagonal of a square with side s: $s\sqrt{2}$.
Length of the diagonal of each 5-inch square: $5\sqrt{2}$.

Length of the window: $l = 3(5\sqrt{2}) = 15\sqrt{2}$
Width of the window: $w = 2(5\sqrt{2}) = 10\sqrt{2}$

Perimeter $= 2l + 2w$

$= 2(15\sqrt{2}) + 2(10\sqrt{2})$ Substitute for length and width.

$= 30\sqrt{2} + 20\sqrt{2}$ Simplify.

$= 50\sqrt{2}$ Distributive Property

≈ 70.7 Use a calculator to approximate.

The perimeter of the window is about 70.7 inches.

Think

Does it make sense that you have a radical expression as the answer?
Yes, because perimeter is a linear measure, and there is no squaring in the calculations.

Got It? **2. a.** Find the perimeter of the window if the side of each small square is 6 in.

b. **Reasoning** Describe a different sequence of steps which you could use to compute the perimeter of the window.

When you have a sum or difference of radical expressions, you should simplify each expression so that you can find all the like radicals.

Problem 3 Simplifying Before Adding or Subtracting

What is the simplest form of the expression? $\sqrt{12} + \sqrt{75} - \sqrt{3}$

Think

To simplify each radical expression, factor each radicand.

These are like radicals. Combine them. Remember $\sqrt{3} = 1\sqrt{3}$.

Write

$\sqrt{12} + \sqrt{75} - \sqrt{3}$

$= \sqrt{4 \cdot 3} + \sqrt{25 \cdot 3} - \sqrt{3}$

$= \sqrt{2^2 \cdot 3} + \sqrt{5^2 \cdot 3} - \sqrt{3}$

$= \sqrt{2^2}\sqrt{3} + \sqrt{5^2}\sqrt{3} - \sqrt{3}$

$= 2\sqrt{3} + 5\sqrt{3} - \sqrt{3}$

$= 6\sqrt{3}$

Got It? **3.** What is the simplest form of the expression? $\sqrt[3]{250} + \sqrt[3]{54} - \sqrt[3]{16}$

You can use the FOIL method to multiply binomials that have radical expressions. Remember that the FOIL method ensures that you multiply each term of one binomial by each term of the other.

Problem 4 Multiplying Binomial Radical Expressions

What is the product of each radical expression?

Plan

How do you multiply two binomials?
Use the FOIL method: **F**irst, **O**uter, **I**nner, **L**ast. Then simplify.

A $(4 + 2\sqrt{2})(5 + 4\sqrt{2})$

$(4 + 2\sqrt{2})(5 + 4\sqrt{2})$

$= 4 \cdot 5 + 4 \cdot 4\sqrt{2} + 2\sqrt{2} \cdot 5 + 2\sqrt{2} \cdot 4\sqrt{2}$ Distribute.

$= 20 + 16\sqrt{2} + 10\sqrt{2} + 16$ Multiply.

$= 36 + 26\sqrt{2}$ Combine like radicals.

B $(3 - \sqrt{7})(5 + \sqrt{7})$

$(3 - \sqrt{7})(5 + \sqrt{7})$

$= 3 \cdot 5 + 3\sqrt{7} - \sqrt{7} \cdot 5 - \sqrt{7} \cdot \sqrt{7}$ Distribute.

$= 15 - 2\sqrt{7} - 7$ Multiply and combine like radicals.

$= 8 - 2\sqrt{7}$ Simplify.

Got It? **4.** What is the product $(3 + 2\sqrt{5})(2 + 4\sqrt{5})$?

Conjugates are expressions, like $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$, that differ only in the signs of the second terms. When a and b are rational numbers, the product of two radical conjugates is a rational number.

Think

Where have you seen conjugates before?
The complex number $a + bi$ has a conjugate, $a - bi$. Multiplying them results in a number with no imaginary part.

Problem 5 Multiplying Conjugates

What is the product $(5 - \sqrt{7})(5 + \sqrt{7})$?

$$(5 - \sqrt{7})(5 + \sqrt{7}) = 5 \cdot 5 + 5\sqrt{7} - 5\sqrt{7} - (\sqrt{7})^2 \quad \text{Distribute.}$$
$$= 25 - 7 \quad \text{Simplify.}$$
$$= 18$$

Got It? **5.** What is each product?

a. $(6 - \sqrt{12})(6 + \sqrt{12})$

b. $(3 + \sqrt{8})(3 - \sqrt{8})$

Sometimes a denominator is a sum or difference involving radicals. If the radical expressions are square roots, you can rationalize the denominator by multiplying the numerator and the denominator by the conjugate of the denominator.

Think

What is a rationalized denominator?
A rationalized denominator contains no radicals.

Problem 6 Rationalizing the Denominator

How can you write the expression with a rationalized denominator?

$$\frac{3\sqrt{2}}{\sqrt{5} - \sqrt{2}}$$

$$\frac{3\sqrt{2}}{\sqrt{5} - \sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{5} - \sqrt{2}} \cdot \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}} \quad \text{Multiply. Use the conjugate of the denominator.}$$
$$= \frac{3\sqrt{2}(\sqrt{5} + \sqrt{2})}{(\sqrt{5})^2 - (\sqrt{2})^2} \quad \text{The radicals in the denominator cancel out.}$$
$$= \frac{3(\sqrt{2} \cdot \sqrt{5} + \sqrt{2} \cdot \sqrt{2})}{5 - 2} \quad \text{Distribute } \sqrt{2} \text{ in the numerator.}$$
$$= \frac{3(\sqrt{10} + 2)}{3} \quad \text{Simplify.}$$
$$= \sqrt{10} + 2$$

Got It? **6.** How can you write the expression with a rationalized denominator?

a. $\frac{2\sqrt{7}}{\sqrt{3} - \sqrt{5}}$

b. $\frac{4x}{3 - \sqrt{6}}$

c. **Reasoning** Suppose you were going to rationalize the denominator of $\frac{1 - \sqrt{8}}{2 - \sqrt{8}}$. Would you simplify $\sqrt{8}$ before or after rationalizing? Explain your answer.

Lesson Check

Do you know HOW?

Simplify if possible.

1. $10\sqrt{6} + 2\sqrt{6}$

2. $3\sqrt{2} + 4\sqrt[3]{2}$

3. $8\sqrt{3x} - 5\sqrt{3x}$

4. $5\sqrt{3} + \sqrt{12}$

Multiply.

5. $(4 + \sqrt{3})(4 - \sqrt{3})$

6. $(5 + 2\sqrt{5})(7 + 4\sqrt{5})$

7. $(2 + 3\sqrt{2})(1 - 3\sqrt{2})$

Do you UNDERSTAND?

8. **Vocabulary** Determine whether each of the following is a pair of like radicals. If so, add them.

a. $3x\sqrt{11}$ and $3x\sqrt{10}$

b. $2\sqrt{3xy}$ and $7\sqrt{3xy}$

c. $12\sqrt{13y}$ and $12\sqrt{6y}$

9. **Compare and Contrast** How are the processes of multiplying radical expressions and multiplying polynomial expressions alike? How are the processes different?

Practice and Problem-Solving Exercises

A Practice

Simplify if possible. **See Problem 1.**

10. $5\sqrt{6} + \sqrt{6}$

11. $6\sqrt[3]{3} - 2\sqrt[3]{3}$

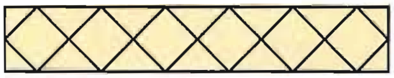
12. $4\sqrt{3} + 4\sqrt[3]{3}$

13. $3\sqrt{x} - 5\sqrt{x}$

14. $14\sqrt{x} + 3\sqrt{y}$

15. $7\sqrt[3]{x^2} - 2\sqrt[3]{x^2}$

16. The design of a garden path uses stone pieces shaped as squares with a side length of 15 in. Find the length of the path. **See Problem 2.**



Simplify. **See Problem 3.**

17. $6\sqrt{18} + 3\sqrt{50}$

18. $14\sqrt{20} - 3\sqrt{125}$

19. $\sqrt{18} + \sqrt{32}$

20. $\sqrt[3]{54} + \sqrt[3]{16}$

21. $3\sqrt[3]{81} - 2\sqrt[3]{54}$

22. $\sqrt[4]{32} + \sqrt[4]{48}$

Multiply. **See Problem 4.**

23. $(3 + \sqrt{5})(1 + \sqrt{5})$

24. $(2 + \sqrt{7})(1 + 3\sqrt{7})$

25. $(3 - 4\sqrt{2})(5 - 6\sqrt{2})$.

26. $(\sqrt{3} + \sqrt{5})^2$

27. $(\sqrt{13} + 6)^2$

28. $(2\sqrt{5} + 3\sqrt{2})^2$

Multiply each pair of conjugates. **See Problem 5.**

29. $(5 - \sqrt{11})(5 + \sqrt{11})$

30. $(4 - 2\sqrt{3})(4 + 2\sqrt{3})$

31. $(2\sqrt{6} + 8)(2\sqrt{6} - 8)$

32. $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$

Rationalize each denominator. Simplify your answer. **See Problem 6.**

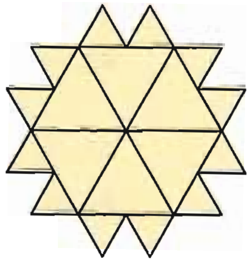
33. $\frac{4}{1 + \sqrt{3}}$

34. $\frac{4}{3\sqrt{3} - 2}$

35. $\frac{5 + \sqrt{3}}{2 - \sqrt{3}}$

36. $\frac{3 + \sqrt{8}}{2 - 2\sqrt{8}}$

B Apply



37. Think About a Plan The design on a parquet floor, shown at the right, is made of equilateral triangles. The side of a large triangle is 6 in., and the side of a small triangle is 3 in. Find the total area of the design to the nearest tenth of a square inch.

- How many large and how many small triangles form the design?
- Can you express the area of an equilateral triangle through its side?

Simplify.

38. $\sqrt{72} + \sqrt{32} + \sqrt{18}$

39. $\sqrt{75} + 2\sqrt{48} - 5\sqrt{3}$

40. $5\sqrt{32x} + 4\sqrt{98x}$

41. $\sqrt{75} - 4\sqrt{18} + 2\sqrt{32}$

42. $4\sqrt{216y^2} + 3\sqrt{54y^2}$

43. $3\sqrt[3]{16} - 4\sqrt[3]{54} + \sqrt[3]{128}$

44. $(1 + \sqrt{72})(5 + \sqrt{2})$

45. $(\sqrt{3} - \sqrt{7})(\sqrt{3} + 2\sqrt{7})$

46. $(\sqrt{y} + \sqrt{2})(\sqrt{y} - 7\sqrt{2})$

47. $(\sqrt{12} + \sqrt{72})^2$

48. $(\sqrt{1.25} - \sqrt{1.8})(\sqrt{5} + \sqrt{0.2})$

49. $(\sqrt{a+1} + \sqrt{a-1})(\sqrt{a+1} - \sqrt{a-1})$

50. Error Analysis Describe and correct the error made while simplifying the expression $\frac{3 + \sqrt{2}}{3 - \sqrt{2}}$.

$$\xcancel{\begin{aligned}\frac{3+\sqrt{2}}{3-\sqrt{2}} &= \frac{3+\sqrt{2}}{3-\sqrt{2}} \cdot \frac{3+\sqrt{2}}{3+\sqrt{2}}\\ &= \frac{3^2+(\sqrt{2})^2}{3^2-(\sqrt{2})^2} = \frac{9+2}{9-2} = \frac{11}{7}\end{aligned}}$$

51. Chemistry A scientist found that x grams of Metal A is completely oxidized in $2x\sqrt{3}$ seconds and x grams of Metal B is completely oxidized in $6x\sqrt{3}$ seconds. How much faster is Metal A oxidized than Metal B?

52. Reasoning Describe the possible values of a such that $\sqrt{72} + \sqrt{a}$ simplifies to a single term.

53. Writing Discuss the advantages and disadvantages of first simplifying $\sqrt{72} + \sqrt{32} + \sqrt{18}$ in order to estimate its decimal value.

54. Geometry Show that a right triangle with legs of lengths $\sqrt{2} - 1$ and $\sqrt{2} + 1$ is similar to a right triangle with legs of lengths $6 - \sqrt{32}$ and 2.

55. Open-Ended Find two pairs of conjugates with a product of 3.

Rationalize the denominators and simplify.

56. $\frac{4 + \sqrt{27}}{2 - 3\sqrt{27}}$

57. $\frac{4 + \sqrt{6}}{\sqrt{2} + \sqrt{3}}$

58. $\frac{5 - \sqrt{21}}{\sqrt{3} - \sqrt{7}}$

59. $\frac{\sqrt{44x^2}}{\sqrt{11} + 3}$

60. $\frac{\sqrt{2} + \sqrt{6}}{\sqrt{1.5} + \sqrt{0.5}}$

61. $\frac{\sqrt{27} - \sqrt{5}}{\sqrt{15} - 3}$

62. $\frac{4 + \sqrt[3]{2}}{\sqrt[3]{2}}$

63. $\frac{5 + \sqrt[4]{x}}{\sqrt[4]{x}}$

64. $\frac{4 - 2\sqrt[3]{6}}{\sqrt[3]{4}}$

Challenge

Add or subtract.

65. $\dfrac{1}{1 - \sqrt{5}} + \dfrac{1}{1 + \sqrt{5}}$

66. $\dfrac{4}{\sqrt{5} - \sqrt{3}} - \dfrac{4}{\sqrt{5} + \sqrt{3}}$

67. For what values of a and b does $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$?

68. In the expression $\sqrt[n]{x^m}$, m and n are positive integers and x is a real number. The expression can be simplified.

- **a.** If $x > 0$, what are the possible values for m and n?
- **b.** If $x < 0$, what are the possible values for m and n?
- **c.** If $x < 0$, and an absolute value symbol is needed in the simplified expression, what are the possible values of m and n?

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.6.2 **69.** What is the value of the expression $(5 - 2\sqrt{3})(5 + 2\sqrt{3})$?

MA.912.A.3.14 **70.** What is the value of z in the solution of the system of equations below?

$$\begin{cases} 2x - 3y + z = 6 \\ -x + y - 2z = -5 \\ 3x - y - 3z = -7 \end{cases}$$

MA.912.A.2.6 **71.** What is the y-value of the y-intercept of the line $5x - 7y = -15$?

MA.912.A.3.10 **72.** What is the slope of a line perpendicular to the line $2x + 5y = 10$?

MA.912.A.7.4 **73.** What is the value of p for which the equation $x^2 - 12x + 4p = 0$ has exactly one real root?

Mixed Review

Simplify each expression. Rationalize all denominators. **See Lesson 6-2.**

74. $\sqrt[3]{3} \cdot \sqrt[3]{18}$

75. $\sqrt[3]{\dfrac{4}{0.5x}}$

76. $\dfrac{\sqrt{32}}{\sqrt{2}}$

77. $\dfrac{\sqrt{216}}{\sqrt{6}}$

78. $\sqrt[3]{2x^2} \cdot \sqrt[3]{4x}$

79. $\sqrt{7x} \cdot \sqrt{14x^3}$

80. $\sqrt{3x} \cdot \sqrt{5x}$

81. $\sqrt{9x^2} \cdot \sqrt{25x^2}$

Solve each equation. **See Lesson 5-3.**

82. $2x^3 - 16 = 0$

83. $x^3 + 1000 = 0$

84. $125x^3 - 1 = 0$

85. $x^4 - 14x^2 + 49 = 0$

86. $25x^4 - 40x^2 + 16 = 0$

87. $81x^4 - 1 = 0$

Get Ready! **To prepare for Lesson 6-4, do Exercises 88–91.**

Simplify. **See p. 680.**

88. $(x^2)^3$

89. $(pq)^5$

90. $(2^4)(2^5)$

91. $(3^{-2})(3^5)$

6-4 Rational Exponents

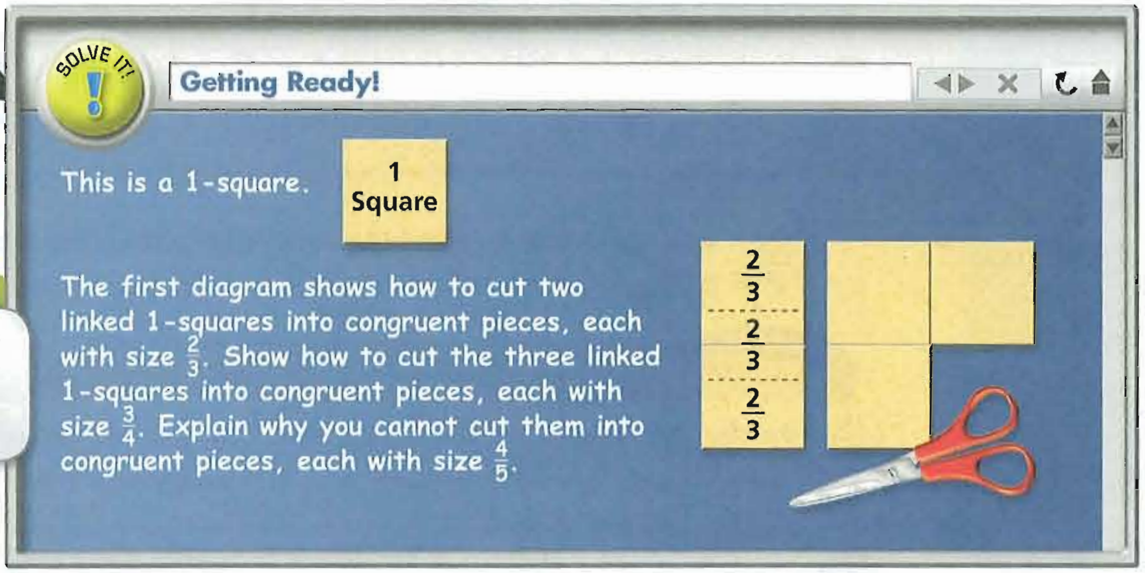
Sunshine State Standards

MA.912.A.6.3 Simplify expressions using properties of rational exponents.

MA.912.A.6.4 Convert between rational exponent and radical forms of expressions.

Objective To simplify expressions with rational exponents

It is easy to cut one 1-square into congruent pieces, each with size $\frac{1}{2}$.



This is a 1-square.

The first diagram shows how to cut two linked 1-squares into congruent pieces, each with size $\frac{2}{3}$. Show how to cut the three linked 1-squares into congruent pieces, each with size $\frac{3}{4}$. Explain why you cannot cut them into congruent pieces, each with size $\frac{4}{5}$.

Lesson Vocabulary
- rational exponent

If $a^x = \sqrt[4]{a^3}$, then by definition, $a^x \cdot a^x \cdot a^x \cdot a^x = a^3$. By adding exponents, $a^{4x} = a^3$, so x is $\frac{3}{4}$. This suggests an alternative notation for radical expressions in which, for example, $\sqrt[4]{a^3} = a^{\frac{3}{4}}$.

Essential Understanding You can write a radical expression in an equivalent form using a fractional (rational) exponent instead of a radical sign.

In general, $\sqrt[n]{x} = x^{\frac{1}{n}}$ for any positive integer n. Like the radical form, the exponent form indicates the principal root.

$$\sqrt{36} = 36^{\frac{1}{2}} \qquad \sqrt[3]{64} = 64^{\frac{1}{3}} \qquad \sqrt[4]{16} = 16^{\frac{1}{4}}$$

Problem 1 Simplifying Expressions with Rational Exponents

Think

What does the denominator of the fractional exponent represent?
The denominator of the fraction is the index of the radical.

What is the simplified form of each expression?

A $216^{\frac{1}{3}}$

$$216^{\frac{1}{3}} = \sqrt[3]{216}$$
$$= \sqrt[3]{6^3}$$
$$= 6$$

Rewrite as radicals.

B $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}}$

$$7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}} = \sqrt{7} \cdot \sqrt{7}$$
$$= \sqrt{7 \cdot 7}$$
$$= \sqrt{7^2}$$
$$= 7$$

You can also solve this problem by adding the exponents.
$7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}} = 7^{\frac{1}{2}+\frac{1}{2}} = 7^1 = 7$

C $5^{\frac{1}{4}} \cdot 125^{\frac{1}{4}}$

$5^{\frac{1}{4}} \cdot 125^{\frac{1}{4}} = \sqrt[4]{5} \cdot \sqrt[4]{125}$	Rewrite as radicals.
$= \sqrt[4]{5 \cdot 125}$	Property for multiplying radical expressions
$= \sqrt[4]{625}$	Multiply.
$= \sqrt[4]{5^4}$	Rewrite the radicand.
$= 5$	Simplify.

Got It? **1.** What is the simplified form of each expression?

a. $64^{\frac{1}{2}}$ **b.** $11^{\frac{1}{2}} \cdot 11^{\frac{1}{2}}$ **c.** $3^{\frac{1}{2}} \cdot 12^{\frac{1}{2}}$

If $\sqrt[n]{x} = x^{\frac{1}{n}}$, it follows from the Laws of Exponents that for all real numbers $\sqrt[n]{x^m} = (x^m)^{\frac{1}{n}} = (x^{\frac{1}{n}})^m = \left(\sqrt[n]{x}\right)^m$. This leads to the definition of a rational exponent.

take note

Key Concept Rational Exponent

If the nth root of a is a real number, m is an integer, and $\frac{m}{n}$ is in lowest terms, then

$a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$. If m is negative, $a \neq 0$.

Problem 2 Converting Between Exponential and Radical Forms

Think

Does the fraction $\frac{3}{7}$ first need to be simplified?
No. The fraction is already in lowest terms.

A **What are $x^{\frac{3}{7}}$ and $y^{-3.5}$ in radical form?**

$x^{\frac{3}{7}} = \sqrt[7]{x^3}$ or $(\sqrt[7]{x})^3$

$$y^{-3.5} = y^{-\frac{7}{2}}$$

$$= \frac{1}{y^{\frac{7}{2}}}$$

$$= \frac{1}{\sqrt{y^7}} = \frac{1}{\sqrt{y^6 y}} = \frac{1}{y^3\sqrt{y}} \text{ or } \frac{\sqrt{y}}{y^4}$$

B **What are $\sqrt{a^5}$ and $(\sqrt[5]{b})^3$ in exponential form?**

$\sqrt{a^5} = (a^5)^{\frac{1}{2}} = a^{\frac{5}{2}}$

$(\sqrt[5]{b})^3 = (b^{\frac{1}{5}})^3 = b^{\frac{3}{5}}$

Got It? **2. a.** What are the expressions $w^{-\frac{5}{8}}$ and $w^{0.2}$ in radical form?

b. What are the expressions $\sqrt[4]{x^3}$ and $(\sqrt[5]{y})^4$ in exponential form?

c. Reasoning Refer to the definition of rational exponent. Explain the need for the restriction that $a \neq 0$ if m is negative.

Problem 3 Using Rational Exponents

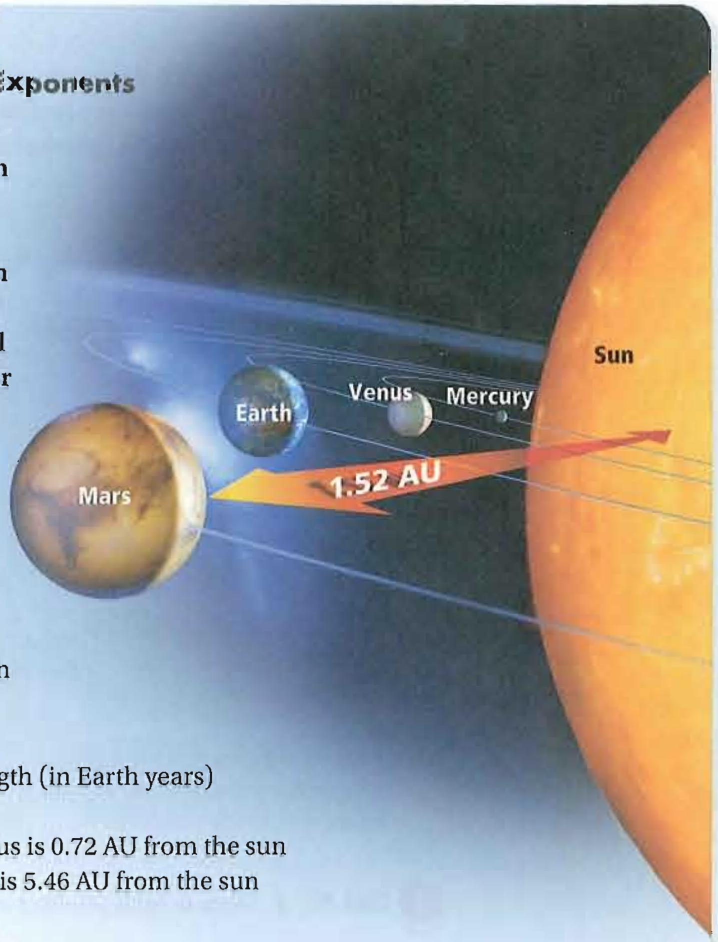
Planetary Motion Kepler's Third Law of Orbital Motion shows how you can approximate the period P (in Earth years) it takes a planet to complete one orbit of the sun. Use the function $P = d^{\frac{3}{2}}$, where d is the distance from the planet to the sun in astronomical units (AU—about 93,000,000 miles or the distance from Earth to the sun). How many Earth years does it take Mars to orbit the sun?

$P = d^{\frac{3}{2}}$ Write the formula.

$= (1.52)^{\frac{3}{2}}$ Substitute for d.

≈ 1.87 Use a calculator.

It takes Mars approximately 1.87 Earth years to orbit the sun.



Plan

How can you find a $\frac{3}{2}$ power on a calculator?
You can use ^ (3 ÷ 2).
You can also cube the number and then take the square root, or take the square root then cube.

Got It? **3.** Find the approximate length (in Earth years) of each planet's year.

a. A Venusian year if Venus is 0.72 AU from the sun

b. A Jovian year if Jupiter is 5.46 AU from the sun

All the properties of integer exponents apply to rational exponents.

take note **Properties** Properties of Rational Exponents

Let m and n represent rational numbers. Assume that no denominator equals 0.

Property	Example	Property	Example
$a^m \cdot a^n = a^{m+n}$	$8^{\frac{1}{3}} \cdot 8^{\frac{2}{3}} = 8^{\frac{1}{3}+\frac{2}{3}} = 8^1 = 8$	$a^{-m} = \frac{1}{a^m}$	$9^{-\frac{1}{2}} = \frac{1}{9^{\frac{1}{2}}} = \frac{1}{3}$
$(a^m)^n = a^{mn}$	$(5^{\frac{1}{2}})^4 = 5^{\frac{1}{2} \cdot 4} = 5^2 = 25$	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{7^{\frac{3}{2}}}{7^{\frac{1}{2}}} = 7^{\frac{3}{2}-\frac{1}{2}} = 7^1 = 7$
$(ab)^m = a^m b^m$	$(4 \cdot 5)^{\frac{1}{2}} = 4^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 2 \cdot 5^{\frac{1}{2}}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{5}{27}\right)^{\frac{1}{3}} = \frac{5^{\frac{1}{3}}}{27^{\frac{1}{3}}} = \frac{5^{\frac{1}{3}}}{3}$

Recall from Lesson 6-2 that you simplified products or quotients involving radical expressions only when they had the same index. However, you can combine radical expressions with different indexes if you convert them to expressions with rational exponents.

Problem 4 Combining Radical Expressions

What is $\frac{\sqrt[4]{x^3}}{\sqrt[8]{x^2}}$ in simplest form?

Think

The radicands are different, but both are powers of the same variable. Write the expressions using exponents.

Use the division property for exponents. Subtract the exponents.

Simplify, and write in either exponential or radical form.

Write

$$\frac{\sqrt[4]{x^3}}{\sqrt[8]{x^2}} = \frac{x^{\frac{3}{4}}}{x^{\frac{2}{8}}}$$

$$= x^{\frac{3}{4}-\frac{2}{8}}$$

$$= x^{\frac{3}{4}-\frac{1}{4}}$$

$$= x^{\frac{1}{2}} = \sqrt{x}$$

Got It? **4.** What is each product or quotient in simplest form?

a. $\sqrt{3}\,(\sqrt[4]{3})$

b. $\frac{\sqrt{x^3}}{\sqrt[3]{x^2}}$

c. $\sqrt{7}\,(\sqrt[3]{7})$

You can simplify a number with a rational exponent by using the properties of exponents or by converting the expression to a radical expression.

Problem 5 Simplifying Numbers With Rational Exponents

What is each number in simplest form?

A $16^{-2.5}$

Plan

What is the first step?
Rewrite the decimal exponent as a fraction in lowest terms.

Method 1

$$16^{-2.5} = 16^{-\frac{5}{2}}$$

$$= (2^4)^{-\frac{5}{2}}$$

$$= 2^{4\cdot-\frac{5}{2}}$$

$$= 2^{-10}$$

$$= \frac{1}{2^{10}} = \frac{1}{1024}$$

Method 2

$$16^{-2.5} = 16^{-\frac{5}{2}}$$

$$= \frac{1}{16^{\frac{5}{2}}}$$

$$= \frac{1}{(\sqrt{16})^5}$$

$$= \frac{1}{4^5}$$

$$= \frac{1}{1024}$$

Think

Does it matter that the base is negative?
No; because the denominator of the exponent is odd, there will be a real root.

B $(-32)^{\frac{4}{5}}$

Method 1

$$\begin{aligned}(-32)^{\frac{4}{5}} &= ((-2)^5)^{\frac{4}{5}}\\ &= (-2)^{5\cdot\frac{4}{5}}\\ &= (-2)^4\\ &= 16\end{aligned}$$

Method 2

$$\begin{aligned}(-32)^{\frac{4}{5}} &= \left(\sqrt[5]{-32}\right)^4\\ &= \left(\sqrt[5]{(-2)^5}\right)^4\\ &= (-2)^4\\ &= 16\end{aligned}$$

Got It? **5.** What is each number in simplest form?

a. $32^{-\frac{3}{5}}$ **b.** $16^{\frac{3}{4}}$ **c.** $9^{-3.5}$

To write an expression with rational exponents in simplest form, write every exponent as a positive number.

Problem 6 Writing Expressions in Simplest Form

Plan

What is the first step in simplifying a radical expression using the properties of exponents?
Rewrite the radicals using rational exponents.

What is each expression in simplest form?

A $\left(-8x\sqrt{xy}\right)^{\frac{2}{3}}$

$$\begin{aligned}\left(-8x\sqrt{xy}\right)^{\frac{2}{3}} &= (-8)^{\frac{2}{3}}\cdot x^{\frac{2}{3}}\cdot\left((xy)^{\frac{1}{2}}\right)^{\frac{2}{3}}\\ &= ((-2)^3)^{\frac{2}{3}}\cdot x^{\frac{2}{3}}\cdot(xy)^{\frac{1}{3}}\\ &= (-2)^2\cdot x^{\frac{2}{3}}\cdot x^{\frac{1}{3}}\cdot y^{\frac{1}{3}}\\ &= 4xy^{\frac{1}{3}},\text{ or }4x\sqrt[3]{y}\end{aligned}$$

B $\left(16y^{-8}\right)^{-\frac{3}{4}}$

$$\begin{aligned}(16y^{-8})^{-\frac{3}{4}} &= 16^{-\frac{3}{4}}\cdot y^{-8\cdot-\frac{3}{4}}\\ &= (2^4)^{-\frac{3}{4}}\cdot y^6\\ &= 2^{-3}y^6\\ &= \frac{y^6}{8}\end{aligned}$$

Got It? **6.** What is each expression in simplest form?

a. $(8x^{15})^{-\frac{1}{3}}$ **b.** $\left(9x\sqrt[4]{y}\right)^{\frac{3}{2}}$

Lesson Check

Do you know HOW?

Simplify each expression.

1. $125^{\frac{1}{3}}$

2. $5^{\frac{1}{2}}\cdot 5^{\frac{1}{2}}$

3. $25^{-\frac{3}{2}}$

4. $4^{-3.5}$

5. $\sqrt{11}(\sqrt[4]{11})$

6. $\dfrac{\sqrt[3]{x}}{\sqrt[6]{x^5}}$

Do you UNDERSTAND?

7. Open-Ended Find a nonzero number q such that $q(1-2^{\frac{1}{2}})$ is a rational number. Explain.

8. Error Analysis Explain why this simplification is incorrect.

$$\begin{aligned}&5(4-5^{\frac{1}{2}})\\ &5(4)-5(5^{\frac{1}{2}})\\ &20-25^{\frac{1}{2}}\\ &15\end{aligned}$$

9. Reasoning Explain why $(-64)^{\frac{1}{3}} = -64^{\frac{1}{3}}$ but $(-64)^{\frac{1}{2}} \neq -64^{\frac{1}{2}}$.

Practice and Problem-Solving Exercises

A Practice

Simplify each expression.

See Problem 1.

10. $36^{\frac{1}{2}}$
11. $27^{\frac{1}{3}}$
12. $49^{\frac{1}{2}}$
13. $10^{\frac{1}{2}} \cdot 10^{\frac{1}{2}}$
14. $(-3)^{\frac{1}{3}} \cdot (-3)^{\frac{1}{3}} \cdot (-3)^{\frac{1}{3}}$
15. $7^{\frac{1}{2}} \cdot 21^{\frac{1}{2}}$
16. $2^{\frac{1}{2}} \cdot 32^{\frac{1}{2}}$
17. $3^{\frac{1}{3}} \cdot 9^{\frac{1}{3}}$
18. $3^{\frac{1}{4}} \cdot 27^{\frac{1}{4}}$

Write each expression in radical form.

See Problem 2.

19. $x^{\frac{1}{6}}$
20. $x^{\frac{1}{5}}$
21. $x^{\frac{2}{7}}$
22. $y^{\frac{2}{5}}$
23. $y^{-\frac{9}{8}}$
24. $t^{-\frac{3}{4}}$
25. $x^{1.5}$
26. $y^{1.2}$

Write each expression in exponential form.

27. $\sqrt{-10}$
28. $\sqrt{7x^3}$
29. $\sqrt{(7x)^3}$
30. $(\sqrt{7x})^3$
31. $\sqrt[3]{a^2}$
32. $(\sqrt[3]{a})^2$
33. $\sqrt[4]{c^2}$
34. $\sqrt[3]{(5xy)^6}$

Optimal Height The optimal height h of the letters of a message printed on pavement is given by the formula $h = \frac{0.00252d^{2.27}}{e}$. Here d is the distance of the driver from the letters and e is the height of the driver's eye above the pavement. All of the distances are in meters. Find h for the given values of d and e.

See Problem 3.

35. $d = 100$ m, $e = 1.2$ m
36. $d = 50$ m, $e = 1.2$ m
37. $d = 50$ m, $e = 2.3$ m
38. $d = 25$ m, $e = 2.3$ m

Find each product or quotient.

See Problem 4.

39. $(\sqrt[4]{6})(\sqrt[3]{6})$
40. $\dfrac{\sqrt[9]{y^3}}{\sqrt[3]{y^9}}$
41. $\sqrt{5} \cdot \sqrt[5]{5}$
42. $\sqrt[7]{7} \cdot \sqrt[3]{7}$
43. $\dfrac{\sqrt[6]{4}}{\sqrt[3]{4}}$
44. $\sqrt[4]{18} \cdot \sqrt{12}$
45. $\dfrac{\sqrt{6}}{\sqrt[3]{36}}$
46. $\dfrac{\sqrt{x^4y}}{\sqrt[4]{x^2y^8}}$

Simplify each number.

See Problem 5.

47. $8^{\frac{2}{3}}$
48. $64^{\frac{2}{3}}64^{\frac{2}{3}}$
49. $(-8)^{\frac{2}{3}}$
50. $(-32)^{\frac{6}{5}}$
51. $(32)^{-\frac{4}{5}}$
52. $4^{1.5}$
53. $16^{1.5}$
54. $10{,}000^{0.75}$

Write each expression in simplest form.

See Problem 6.

55. $\left(x^{\frac{2}{3}}\right)^{-3}$
56. $\left(x^{-\frac{4}{7}}\right)^{7}$
57. $\left(3x^{\frac{2}{3}}\right)^{-1}$
58. $5\left(x^{\frac{2}{3}}\right)^{-1}$
59. $\left(-27x^{-9}\right)^{\frac{1}{3}}$
60. $\left(-32y^{15}\right)^{\frac{1}{5}}$
61. $\left(x^{\frac{1}{2}}y^{-\frac{2}{3}}\right)^{-6}$
62. $\left(x^{\frac{2}{3}}y^{-\frac{1}{6}}\right)^{-12}$
63. $\left(\dfrac{x^3}{x^{-1}}\right)^{-\frac{1}{4}}$
64. $\left(\dfrac{x^2}{x^{-11}}\right)^{\frac{1}{3}}$
65. $\left(\dfrac{x^{\frac{1}{4}}}{y^{-\frac{3}{4}}}\right)^{12}$
66. $\left(\dfrac{x^{-\frac{2}{3}}}{y^{-\frac{1}{3}}}\right)^{15}$

B Apply

67. **Think About a Plan** The ratio R of radioactive carbon to nonradioactive carbon left in a sample of an organism that died T years ago can be approximated by the formula $R = A(2.7)^{-\frac{T}{8033}}$. Here A is the ratio of radioactive carbon to nonradioactive carbon in the living organism. What percent of A is left after 2000 years? After 4000 years? After 8000 years?

- What are the known and unknown values?
- How can you use the properties of exponents to solve this problem?

68. The expression $0.036m^{\frac{3}{4}}$ is used in the study of fluids. Which best represents the value of the expression for $m = 46 \times 10^4$?

Ⓐ 636 Ⓑ 1460 Ⓒ 1660 Ⓓ 16,600

Simplify each number.

69. $(-343)^{\frac{1}{3}}$

70. $(-243)^{\frac{1}{5}}$

71. $32^{1.2}$

72. $243^{1.2}$

73. $64^{3.5}$

74. $100^{4.5}$

75. $-(-27)^{-\frac{4}{3}}$

76. $\dfrac{1000^{\frac{4}{3}}}{100^{\frac{3}{2}}}$

77. $25^{\frac{3}{2}}$

78. **Science** A desktop world globe has a volume of about 1386 cubic inches. The radius of Earth is approximately equal to the radius of the globe raised to the 10th power. Find the radius of Earth. (*Hint:* Use the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere.)

Simplify each expression.

79. $x^{\frac{2}{7}} \cdot x^{\frac{3}{14}}$

80. $y^{\frac{1}{2}} \cdot y^{\frac{3}{10}}$

81. $x^{\frac{3}{5}} \div x^{\frac{3}{10}}$

82. $y^{\frac{5}{7}} \div y^{\frac{3}{14}}$

83. $\dfrac{x^{\frac{2}{3}}y^{-\frac{1}{4}}}{x^{\frac{1}{2}}y^{-\frac{1}{2}}}$

84. $\dfrac{x^{\frac{1}{2}}y^{-\frac{1}{3}}}{x^{\frac{3}{4}}y^{\frac{1}{2}}}$

85. $\left(\dfrac{16x^{14}}{81y^{18}}\right)^{\frac{1}{2}}$

86. $\left(\dfrac{81y^{16}}{16x^{12}}\right)^{\frac{1}{2}}$

87. $\left(\dfrac{8x^{6}}{27y^{9}}\right)^{\frac{1}{3}}$

88. **Open-Ended** Find three nonzero numbers a such that $a\left(4 + 5^{\frac{1}{2}}\right)$ is a rational number. Can a itself be a rational number? Explain.

89. a. **Reasoning** Show that $\sqrt[4]{x^2} = \sqrt{x}$ by using the definition of fourth root.

b. **Reasoning** Show that $\sqrt[4]{x^2} = \sqrt{x}$ by rewriting $\sqrt[4]{x^2}$ in exponential form.

90. Simplify $4^{\frac{1}{2}} \cdot 4^{\frac{1}{2}}$ using the following methods. Show all your work.

a. Use the properties of exponents.

b. Simplify each term in the product, then multiply.

c. Convert to radical form, then simplify.

Challenge You can define the rules for irrational exponents so that they have the same properties as rational exponents . Use those properties to simplify each expression.

91. $\left(7^{\sqrt{2}}\right)^{\sqrt{2}}$

92. $\dfrac{3^{3+\sqrt{5}}}{3^{1+\sqrt{5}}}$

93. $\dfrac{x^{4\pi}}{x^{2\pi}}$

94. $5^{2\sqrt{3}} \cdot 25^{-\sqrt{3}}$

95. $9^{\frac{1}{\sqrt{2}}}$

96. $\left(3^{2+\sqrt{2}}\right)^{2-\sqrt{2}}$

97. **Weather** Using data for the effect of temperature and wind on an exposed face, the National Weather Service uses the following formula to determine wind chill.

$$\text{Wind Chill Index} = 35.74 + 0.6215T - 35.75V^{0.16} + 0.4275TV^{0.16}$$

T is the temperature in degrees Fahrenheit and V is the velocity of the wind in miles per hour. Frostbite occurs in about 15 minutes when the wind chill index is -20. Find the wind velocity that produces a wind chill index of -20 when the temperature is 5°F.

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.6.4 **98.** What is the simplified value of $\left(\frac{1}{64}\right)^{-\frac{1}{6}}$?

MA.912.A.7.3 **99.** What positive value of b makes $9x^2 - bx + 4$ a perfect square trinomial?

MA.912.A.4.6 **100.** How many real roots does the cubic polynomial equation $x^3 - 7x^2 + 13x - 4 = 0$ have?

MA.912.A.2.10 **101.** What is the y-value of the y-intercept of the graph of $f(x) = 4|x - 2| - 5$?

Mixed Review

Simplify. **See Lesson 6-3.**

102. $6\sqrt[3]{3} - 2\sqrt[3]{3}$

103. $3\sqrt{18} + 2\sqrt{72}$

104. $(\sqrt{5} - 1)(\sqrt{5} + 4)$

105. $(1 - 2\sqrt{2})(1 + 2\sqrt{2})$

106. $2\sqrt{12} - 4\sqrt{27}$

107. $\sqrt[4]{162} + 3\sqrt[4]{32}$

Factor each expression completely. **See Lesson 4-4.**

108. $4x^3 - 8x^2 + 16x$

109. $x^2 + 4x + 4$

110. $x^2 - 18x + 81$

111. $16a^2 - 9b^2$

112. $25x^2 - 40xy + 16y^2$

113. $9x^2 + 48x + 64$

Get Ready! **To prepare for Lesson 6-5, do Exercises 114–119.**

Solve by factoring. Check your answers. **See Lesson 4-5.**

114. $x^2 = -x + 6$

115. $x^2 = 5x + 14$

116. $2x^2 + x = 3$

117. $3x^2 - 2 = 5x$

118. $4x^2 = -8x + 5$

119. $6x^2 = 5x + 6$

6 Mid-Chapter Quiz

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

Find all the real square roots of each number.

1. 100 **2.** 0.49

Simplify each radical expression. Use absolute value symbols when needed.

3. $\sqrt{36x^2}$ **4.** $\sqrt[3]{0.008y^3x^6}$

Simplify.

5. $\sqrt{50x^4y^8}$ **6.** $\sqrt[4]{32m^7n^9}$

Multiply and simplify.

7. $6\sqrt{4x^2} \cdot 2\sqrt{9x^2y^2}$

8. $\sqrt[3]{9} \cdot \sqrt[3]{9}$

9. $\sqrt[4]{16x^8} \cdot \sqrt[4]{x^{14}}$

Divide and simplify.

10. $\dfrac{\sqrt{36x^4}}{\sqrt{9x^6}}$ **11.** $\dfrac{\sqrt[3]{64x^9y^3}}{\sqrt[3]{8x^3}}$

Simplify. Rationalize all denominators.

12. $10\sqrt[3]{81} - 8\sqrt[3]{24}$

13. $\dfrac{4 + \sqrt{12}}{4 - \sqrt{12}}$

14. $\sqrt{48} - 3\sqrt{27} + 2\sqrt{75}$

15. $(3 + \sqrt{63})(1 + \sqrt{7})$

16. $\dfrac{\sqrt{x}}{\sqrt{6y^3}}$

Write each expression in exponential form.

17. $-\sqrt{17}$ **18.** $\sqrt[3]{y^8}$

Write each expression in radical form.

19. $m^{\frac{3}{7}}$ **20.** $y^{-\frac{4}{3}}$

Simplify each expression.

21. $(-27)^{\frac{2}{3}}$ **22.** $(16)^{\frac{3}{4}}$

Write each expression in simplest form.

23. $7\sqrt[3]{2x} - 3\sqrt[3]{2x}$

24. $2\sqrt{32x^2} + 3\sqrt{72x^2}$

25. $\sqrt[3]{125x^6} - \sqrt[3]{27x^6}$

26. $\sqrt[4]{7} - \sqrt[3]{7}$

27. $(\sqrt{y} - \sqrt{3})(\sqrt{y} + 2\sqrt{3})$

28. $\left(16x^{\frac{1}{4}}y^{\frac{3}{4}}\right)^{-4}$

29. $\left(\dfrac{x^{\frac{1}{3}}}{y^{\frac{2}{3}}}\right)^{9}$

30. $\left(\dfrac{x^{-10}}{x^5}\right)^{\frac{2}{5}}$

31. The radius of a circle can be expressed as $r = \sqrt{\frac{A}{\pi}}$ inches where r is the radius and A is the area of the circle. If the area of a circle is 169π in.^{2}, what is its radius?

Do you UNDERSTAND?

32. What are the real roots of $\sqrt{-16}$? Explain.

33. **Error Analysis** Identify the error in this statement.

$$\frac{\sqrt[3]{x}}{\sqrt[3]{y}} \cdot \frac{\sqrt[3]{y}}{\sqrt[3]{y}} = \frac{\sqrt[3]{xy}}{y}$$

34. **Reasoning** If $0^{\frac{2}{3}} = 0$, why is $0^{-\frac{2}{3}}$ undefined?

35. Given that x and y are integers, explain why the product of $x + \sqrt{y}$ and its conjugate will always be an integer.

36. **Reasoning** Explain why $(-8)^{\frac{1}{2}} \neq -(8)^{\frac{1}{2}}$, but $(-27)^{\frac{1}{3}} = -(27)^{\frac{1}{3}}$.

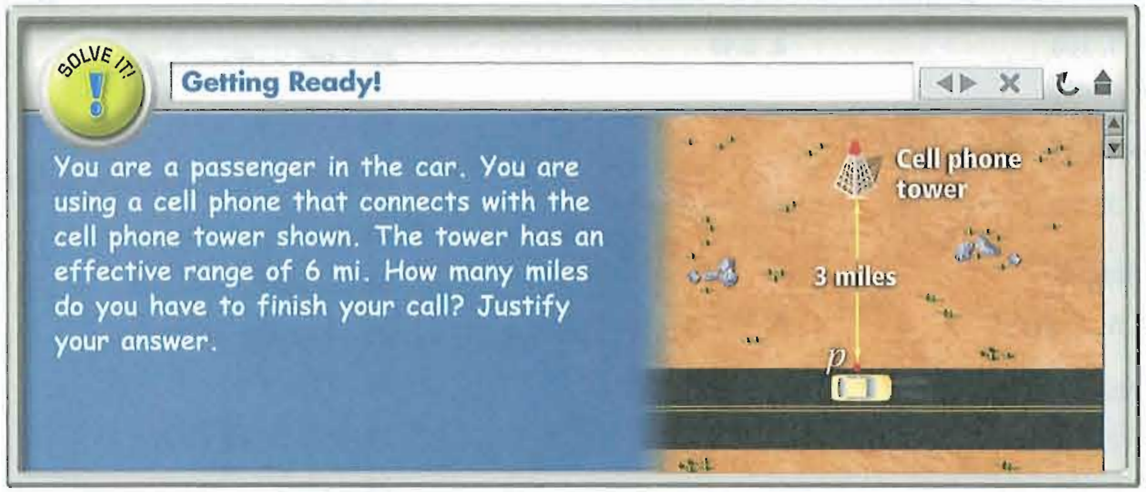
6-5 Solving Square Root and Other Radical Equations

Sunshine State Standards

MA.912.A.6.5 Solve equations that contain radical expressions.

MA.912.A.6.4 Convert between rational exponent and radical forms of expressions.

Objective To solve square root and other radical equations



Getting Ready!

You are a passenger in the car. You are using a cell phone that connects with the cell phone tower shown. The tower has an effective range of 6 mi. How many miles do you have to finish your call? Justify your answer.

Lesson Vocabulary
- radical equation
- square root equation

A **radical equation** is an equation that has a variable in a radicand or a variable with a rational exponent. If the radical has index 2, the equation is a **square root equation**. In this lesson, assume that all radicals and expressions with rational exponents represent real numbers.

Essential Understanding Solving a square root equation may require that you square each side of the equation. This can introduce extraneous solutions.

To solve a radical equation, isolate the radical on one side of the equation. Then raise each side to the power suggested by the index.

Problem 1 Solving a Square Root Equation

Think

Do you need to introduce a ± sign here?
No, when you take the square root of each side of an equation you do, but here you are squaring both sides of the equation.

What is the solution of $3 + \sqrt{2x - 3} = 8$?

$$\begin{aligned}
3 + \sqrt{2x - 3} &= 8 \\
\sqrt{2x - 3} &= 5 && \text{Isolate the radical expression.} \\
(\sqrt{2x - 3})^2 &= 5^2 && \text{Square each side.} \\
2x - 3 &= 25 \\
2x &= 28 && \text{Add 3 to each side.} \\
x &= 14 && \text{Divide each side by 2.}
\end{aligned}$$

Check

$3 + \sqrt{2x - 3} = 8$ Write the original equation.

$3 + \sqrt{2(14) - 3} \stackrel{?}{=} 8$ Substitute 14 for *x*.

$3 + \sqrt{25} \stackrel{?}{=} 8$ Simplify.

$3 + 5 \stackrel{?}{=} 8$

$8 = 8$ ✔

Got It? **1.** What is the solution of $\sqrt{4x + 1} - 5 = 0$?

To solve equations of the form $x^{\frac{m}{n}} = k$, raise each side of the equation to the power $\frac{n}{m}$, the reciprocal of $\frac{m}{n}$. If either *m* or *n* is even, then $\left(x^{\frac{m}{n}}\right)^{\frac{n}{m}} = |x|$.

Problem 2 Solving Other Radical Equations

A **What is the solution of $3(x + 1)^{\frac{2}{3}} = 12$?**

Know	Need	Plan
• The equation • The power of the exponential expression	Solution of the equation	• Isolate the exponential expression. • Use the inverse of the power to simplify and solve the equation.

Think

How can you get rid of the rational exponent?

Raise each side to the reciprocal power.

$3(x + 1)^{\frac{2}{3}} = 12$

$(x + 1)^{\frac{2}{3}} = 4$ Divide each side by 3.

$\left((x + 1)^{\frac{2}{3}}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}}$ Raise each side to the $\frac{3}{2}$ power.

$(x + 1)^{\frac{6}{6}} = 4^{\frac{3}{2}}$

$|x + 1| = 8$

Since the numerator of $\frac{2}{3}$ is even, $(x^{\frac{2}{3}})^{\frac{3}{2}} = |x|$.

$x + 1 = \pm 8$

$x = 7 \text{ or } x = -9$

The solutions are 7 and −9.

Check

$3(x + 1)^{\frac{2}{3}} = 12$

$3(7 + 1)^{\frac{2}{3}} \stackrel{?}{=} 12$

$3(2^3)^{\frac{2}{3}} \stackrel{?}{=} 12$

$3(2)^2 \stackrel{?}{=} 12$

$12 = 12$ ✔

$3(x + 1)^{\frac{2}{3}} = 12$

$3(-9 + 1)^{\frac{2}{3}} \stackrel{?}{=} 12$

$3((-2)^3)^{\frac{2}{3}} \stackrel{?}{=} 12$

$3(-2)^2 \stackrel{?}{=} 12$

$12 = 12$ ✔

Think

Why is isolating the variable important?
If you raise each side of $3\sqrt[5]{(x+1)^3} + 1 = 25$ to the $\frac{5}{3}$ power you will end up with a more complicated equation, not a simpler one.

B **What is the solution of $3\sqrt[5]{(x+1)^3} + 1 = 25$?**

$$3\sqrt[5]{(x+1)^3} + 1 = 25$$

$$3(x+1)^{\frac{3}{5}} + 1 = 25 \quad \text{Rewrite the radical using a rational exponent.}$$

$$3(x+1)^{\frac{3}{5}} = 24 \quad \text{Subtract 1 from each side.}$$

$$(x+1)^{\frac{3}{5}} = 8 \quad \text{Divide each side by 3.}$$

$$\left((x+1)^{\frac{3}{5}}\right)^{\frac{5}{3}} = 8^{\frac{5}{3}} \quad \text{Raise each side to the } \tfrac{5}{3} \text{ power.}$$

$$x + 1 = 32 \quad \text{Simplify.}$$

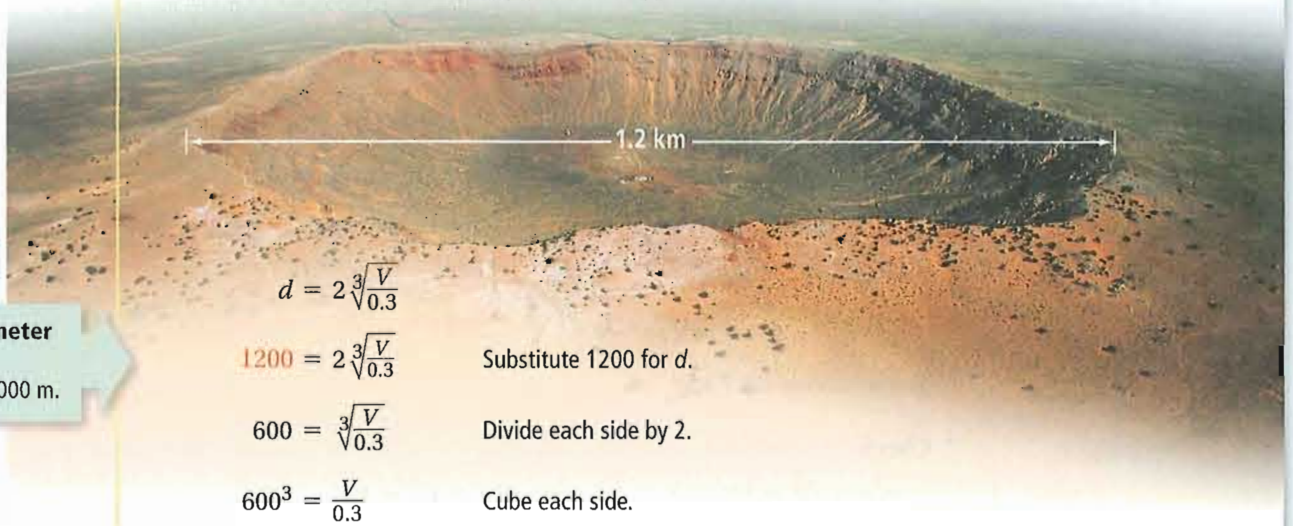
$$x = 31 \quad \text{Subtract 1 from each side.}$$

The solution is 31.

Got It? **2.** What are the solution(s) of $2(x+3)^{\frac{2}{3}} = 8$?

Problem 3 Using Radical Equations

Earth Science **For Meteor Crater in Arizona, the formula $d = 2\sqrt[3]{\frac{V}{0.3}}$ relates the diameter d of the rim (in meters) to the volume V (in cubic meters). What is the volume of Meteor Crater? (All values are approximate.)**



Think

What is the diameter in meters?
1.2 km = 1.2 × 1000 m.

$$d = 2\sqrt[3]{\frac{V}{0.3}}$$

$$1200 = 2\sqrt[3]{\frac{V}{0.3}} \quad \text{Substitute 1200 for } d.$$

$$600 = \sqrt[3]{\frac{V}{0.3}} \quad \text{Divide each side by 2.}$$

$$600^3 = \frac{V}{0.3} \quad \text{Cube each side.}$$

$$0.3 \cdot 600^3 = V \quad \text{Multiply each side by 0.3.}$$

$$V = 64{,}800{,}000 \quad \text{Simplify.}$$

The volume of Meteor Crater is about 64,800,000 m^3.

Got It? **3.** Suppose the diameter of a similarly shaped crater is 1 km. What is the volume of the crater? Use the formula given in Problem 3.

When you raise each side of an equation to a power, it is possible to introduce extraneous solutions. Therefore, it becomes very important that you check all solutions in the original equation. A correct solution will give a true statement. An extraneous solution will give a false statement.

Problem 4 Checking for Extraneous Solutions

What is the solution of $\sqrt{x+7} - 5 = x$? Check your results.

Think

How do you square a binomial?
Use the formula,
$(a + b)^2 = a^2 + 2ab + b^2$.

$\sqrt{x+7} - 5 = x$	
$\sqrt{x+7} = x + 5$	Isolate the radical.
$\left(\sqrt{x+7}\right)^2 = (x+5)^2$	Square each side.
$x + 7 = x^2 + 10x + 25$	Simplify.
$0 = x^2 + 9x + 18$	Combine like terms.
$0 = (x+3)(x+6)$	Factor.
$x = -3$ or $x = -6$	Zero-Product Property

Check

$\sqrt{x+7} - 5 = x$	$\sqrt{x+7} - 5 = x$
$\sqrt{-3+7} - 5 \stackrel{?}{=} -3$	$\sqrt{-6+7} - 5 \stackrel{?}{=} -6$
$\sqrt{4} - 5 \stackrel{?}{=} -3$	$\sqrt{1} - 5 \stackrel{?}{=} -6$
$2 - 5 \stackrel{?}{=} -3$	$1 - 5 \stackrel{?}{=} -6$
$-3 = -3$ ✓	$-4 \neq -6$
	false

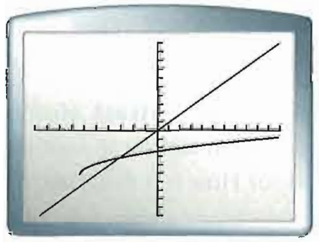
The only solution is -3.

Got It? **4. a.** What is the solution of $\sqrt{5x - 1} + 3 = x$? Check your results.

b. Reasoning When should you check for extraneous solutions? Explain.

In this lesson you studied algebraic methods of solving square root and radical equations. In Lesson 6-8 you will study the graphs of square root functions. These graphs can help you find solutions and identify extraneous solutions.

The calculator screen shows the graphs **Y1 = √(x + 7) − 5** and **Y2 = x.** From the graph, it is clear that -3 is a solution of $\sqrt{x+7} - 5 = x$, and -6 is not a solution.



If an equation contains two radical expressions (or two terms with rational exponents), isolate one of the radicals (or one of the terms), then eliminate it (or its rational exponent). Isolate the more complicated radical expression first. In the resulting equation, simplify the expressions before you eliminate the second radical.

Plan

Which radical expression should you isolate first? Isolate the more complicated radical first, $\sqrt{2x+1}$.

Problem 5 Solving an Equation With Two Radicals

What is the solution of $\sqrt{2x+1} - \sqrt{x} = 1$?

$\sqrt{2x+1} - \sqrt{x} = 1$

$\sqrt{2x+1} = \sqrt{x} + 1$ Isolate the more complicated radical.

$(\sqrt{2x+1})^2 = (\sqrt{x}+1)^2$ Square each side.

$2x + 1 = x + 2\sqrt{x} + 1$

$x = 2\sqrt{x}$ Isolate $2\sqrt{x}$.

$x^2 = (2\sqrt{x})^2$ Square each side.

$x^2 = 4x$

$x^2 - 4x = 0$ Subtract $4x$ from each side.

$x(x-4) = 0$ Factor.

$x = 0$ or $x = 4$ Zero-Product Property

Check

$$\begin{aligned}\sqrt{2x+1} - \sqrt{x} &= 1\\ \sqrt{2(0)+1} - \sqrt{0} &\stackrel{?}{=} 1\\ \sqrt{1} - 0 &\stackrel{?}{=} 1\\ 1 - 0 &\stackrel{?}{=} 1\\ 1 &= 1 \checkmark\end{aligned}$$

$$\begin{aligned}\sqrt{2x+1} - \sqrt{x} &= 1\\ \sqrt{2(4)+1} - \sqrt{4} &\stackrel{?}{=} 1\\ \sqrt{9} - \sqrt{4} &\stackrel{?}{=} 1\\ 3 - 2 &\stackrel{?}{=} 1\\ 1 &= 1 \checkmark\end{aligned}$$

The solutions are 0 and 4.

Got It? **5.** What is the solution of $\sqrt{5x+4} - \sqrt{x} = 4$?

Lesson Check

Do you know HOW?

Solve. Check for extraneous solutions.

1. $\sqrt{4x-23} - 3 = 2$

2. $-\sqrt[3]{x} + 3 = 0$

3. $5\sqrt{x} + 7 = 8$

4. $3\sqrt{x} = 6$

5. $5 - 2\sqrt{x} = 3$

6. $\sqrt[3]{x} = 8$

Do you UNDERSTAND?

7. **Vocabulary** Which value, 12 or 3, is an extraneous solution of $x - 6 = \sqrt{3x}$? Explain your reasoning.

8. **Compare and Contrast** How is solving a square root equation similar to solving an absolute value equation? How is it different?

Practice and Problem-Solving Exercises

A Practice

Solve.

See Problem 1.

9. $3\sqrt{x} + 3 = 15$
10. $4\sqrt{x} - 1 = 3$
11. $\sqrt{x + 3} = 5$
12. $\sqrt{x + 1} = 4$
13. $\sqrt{2x - 1} = 3$
14. $\sqrt{x + 2} - 2 = 0$
15. $\sqrt{3x + 4} = 4$
16. $\sqrt{2x + 3} - 7 = 0$
17. $\sqrt{6 - 3x} - 2 = 0$

Solve.

See Problem 2.

18. $(x + 5)^{\frac{2}{3}} = 4$
19. $(x + 2)^{\frac{2}{3}} = 9$
20. $3(x - 2)^{\frac{3}{4}} = 24$
21. $3(x + 3)^{\frac{3}{4}} = 81$
22. $(x + 1)^{\frac{3}{2}} - 2 = 25$
23. $3 + (4 - x)^{\frac{3}{2}} = 11$

See Problem 3.

24. Volume A spherical water tank holds 9000 ft^3 of water. What is the diameter of the tank? $\left(Hint\text{: } d = \sqrt[3]{\frac{6V}{\pi}}\right)$

25. Hydraulics The formula $d = \sqrt{\frac{4Q}{\pi v}}$ models the diameter of a pipe where Q is the maximum flow of water in a pipe, and v is the velocity of the water. What is the diameter of a pipe that allows a maximum flow of 30 ft^3/min of water flowing at a velocity of 400 ft/min? Round your answer to the nearest inch.

Solve. Check for extraneous solutions.

See Problem 4.

26. $\sqrt{3x + 7} = x - 1$
27. $(5 - x)^{\frac{1}{2}} = x + 1$
28. $\sqrt{-3x - 5} = x + 3$
29. $\sqrt{11x + 3} - 2x = 0$
30. $(5x - 4)^{\frac{1}{2}} - x = 0$
31. $\sqrt{3x + 13} - 5 = x$
32. $\sqrt{x + 7} + 5 = x$
33. $(x + 3)^{\frac{1}{2}} - 1 = x$
34. $\sqrt{x + 7} - x = 1$

Solve. Check for extraneous solutions.

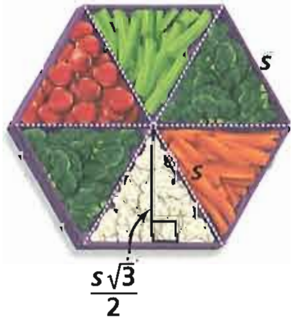
See Problem 5.

35. $\sqrt{3x} = \sqrt{x + 6}$
36. $(2x)^{\frac{1}{2}} = (x + 5)^{\frac{1}{2}}$
37. $(7x + 6)^{\frac{1}{2}} - (9 + 4x)^{\frac{1}{2}} = 0$
38. $\sqrt{3x + 2} - \sqrt{2x + 7} = 0$
39. $(x + 5)^{\frac{1}{2}} - (5 - 2x)^{\frac{1}{4}} = 0$
40. $(x - 2)^{\frac{1}{2}} - (28 - 2x)^{\frac{1}{4}} = 0$
41. $\sqrt{5 - x} - \sqrt{x} = 1$
42. $\sqrt{3x + 1} - \sqrt{x + 1} = 2$
43. $\sqrt{2x + 6} - \sqrt{x - 1} = 2$
44. $\sqrt{3 - x} + \sqrt{x + 2} = 3$

B Apply

45. Think About a Plan A hexagonal tray of vegetables has an area of 450 cm^2. What is the length of each side of the hexagon?

- What is the area of the triangle at the bottom in terms of the side length?
- How can you use the diagram at the right to find the formula for the area of the hexagon? (*Hint:* Six triangles make one hexagon.)



46. Traffic Signs A stop sign is a regular octagon, formed by cutting triangles off the corners of a square. If a stop sign measures 36 in. from top to bottom, what is the length of each side?

47. **Mental Math** What is the solution? $\sqrt{x + 11} = 4$

48. You can find the area A of a square whose side is s units with the formula $A = s^2$. What is the best estimate for the side of a square with an area of 32 m^2?

Ⓐ 4.2 m

Ⓑ 5.7 m

Ⓒ 8.0 m

Ⓓ 16 m

Solve. Check for extraneous solutions.

49. $3\sqrt{2x} - 3 = 9$

50. $2(2x)^{\frac{1}{3}} + 1 = 5$

51. $\sqrt{2x - 1} - 3 = 0$

52. $(2x + 3)^{\frac{1}{2}} - 7 = 0$

53. $\sqrt{x^2 + 3} = x + 1$

54. $(2x + 3)^{\frac{3}{4}} - 3 = 5$

55. $2(x - 1)^{\frac{4}{3}} + 4 = 36$

56. $x^{\frac{1}{2}} - (x - 5)^{\frac{1}{2}} = 2$

57. $\sqrt{x} = \sqrt{x - 8} + 2$

58. $(x - 3)^{\frac{2}{3}} = x - 7$

59. **Error Analysis** A student said that 4 and 1 are the solutions of the problem shown. Describe and correct the student's error.

~~$\sqrt{x} + 2 = x$~~

~~$\sqrt{x} = x - 2$~~

~~$(\sqrt{x})^2 = (x - 2)^2$~~

~~$x = x^2 - 4x + 4$~~

~~$0 = x^2 - 5x + 4$~~

~~$0 = (x - 4)(x - 1)$~~

60. **Physics** The velocity v of an object dropped from a tall building is given by the formula $v = \sqrt{64d}$, where d is the distance the object has dropped. Solve the formula for d.

61. **Open-Ended** Write an equation that has two radical expressions and no real roots.

62. **Reasoning** You have solved equations containing square roots by squaring each side. You were using the property that if $a = b$ then $a^2 = b^2$. Show that the following statements are *not* true for all real numbers.

a. If $a^2 = b^2$ then $a = b$.

b. If $a \leq b$ then $a^2 \leq b^2$.

63. A teacher asked students why it is necessary to check for extraneous roots when squaring both sides of the equation. Which of the following answers is the best? Is this answer complete? Explain.

Ⓐ Because the squared equation can have negative roots.

Ⓑ Because squaring is multiplication, and any multiplication is a potential source of extraneous roots.

Ⓒ Because when you square both sides of the equation $a = b$, you add to the solution set the roots of the equation $a = -b$.

Ⓓ Because any operation with an equation may result in extraneous roots.

Solve. Check for extraneous solutions.

64. $\sqrt{x + 1} + \sqrt{2x} = \sqrt{5x + 3}$

65. $\sqrt{x + \sqrt{2x}} = \sqrt{2x}$

66. $\sqrt{x + \sqrt{2x}} = 2$

67. $\sqrt{\sqrt{x + 25}} = \sqrt{x + 5}$

Challenge

68. **Reasoning** Devise a plan to find the value of x.

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$

For each set of values, determine which is greater without using a calculator.

69. $\sqrt{6}$ or $\sqrt{2} + 1$

70. $\sqrt{3} + \sqrt{11}$ or 5

71. $\sqrt{10}$ or $\sqrt{2} + \sqrt{3}$

72. $\sqrt{19} + \sqrt{3}$ or $\sqrt{5} + \sqrt{13}$

Sunshine State Standards Practice

MA.912.A.6.5 **73.** What is the solution of $(x + 2)^{\frac{3}{4}} = 27$?

Ⓐ $x = 27$ Ⓑ $x = 79$ Ⓒ $x = 81$ Ⓓ $x = 83$

MA.912.A.4.6 **74.** A problem on a test asked students to solve a fifth-degree polynomial equation with rational coefficients. Adam found the following roots: -11.5, $\sqrt{2}$, $\frac{2i + 6}{2}$, $-\sqrt{2}$ and $3 - i$. His teacher wrote that four of these roots are correct, and one is incorrect. Which root is incorrect?

Ⓕ -11.5 Ⓖ $\sqrt{2}$ Ⓗ $\frac{2i + 6}{2}$ Ⓘ $3 - i$

MA.912.A.3.3 **75.** Which expression represents the solution of the equation $\frac{x}{y} = \frac{c}{a + b}$ solved for a?

Ⓐ $\frac{c}{b} - \frac{x}{y}$ Ⓑ $\frac{yc}{a + b}$ Ⓒ $\frac{yc}{x} + b$ Ⓓ $\frac{yc - xb}{x}$

MA.912.A.6.2 **76.** **Short Response** To rationalize the denominator of $\sqrt[4]{\frac{4}{25}}$, by what number would you multiply the numerator and denominator of the fraction?

Mixed Review

Simplify each number. See Lesson 6-4.

77. $81^{\frac{1}{4}}$

78. $4^{\frac{1}{2}}$

79. $125 \cdot 125^{\frac{1}{3}}$

80. $32 \cdot 256^{\frac{1}{2}}$

81. $100^{\frac{-3}{2}}$

82. $64^{\frac{2}{3}}$

83. $25^{1.5}$

84. $6^{\frac{1}{2}} \cdot 12^{\frac{1}{2}}$

Solve each equation by factoring. Check your answers. See Lesson 4-5.

85. $x^2 - 7x + 12 = 0$

86. $x^2 - 8x + 15 = 0$

87. $x^2 + 9x + 20 = 0$

88. $3x^2 + 8x + 4 = 0$

89. $9x^2 + 15x + 4 = 0$

90. $4x^2 + 11x + 6 = 0$

Get Ready! **To prepare for Lesson 6-6, do Exercises 91–96.**

Find the domain and range of each relation, and determine whether it is a function. See Lesson 2-1.

91. $\{(0, -5), (2, -3), (4, -1)\}$

92. $\{(-1, 2), (0, 0), (1, 1)\}$

93. $\{(-2, -2), (0, 0), (1, 1)\}$

94. $\{(3, -1), (4, -1), (5, -1)\}$

95. $\{(0, 0), (1, 0), (2, 1), (2, 2)\}$

96. $\{(0, -2), (0, 0), (0, 2)\}$

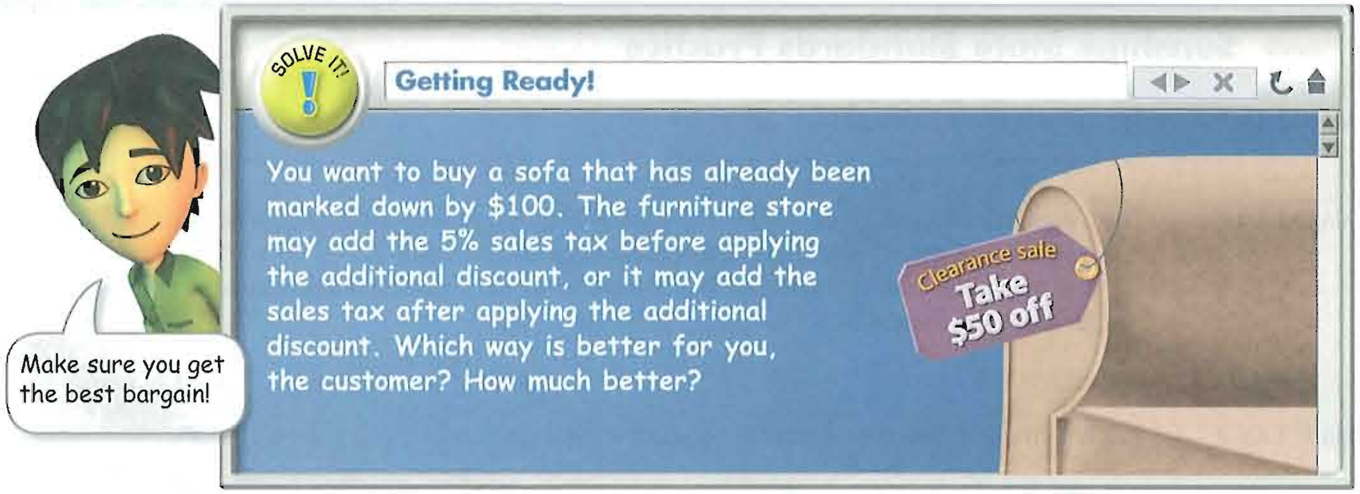
6-6 Function Operations

Sunshine State Standards

MA.912.A.2.7 Perform operations (addition, subtraction, division and multiplication) of functions algebraically, numerically, and graphically.

MA.912.A.2.8 Determine the composition of functions.

Objectives To add, subtract, multiply, and divide functions
To find the composite of two functions



Getting Ready!

You want to buy a sofa that has already been marked down by $100. The furniture store may add the 5% sales tax before applying the additional discount, or it may add the sales tax after applying the additional discount. Which way is better for you, the customer? How much better?

Lesson Vocabulary
- composite function

The final cost of the sofa in the Solve It involves two functions: one that gives an additional discount and one that multiplies to find the sales tax.

Essential Understanding You can add, subtract, multiply, and divide functions based on how you perform these operations for real numbers. One difference, however, is that you must consider the domain of each function.

take note

Key Concepts Function Operations

Addition	$(f + g)(x) = f(x) + g(x)$
Subtraction	$(f - g)(x) = f(x) - g(x)$
Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

The domains of the sum, difference, product, and quotient functions consist of the x-values that are in the domains of *both* f and g. Also, the domain of the quotient function does not contain any x-value for which $g(x) = 0$.

Problem 1 Adding and Subtracting Functions

Let $f(x) = 4x + 7$ and $g(x) = \sqrt{x} + x$. What are $f + g$ and $f - g$? What are their domains?

Think

What determines the domain of g?
Because there is a square root of x, x must be ≥ 0.

$$(f + g)(x) = f(x) + g(x) = (4x + 7) + (\sqrt{x} + x) = 5x + \sqrt{x} + 7$$

$$(f - g)(x) = f(x) - g(x) = (4x + 7) - (\sqrt{x} + x) = 3x - \sqrt{x} + 7$$

The domain of f is the set of all real numbers. The domain of g is all $x \geq 0$. The domain of both $f + g$ and $f - g$ is the set of numbers common to the domains of both f and g, which is all $x \geq 0$.

Got It? **1.** Let $f(x) = 2x^2 + 8$ and $g(x) = x - 3$. What are $f + g$ and $f - g$? What are their domains?

Problem 2 Multiplying and Dividing Functions

Let $f(x) = x^2 - 9$ and $g(x) = x + 3$. What are $f \cdot g$ and $\frac{f}{g}$ and their domains?

Think

Is the domain of $\frac{f}{g}$ the domain of $x - 3$?
No; The fraction can only be simplified and the function is only defined when $g(x) \neq -3$.

$$(f \cdot g)(x) = f(x) \cdot g(x) = (x^2 - 9)(x + 3)$$

$$= x^3 + 3x^2 - 9x - 27$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 9}{x + 3} = \frac{(x - 3)(x + 3)}{x + 3} = x - 3, x \neq -3$$

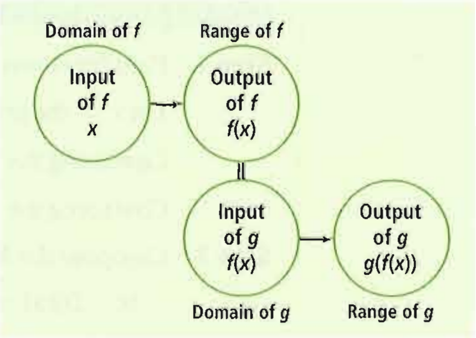
The domain of both f and g is the set of real numbers, so the domain of $f \cdot g$ is also the set of real numbers.

The domain of $\frac{f}{g}$ is the set of all real numbers except $x \neq -3$, because $g(-3) = 0$. The definition of $\frac{f}{g}$ requires that you consider the zero denominator in the *original* expression for $\frac{f(x)}{g(x)}$ despite the fact that the simplified form has the domain all real numbers.

Got It? **2.** Let $f(x) = 3x^2 - 11x - 4$ and $g(x) = 3x + 1$. What are $f \cdot g$ and $\frac{f}{g}$ and their domains?

The diagram shows what happens when you apply one function $g(x)$ after another function $f(x)$.

The output from the first function becomes the input for the second function. When you combine two functions as in the diagram, you form a **composite function**.



take note

Key Concept Composition of Functions

The composition of function g with function f is written as $g \circ f$ and is defined as $(g \circ f)(x) = g(f(x))$. The domain of $g \circ f$ consists of the x-values in the domain of f for which $f(x)$ is in the domain of g.

$(g \circ f)(x) = \underbrace{g(\underbrace{f(x)}_{1})}_{2}$

1. Evaluate $f(x)$ first.
2. Then use $f(x)$ as the input for g.

Function composition is not commutative since $f(g(x))$ does not always equal $g(f(x))$.

Problem 3 Composing Functions

GRIDDED RESPONSE

Let $f(x) = x - 5$ and $g(x) = x^2$. What is $(g \circ f)(-3)$?

Think

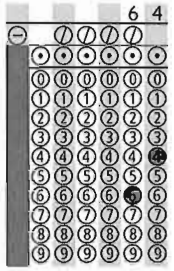
Which function is substituted into the other?
Use $f(x)$ as the input for g.

Method 1

$$(g \circ f)(x) = g(f(x))$$
$$= g(x - 5) = (x - 5)^2$$
$$(g \circ f)(-3) = (-3 - 5)^2$$
$$= (-8)^2$$
$$= 64$$

Method 2

$$(g \circ f)(-3) = g(f(-3))$$
$$= g(-3 - 5)$$
$$= g(-8)$$
$$= (-8)^2$$
$$= 64$$



Got It? **3.** What is $(f \circ g)(-3)$ for the functions f and g defined in Problem 3?

Problem 4 Using Composite Functions

You have a coupon good for \$5 off the price of any large pizza. You also get a 10% discount on any pizza if you show your student ID. How much more would you pay for a large pizza if the cashier applies the coupon first?

Know	Need	Plan
The coupon value and the discount rate	The difference between the results of applying the discount or coupon first	• Compose two functions in two ways. • Then find the difference in their results.

Step 1 Find functions C and D that model the cost of a large pizza.

Let x = the price of a large pizza.

Cost using the coupon: $C(x) = x - 5$

Cost using the 10% discount: $D(x) = x - 0.1x = 0.9x$

Step 2 Compose the functions to apply the discount and then the coupon.

$(C \circ D)(x) = C(D(x))$ Apply the discount, $D(x)$, first.

$= C(0.9x) = 0.9x - 5$

Step 3 Compose the functions to apply the coupon and then the discount.

$(D \circ C)(x) = D(C(x))$ Apply the coupon, $C(x)$, first.

$= D(x - 5) = 0.9(x - 5) = 0.9x - 4.5$

Step 4 Subtract the functions to find how much more you would pay if the cashier applies the coupon first.

$$\begin{aligned}(D \circ C)(x) - (C \circ D)(x) &= (0.9x - 4.5) - (0.9x - 5)\\ &= -4.5 + 5\\ &= 0.5\end{aligned}$$

You pay $.50 more if the cashier applies the coupon first.

Got It? **4.** A store is offering a 15% discount on all items. Also, employees get a 20% employee discount. Write composite functions

a. to model taking the 15% discount and then the 20% discount.
b. to model taking the 20% discount and then the 15% discount.
c. **Reasoning** If you were an employee, which discount would you take first? Why?

Lesson Check

Do you know HOW?

Let $f(x) = 3x - 2$ and $g(x) = x^2 + 1$. Perform each function operation.

1. $(f \cdot g)(x)$
2. $(f - g)(x)$
3. $(f \circ g)(x)$
4. $f(x) + g(x)$
5. $g(x) - f(x)$
6. $f(x) - g(x)$

Do you UNDERSTAND?

7. **Error Analysis** Your friend used some simple functions and found that $(f \circ g)(x) = (g \circ f)(x)$, and concluded that function composition is commutative. Give an example to show that your friend is mistaken.

8. **Open-Ended** Find two functions f and g such that $f(g(x)) = x$ for all real numbers x.

Practice and Problem-Solving Exercises

A Practice

Let $f(x) = 7x + 5$ and $g(x) = x^2$. Perform each function operation and then find the domain of the result.

See Problems 1 and 2.

9. $(f + g)(x)$
10. $(f - g)(x)$
11. $(g - f)(x)$
12. $(f \cdot g)(x)$
13. $\frac{f}{g}(x)$
14. $\frac{g}{f}(x)$

Let $f(x) = 2 - x$ and $g(x) = \frac{1}{x}$. Perform each function operation and then find the domain of the result.

15. $(f + g)(x)$
16. $(f - g)(x)$
17. $(g - f)(x)$
18. $(f \cdot g)(x)$
19. $\frac{f}{g}(x)$
20. $\frac{g}{f}(x)$

Let $f(x) = 2x^2 + x - 3$ and $g(x) = x - 1$. Perform each function operation and then find the domain.

21. $(f + g)(x)$

22. $(f - g)(x)$

23. $(g - f)(x)$

24. $(f \cdot g)(x)$

25. $\frac{f}{g}(x)$

26. $\frac{g}{f}(x)$

Let $g(x) = 2x$ and $h(x) = x^2 + 4$. Find each value or expression.

See Problem 3.

27. $(h \circ g)(1)$

28. $(h \circ g)(-5)$

29. $(h \circ g)(-2)$

30. $(g \circ h)(-2)$

31. $(g \circ h)(0)$

32. $(g \circ h)(a)$

33. $(g \circ g)(a)$

34. $(h \circ h)(a)$

35. $(h \circ g)(a)$

Let $f(x) = x^2$ and $g(x) = x - 3$. Find each value or expression.

36. $(g \circ f)(-2)$

37. $(f \circ g)(-2)$

38. $(g \circ f)(0)$

39. $(f \circ g)(0)$

40. $(g \circ f)(3.5)$

41. $(f \circ g)(3.5)$

42. $(f \circ g)(a)$

43. $(g \circ f)(-a)$

44. $(f \circ g)(-a)$

45. **Sales** A computer store offers a 5% discount off the list price x for any computer bought with cash, rather than put on credit. At the same time, the manufacturer offers a \$200 rebate for each purchase of a computer.

See Problem 4.

a. Write a function $f(x)$ to represent the price after the cash discount.

b. Write a function $g(x)$ to represent the price after the \$200 rebate.

c. Suppose the list price of a computer is \$1500. Use a composite function to find the price of the computer if the discount is applied before the rebate.

d. Suppose the list price of a computer is \$1500. Use a composite function to find the price of the computer if the rebate is applied before the discount.

46. **Economics** Suppose the function $f(x) = 0.15x$ represents the number of U.S. dollars equivalent to x Chinese yuan and the function $g(y) = 14.07y$ represents the number of Mexican pesos equivalent to y U.S. dollars.

a. Write a composite function that represents the number of Mexican pesos equivalent to x Chinese yuan.

b. Find the value in Mexican pesos of an item that costs 15 Chinese yuan.

Let $f(x) = 2x + 5$ and $g(x) = x^2 - 3x + 2$. Perform each function operation and then find the domain.

47. $f(x) + g(x)$

48. $3f(x) - 2$

49. $g(x) - f(x)$

50. $-2g(x) + f(x)$

51. $f(x) - g(x) + 10$

52. $4f(x) + 2g(x)$

53. $-f(x) + 4g(x)$

54. $f(x) - 2g(x)$

55. $f(x) \cdot g(x)$

56. $-3f(x) \cdot g(x)$

57. $\frac{f(x)}{g(x)}$

58. $\frac{5f(x)}{g(x)}$

B Apply

59. **Think About a Plan** A craftsman makes and sells violins. The function $I(x) = 5995x$ represents the income in dollars from selling x violins. The function $P(y) = y - 100{,}000$ represents his profit in dollars if he makes an income of y dollars. What is the profit from selling 30 violins?

- How can you write a composite function to represent the craftsman's profit?
- How can you use the composite function to find the profit earned when he sells 30 violins?

60. Suppose your teacher offers to give the whole class a bonus if everyone passes the next math test. The teacher says she will give everyone a 10-point bonus and increase everyone's grade by 9% of their score.

a. You earned a 75 on the test. Would you rather have the 10-point bonus first and then the 9% increase, or the 9% increase first and then the 10-point bonus?

b. **Reasoning** Is this the best plan for all students? Explain.

61. **Sales** A salesperson earns a 3% bonus on weekly sales over \$5000. Consider the following functions.

$g(x) = 0.03x \qquad h(x) = x - 5000$

a. Explain what each function above represents.

b. Which composition, $(h \circ g)(x)$ or $(g \circ h)(x)$, represents the weekly bonus? Explain.

62. If $(f \circ g)(x) = x^2 - 6x + 8$ and $g(x) = x - 3$, what is $f(x)$?

Let $g(x) = 3x + 2$ and $f(x) = \frac{x-2}{3}$. Find each value.

63. $f(g(1))$

64. $g(f(-4))$

65. $f(g(0))$

66. $g(f(2))$

67. $g(g(0))$

68. $(g \circ g)(1)$

69. $(f \circ g)(-2)$

70. $(f \circ f)(0)$

71. **Geometry** You toss a pebble into a pool of water and watch the circular ripples radiate outward. You find that the function $r(x) = 12.5x$ describes the radius r, in inches, of a circle x seconds after it was formed. The function $A(x) = \pi x^2$ describes the area A of a circle with radius x.

a. Find $(A \circ r)(x)$ when $x = 2$. Interpret your answer.

b. Find the area of a circle 4 seconds after it was formed.

For each pair of functions, find $f(g(x))$ and $g(f(x))$.

72. $f(x) = 3x, g(x) = x^2$

73. $f(x) = x + 3, g(x) = x - 5$

74. $f(x) = 3x^2 + 2, g(x) = 2x$

75. $f(x) = \frac{x-3}{2}, g(x) = 2x - 3$

76. $f(x) = -x - 7, g(x) = 4x$

77. $f(x) = \frac{x+5}{2}, g(x) = x^2$

78. **Open-Ended** Write a function rule that approximates each value.

a. The amount you save is a percent of what you earn. (You choose the percent.)

b. The amount you earn depends on how many hours you work. (You choose the hourly wage.)

c. Write and simplify a composite function that expresses your savings as a function of the number of hours you work. Interpret your results.

Challenge Let $f(x) = x^4 + 2x^3 - 5x^2 - 10x$ and $g(x) = x^3 - 3x^2 - 5x + 15$. Perform each function operation and simplify, and then find the domain.

79. $f(x) \cdot g(x)$

80. $\frac{f(x)}{g(x)}$

81. $\frac{g(x)}{f(x)}$

Find each composition of functions. Simplify your answer.

82. Let $f(x) = \frac{1}{x}$. Find $f(f(f(x)))$.

83. Let $f(x) = 2x - 3$. Find $\frac{f(1 + h) - f(1)}{h}$, $h \neq 0$.

84. Let $f(x) = 4x - 1$. Find $\frac{f(a + h) - f(a)}{h}$, $h \neq 0$.

85. Let $f(x) = 4x^2 - 1$. Find $\frac{f(a + h) - f(a)}{h}$, $h \neq 0$.

Sunshine State Standards Practice

MA.912.A.2.4

86. Let $f(x) = x + 5$ and $g(x) = x^2 - 25$. What is the domain of $\frac{f}{g}(x)$?

Ⓐ All real numbers

Ⓑ All real numbers except 5

Ⓒ All real numbers except −5

Ⓓ All real numbers except −5 and 5

MA.912.A.2.8

87. Let $g(x) = x - 3$ and $h(x) = x^2 + 6$. What is $(h \circ g)(1)$?

Ⓕ −14 Ⓖ 4 Ⓗ 10 Ⓘ 15

MA.912.A.3.6

88. Which number is a solution of $|3 - 2x| < 5$?

Ⓐ −6 Ⓑ −1 Ⓒ 2 Ⓓ 4

MA.912.A.4.12

89. Short Response What is the coefficient of the x^3y^4 term in the expansion of $(3x - y)^7$? Show your work.

Mixed Review

Solve. Check for extraneous solutions. See Lesson 6-5.

90. $\sqrt{x^2 + 3} = x + 1$

91. $x + 8 = (x^2 + 16)^{\frac{1}{2}}$

92. $\sqrt{x^2 + 9} = x + 1$

93. $(x^2 - 9)^{\frac{1}{2}} - x = -3$

94. $\sqrt{x^2 + 12} - 2 = x$

95. $(3x)^{\frac{1}{2}} = (x + 6)^{\frac{1}{2}}$

Expand each binomial. See Lesson 5-7.

96. $(x + 4)^8$

97. $(x + y)^6$

98. $(2x - y)^4$

99. $(2x - 3y)^7$

100. $(9 - 2x)^5$

101. $(4x - y)^5$

102. $(x^2 + x)^4$

103. $(x^2 + 2y^3)^6$

Get Ready! **To prepare for Lesson 6-7, do Exercises 104–106.**

Graph and solve each system. See Lesson 3-1.

104. $\begin{cases} y = x - 6 \\ y = x + 6 \end{cases}$

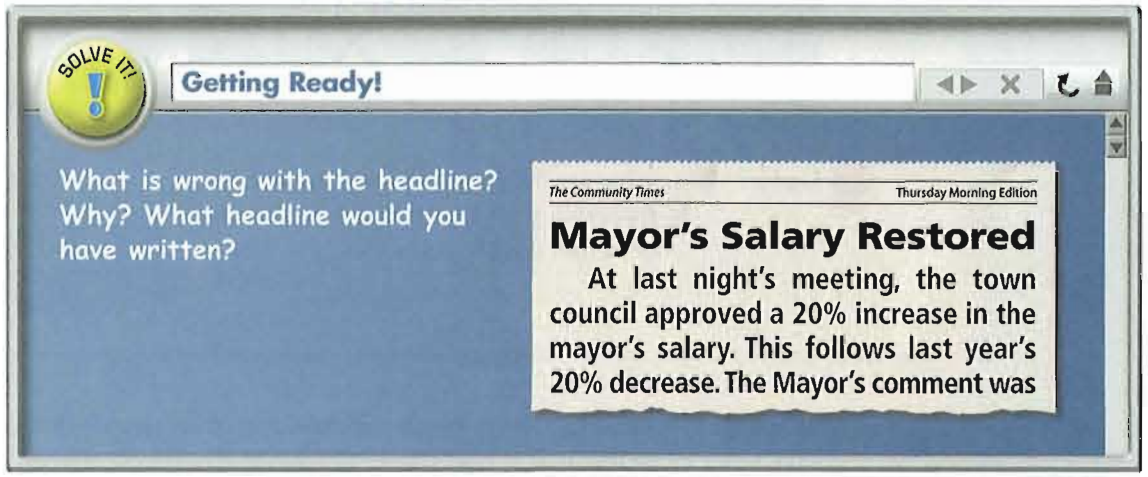
105. $\begin{cases} y = 0.5x + 1 \\ y = 2x - 2 \end{cases}$

106. $\begin{cases} y = \frac{x + 4}{5} \\ y = 5x - 4 \end{cases}$

6-7 Inverse Relations and Functions

Sunshine State Standard
MA.912.A.2.11 Solve problems involving functions and their inverses.

Objective To find the inverse of a relation or function


SOLVE IT! **Getting Ready!**

What is wrong with the headline? Why? What headline would you have written?

The Community Times Thursday Morning Edition

Mayor's Salary Restored

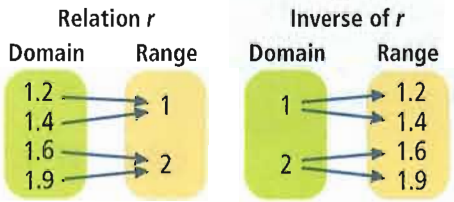
At last night's meeting, the town council approved a 20% increase in the mayor's salary. This follows last year's 20% decrease. The Mayor's comment was

Lesson Vocabulary
- inverse relation
- inverse function
- one-to-one function

If a relation pairs element a of its domain to element b of its range, the **inverse relation** pairs b with a. So, if (a, b) is an ordered pair of a relation, then (b, a) is an ordered pair of its inverse. If both a relation and its inverse happen to be functions, they are **inverse functions.**

Essential Understanding The inverse of a function may or may not be a function.

This diagram shows a relation r (a function) and its inverse (not a function). The range of the relation is the domain of the inverse. The domain of the relation is the range of the inverse.



Problem 1 Finding the Inverse of a Relation

Think
(0, −1) is in *s*. How do you find the corresponding pair in the inverse of *s*?
Switch the coordinates. (−1, 0) is in the inverse of *s*.

A What is the inverse of relation s?

Relation *s*

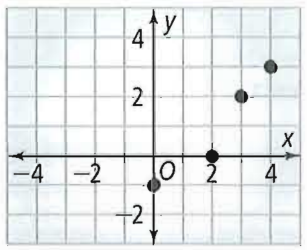
x	y
0	−1
2	0
3	2
4	3

Switch the x and y values to get the inverse. →

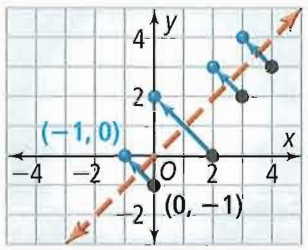
Inverse of Relation *s*

x	y
−1	0
0	2
2	3
3	4

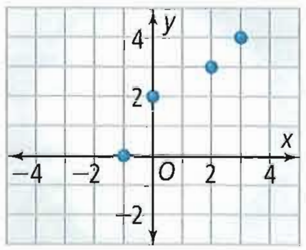
B What are the graphs of *s* and its inverse?



Relation *s*



Reversing the Ordered Pairs



Inverse of *s*

Got It? **1. a.** What are the graphs of *t* and its inverse?

b. Reasoning Is *t* a function? Is the inverse of *t* a function? Explain.

Relation *t*

x	0	1	2	3
y	−5	−4	−3	−3

As shown in Problem 1, the graphs of a relation and its inverse are the reflections of each other in the line $y = x$. If you describe a relation or function by an equation in *x* and *y*, you can switch *x* and *y* to get an equation for the inverse.

Think

Why do you solve for *y*?
If you solve the equation for *y*, you can use it to easily generate ordered pairs that are part of the inverse relation.

Problem 2 Finding an Equation for the Inverse

What is the inverse of the relation described by $y = x^2 - 1$?

$$y = x^2 - 1$$

$$x = y^2 - 1 \quad \text{Switch } x \text{ and } y.$$

$$x + 1 = y^2 \quad \text{Add 1 to each side.}$$

$$\pm\sqrt{x + 1} = y \quad \text{Find the square root of each side to solve for } y.$$

Got It? **2.** What is the inverse of $y = 2x + 8$?

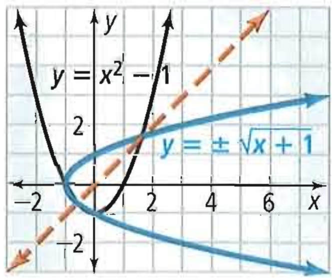
Think

What does the graph of $y = x^2 - 1$ look like?
The graph of $y = x^2 - 1$ is a translation of $y = x^2$ down one unit.

Problem 3 Graphing a Relation and Its Inverse

What are the graphs of $y = x^2 - 1$ and its inverse, $y = \pm\sqrt{x + 1}$?

The graph of $y = x^2 - 1$ is a parabola that opens upward with vertex $(0, -1)$. The graph of the inverse is the reflection of the parabola in the line $y = x$.



Got It? **3.** What are the graphs of $y = 2x + 8$ and its inverse?

The inverse of a function f is denoted by f^{-1}. You read f^{-1} as "the inverse of f" or as "f inverse." The notation $f(x)$ is used for functions, but the relation f^{-1} may not even be a function.

Problem 4 Finding an Inverse Function

Consider the function $f(x) = \sqrt{x - 2}$.

A What are the domain and range of f?

The radicand cannot be negative, so the numbers $x \geq 2$ make up the domain. The principal square root is nonnegative, so the numbers $y \geq 0$ make up the range.

B What is f^{-1}, the inverse of f?

$f(x) = \sqrt{x - 2}$

$y = \sqrt{x - 2}$ Rewrite the equation using y.

$x = \sqrt{y - 2}$ Switch x and y. Since x equals a principal square root, $x \geq 0$.

$x^2 = y - 2$ Square both sides.

$y = x^2 + 2$ Solve for y.

So, $f^{-1}(x) = x^2 + 2$, for $x \geq 0$.

C What are the domain and range of f^{-1}?

Part (b) shows that the domain of f^{-1} is the range of f—the numbers $x \geq 0$. Since $x^2 \geq 0$, $x^2 + 2 \geq 2$. Therefore, the numbers $y \geq 2$ make up the range of f^{-1}. Note that the range of f^{-1} is the same as the domain of f.

D Is f^{-1} a function? Explain.

For each x in the domain ($x \geq 0$) of f^{-1}, there is only one value of y in the range. So $f^{-1}(x) = x^2 + 2, x \geq 0$, is a function.

Think

How could a graph help you check your answer?

You could graph f^{-1} and see whether the graph passes the vertical line test. If it does, f^{-1} is a function.

Got It? 4. Let $g(x) = 6 - 4x$.

a. What are the domain and range of g?
b. What is the inverse of g?
c. What are the domain and range of g^{-1}?
d. Is g^{-1} a function? Explain.

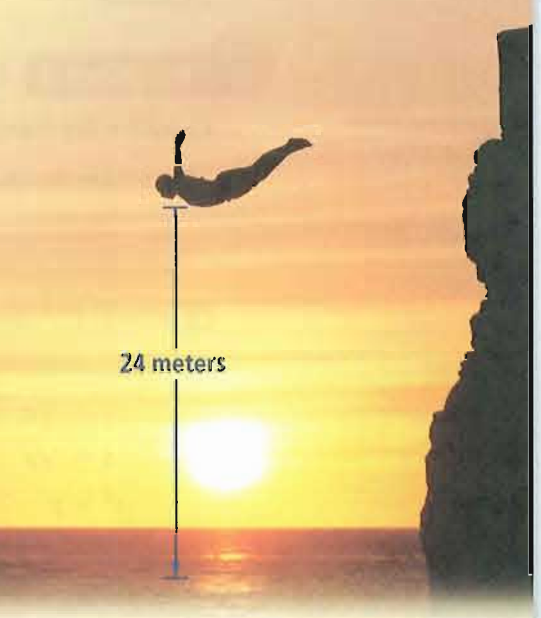
Functions that model real-world behavior are often expressed as formulas with meaningful variables, like $A = \pi r^2$ for the area of a circle. Strictly speaking, the inverse formula would be $r = \pi A^2$, but this expresses a false relationship between A and r. It is better to leave the variables in place and solve for r as a function of A.

$A = \pi r^2$ Original formula.

$r = \sqrt{\frac{A}{\pi}}$ Same formula, but inversely expressed.

Problem 5 Finding the Inverse of a Formula

The function $d = 4.9t^2$ represents the distance d, in meters, that an object falls in t seconds due to Earth's gravity. Find the inverse of this function. How long, in seconds, does it take for the cliff diver shown to reach the water below?



$d = 4.9t^2$

$t^2 = \frac{d}{4.9}$ Solve for t. Do not switch the variables.

$t = \sqrt{\frac{d}{4.9}}$ Time must be nonnegative.

$= \sqrt{\frac{24}{4.9}}$ Substitute 24 for d.

≈ 2.2 Use a calculator.

It will take about 2.2 seconds for the diver to reach the water.

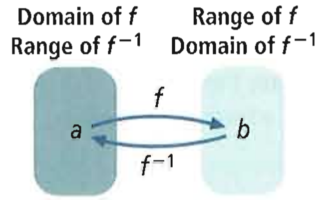
Think

Why shouldn't you interchange the variables?

Interchanging the variables leads to a false relationship between distance and time.

Got It? 5. The function $d = \frac{v^2}{19.6}$ relates the distance d, in meters, that an object has fallen to its velocity v, in meters per second. Find the inverse of this function. What is the velocity of the cliff diver in meters per second as he enters the water?

You know that for any function f, each x-value in the domain corresponds to exactly one y-value in the range. For a **one-to-one function**, it is also true that each y-value in the range corresponds to exactly one x-value in the domain. A one-to-one function f has an inverse f^{-1} that is also a function. If f maps a to b, then f^{-1} must map b to a.



Key Concept Composition of Inverse Functions

take note

If f and f^{-1} are inverse functions, then

$(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$ for x in the domains of f and f^{-1}, respectively.

This says that the composition of a function and its inverse is essentially the identity function, $id(x) = x$, or $y = x$.

Problem 6 Composing Inverse Functions

For $f(x) = \frac{1}{x-1}$, what is each of the following?

A $f^{-1}(x)$

$f(x) = \frac{1}{x-1}$

$y = \frac{1}{x-1}$ Rewrite the equation using y.

$x = \frac{1}{y-1}$ Switch x and y.

$x(y-1) = 1$ Solve for y.

$y - 1 = \frac{1}{x}$

$y = \frac{1}{x} + 1$

So $f^{-1}(x) = \frac{1}{x} + 1$.

Think

Is this a function?
Yes. For each value of x, there is only one value for y.

B $(f \circ f^{-1})(1)$

$$\begin{aligned}(f \circ f^{-1})(1) &= f(f^{-1}(1)) \\ &= f\left(\frac{1}{1} + 1\right) \\ &= f(2) \\ &= \frac{1}{2-1} = 1\end{aligned}$$

C $(f^{-1} \circ f)(1)$

$$\begin{aligned}(f^{-1} \circ f)(1) &= f^{-1}(f(1)) \\ &= f^{-1}\left(\frac{1}{1-1}\right) \\ &= f^{-1}\left(\frac{1}{0}\right)\end{aligned}$$

undefined

1 is not in the domain of f. Therefore $(f^{-1} \circ f)(1)$ does not exist.

Got It? **6.** Let $g(x) = \frac{4}{x+2}$. What is each of the following?
a. $g^{-1}(x)$ **b.** $(g \circ g^{-1})(0)$ **c.** $(g^{-1} \circ g)(0)$

Lesson Check

Do you know HOW?

Find the inverse of each function. Is the inverse a function?

1. $f(x) = 4x + 3$

2. $f(x) = x^2 - 1$

3. $f(x) = (x + 1)^2$

4. For $h(x) = -\frac{1}{x+2}$, find:
  a. $h^{-1}(x)$
  b. $h^{-1}(4)$
  c. Value of x for which the equality $(h \circ h^{-1})(x) = x$ does not hold.

Do you UNDERSTAND?

5. Vocabulary Does every function have an inverse which is a function? Does every relation have an inverse which is a relation?

6. Reasoning A function consists of the pairs (2, 3), (x, 4), and (5, 6). What values, if any, may x not assume?

7. Error Analysis A classmate says that $(f \circ g)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$. Show that this is incorrect by finding examples of $f(x)$ and $g(x)$ for which the equation does not hold.

Practice and Problem-Solving Exercises

A Practice

Find the inverse of each relation. Graph the given relation and its inverse. **See Problem 1.**

8.

x	y
1	0
2	1
3	0
4	2

9.

x	y
1	0
2	1
3	2
4	3

10.

x	y
0	0
1	1
2	4
3	9

11.

x	y
−3	2
−2	2
−1	2
0	2

Find the inverse of each function. Is the inverse a function? **See Problem 2.**

12. $y = 3x + 1$

13. $y = 2x - 1$

14. $y = 4 - 3x$

15. $y = 5 - 2x^2$

16. $y = x^2 + 4$

17. $y = 3x^2 - 5$

18. $y = (x - 8)^2$

19. $y = (3x - 4)^2$

20. $y = (1 - 2x)^2 + 5$

Graph each relation and its inverse **See Problem 3.**

21. $y = 2x - 3$

22. $y = 3 - 7x$

23. $y = -x$

24. $y = 3x^2$

25. $y = -x^2$

26. $y = 4x^2 - 2$

27. $y = (x - 1)^2$

28. $y = (2 - x)^2$

29. $y = (3 - 2x)^2 - 1$

For each function, find the inverse and the domain and range of the function and its inverse. Determine whether the inverse is a function. **See Problem 4.**

30. $f(x) = 3x + 4$

31. $f(x) = \sqrt{x - 5}$

32. $f(x) = \sqrt{x + 7}$

33. $f(x) = \sqrt{-2x + 3}$

34. $f(x) = 2x^2 + 2$

35. $f(x) = -x^2 + 1$

36. Temperature The formula for converting from Celsius to Fahrenheit temperatures is $F = \frac{9}{5}C + 32$. **See Problem 5.**

a. Find the inverse of the formula. Is the inverse a function?

b. Use the inverse to find the Celsius temperature that corresponds to 25°F.

37. Geometry The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

a. Find the inverse of the formula. Is the inverse a function?

b. Use the inverse to find the radius of a sphere that has a volume of 35,000 ft ^{3}.

For Exercises 38–41, $f(x) = 10x - 10$. Find each value. **See Problem 6.**

38. $(f^{-1} \circ f)(10)$

39. $(f \circ f^{-1})(-10)$

40. $(f^{-1} \circ f)(0.2)$

41. $(f \circ f^{-1})(d)$

B Apply

Find the inverse of each function. Is the inverse a function?

42. $f(x) = x^3$

43. $f(x) = x^4$

44. $f(x) = \frac{2x^2}{5} + 1$

45. $f(x) = 1.5x^2 - 4$

46. $f(x) = \frac{3x^2}{4}$

47. $f(x) = \sqrt{2x - 1} + 3$

48. Think About a Plan The velocity of the water that flows from an opening at the base of a tank depends on the height of water above the opening. The function $v(x) = \sqrt{2gx}$ models the velocity v in feet per second where g, the acceleration due to gravity, is about 32 ft/s^2 and x is the height in feet of the water. What is the depth of water when the flow is 40 ft/s, and when the flow is 20 ft/s?

- How can you use inverse functions to help you find the answer?
- What restrictions are on the domain of $v(x)$? of $v^{-1}(x)$?

49. Let $f(x) = 3x^2 - 4$ and $g(x) = x - 2$. Calculate $(f \circ g^{-1})(x)$ for $x = -3$.

50. Writing Explain how you can find the range of the inverse of $f(x) = \sqrt{x - 1}$ without finding the inverse itself.

For each function, find the inverse and the domain and range of the function and its inverse. Determine whether the inverse is a function.

51. $f(x) = -\sqrt{x}$

52. $f(x) = \sqrt{x} + 3$

53. $f(x) = \sqrt{-x + 3}$

54. $f(x) = \sqrt{x + 2}$

55. $f(x) = \frac{x^2}{2}$

56. $f(x) = \frac{1}{x^2}$

57. $f(x) = (x - 4)^2$

58. $f(x) = (7 - x)^2$

59. $f(x) = \frac{1}{(x + 1)^2}$

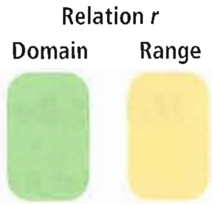
60. $f(x) = 4 - 2\sqrt{x}$

61. $f(x) = \frac{3}{\sqrt{x}}$

62. $f(x) = \frac{1}{\sqrt{-2x}}$

63. a. Open-Ended Copy the mapping diagram at the right. Complete it by writing members of the domain and range and connecting them with arrows so that r is a function and r^{-1} is not a function.

b. Repeat part (a) so that r is not a function and r^{-1} is a function.



64. Reasoning Relation r has one element in its domain and two elements in its range. Is r a function? Is the inverse of r a function? Explain.

65. Geometry Write a function that gives the length of the hypotenuse of an isosceles right triangle with side length s. Evaluate the inverse of the function to find the side length of an isosceles right triangle with a hypotenuse of 6 in.

66. Open-Ended Write a function f such that the graph of f^{-1} lies only in Quadrants III and IV.

67. Reasoning To determine if the inverse of function f is also a function, you can use a *horizontal line test.* It says that if no horizontal line intersects the graph of the function f in more than one point, then the inverse of f is a function.

a. Explain why the horizontal line test works.

b. The graph of a polynomial function passes through the points $(-1, 1)$, $(0, 4)$ and $(2, 3)$. Can its inverse be a function?

Challenge **Find the inverse of each function. Is the inverse a function?**

68. $f(x) = \frac{1}{5}x^3$

69. $f(x) = \sqrt[3]{x - 5}$

70. $f(x) = \frac{\sqrt[3]{x}}{3}$

71. $f(x) = (x - 2)^3$

72. $f(x) = \sqrt[4]{x}$

73. $f(x) = 1.2x^4$

74. Function $f(x)$ is defined the following way:

- if x is an integer, then $f(x) = x + 1$;
- for all other x, $f(x) = x + 2$.

Is the inverse of $f(x)$ a function? Explain.

Sunshine State Standards Practice

MA.912.A.1.6

75. Which pair of words make this sentence FALSE?
The product of two ___(I)___ numbers is always a (n) ___(II)___ number.

Ⓐ (I) complex; (II) complex
Ⓑ (I) real; (II) complex
Ⓒ (I) rational; (II) real
Ⓓ (I) imaginary; (II) imaginary

MA.912.A.2.8

76. If $f(x) = x + 1$ and $g(x) = x^2 - 3x - 4$, what is $(f \circ g)(x)$?

Ⓕ $x^2 - 3x - 3$
Ⓖ $x^2 - x - 6$
Ⓗ $x^2 - x$
Ⓘ $x^2 - x - 3$

MA.912.A.6.3

77. What is the simplified form of $\left(a^{\frac{2}{3}}b^{\frac{3}{4}}\right)^2$?

Ⓐ $a^{\frac{4}{9}}b^{\frac{9}{16}}$
Ⓑ $a^{\frac{4}{3}}b^{\frac{3}{2}}$
Ⓒ ab
Ⓓ $(ab)^{\frac{17}{6}}$

MA.912.A.2.11

78. Extended Response Let $f(x) = (x + 1)^2 - 2$. Find the x- and y-intercepts of $f(x)$ and the inverse of $f(x)$. Is the inverse a function?

Mixed Review

Let $f(x) = 4x$, $g(x) = \frac{1}{2}x + 7$, and $h(x) = -2x + 4$. Perform each function operation. **See Lesson 6-6.**

79. $(g \circ f)(x)$

80. $(h \circ g)(x)$

81. $h(x) + g(x)$

82. $f(x) \cdot g(x)$

83. $(f \circ g)(x) + h(x)$

84. $(f \circ g)(x)$

Find each real root. **See Lesson 6-1.**

85. $-\sqrt[4]{16}$

86. $\sqrt[4]{-16}$

87. $\sqrt[5]{243}$

88. $-\sqrt[5]{243}$

89. $\sqrt[5]{-243}$

90. $\sqrt[3]{0.064}$

91. $\sqrt[4]{810{,}000}$

92. $\sqrt[4]{\frac{1}{160{,}000}}$

Get Ready! **To prepare for Lesson 6-8, do Exercises 93–95.**

Graph each function. **See Lesson 4-1.**

93. $y = -x^2 - 1$

94. $y = -(x + 1)^2 + 1$

95. $y = 3x^2 + 3$

Concept Byte

For Use With Lesson 6-7

TECHNOLOGY

Graphing Inverses

Sunshine State Standard

MA.912.A.4.5 Graph polynomial functions with the use of graphing technology.

You can graph inverses of functions on a graphing calculator by using the **DrawInv** feature or by using parametric equations. It takes more keystrokes to set up parametric equations, but once you do you can easily change from one function to another and quickly see the graphs of the new function and its inverse.

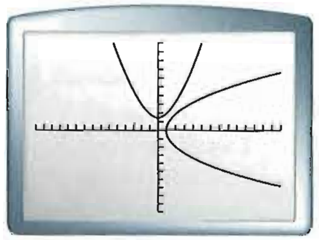
Activity

Graph $y = 0.3x^2 + 1$ and its inverse.

Method 1 Use the **DrawInv** feature.

Step 1 Press y= and enter the equation. Press zoom **5** to see a graph of the function with equal x- and y-intervals.

Step 2 Press 2nd draw **8.** You will see **DrawInv** followed by a flashing cursor. Select equation Y_1 by pressing vars ▷ **1 1.** Press enter to see the graph of the function and its inverse.



Method 2 Use parametric equations.

Step 1 Set to parametric mode. Press mode, select **Par**, and press 2nd quit.

Step 2 Enter the given equation in parametric form. Press y= and enter the equations $X_{1T} = T$ and $Y_{1T} = .3T^2 + 1$.

Step 3 Now use $X_{2T} = Y_{1T}$ and $Y_{2T} = X_{1T}$ to interchange the x- and y-values of the first parametric equation. Press y= and move the cursor to follow $X_{2T} =$. Select Y_{1T} by pressing vars ▷ **2 2.** Enter the equation $Y_{2T} = X_{1T}$ in a similar fashion.

Step 4 Press zoom **5.** Adjust the **Window** so that **Tmin** and **Tmax** approximately agree with **Xmin** and **Xmax**. Press graph to see the graph of the function and its inverse.

```
Plot1     Plot2     Plot3
\X1T =T
 Y1T =.3T² + 1
\X2T =Y1T
 Y2T =X1T
\X3T =
 Y3T =
\X4T =
```

Exercises

Graph each function and its inverse with a graphing calculator. Then sketch the graphs.

1. $y = x^2 - 5$

2. $y = (x - 3)^2$

3. $y = 0.01x^4$

4. $y = 0.5x^3 - 3$

5. **Writing** Change the parametric equation $X_{2T} = Y_{1T}$ in Method 2, Step 3 to $X_{2T} = -Y_{1T}$. Describe the graph that results.

6. Explain how once you set up parametric equations, you can change from one function to another and quickly see the graphs of the new function and its inverse.

6-8 Graphing Radical Functions

Sunshine State Standards

MA.912.A.2.6 Identify and graph radical functions.

MA.912.A.2.10 Describe and graph transformations of functions.

Objective To graph square root and other radical functions

SOLVE IT! **Getting Ready!**

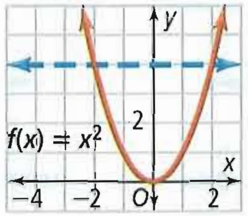
A red plastic strip binds the three identical cylinders. The cross-sectional area enclosed by the strip is 115 cm². What is the radius of each cylinder? What is the length of the plastic strip?

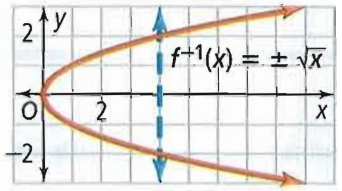
Lesson Vocabulary
- radical function
- square root function

The formula $A = \pi r^2$ shows that area is a quadratic function of the radius of a circle. The formula $r = \frac{1}{\sqrt{\pi}}\sqrt{A}$ shows that the radius of a circle is a square root function of the area.

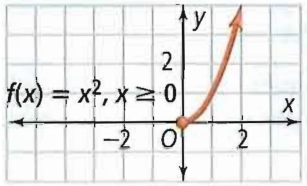
Essential Understanding A square root function is the inverse of a quadratic function that has a restricted domain.

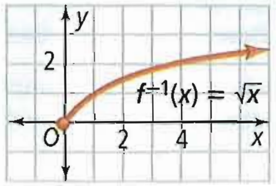
A horizontal line can intersect the graph of $f(x) = x^2$ in two points—where $f(-2) = f(2)$, for example. Thus, a vertical line can intersect the graph of f^{-1} in two points. f^{-1} is *not* a function because it fails the vertical line test.





However, you can restrict the domain of f so that the inverse of the restricted function is a function.





Inverses of the power functions $y = x^n$ (with domains restricted as needed) form parent functions $y = \sqrt[n]{x}$ for families of **radical functions**. In particular, $f(x) = \sqrt{x}$ is the parent for the family of **square root functions**. Members of this family have the general form $f(x) = a\sqrt{x - h} + k$.

Dynamic Activity
Radical Functions

take note

Key Concepts Families of Radical Functions

	Square Root	Radical
Parent function:	$y = \sqrt{x}$	$y = \sqrt[n]{x}$
Reflection in x-axis:	$y = -\sqrt{x}$	$y = -\sqrt[n]{x}$
Stretch $(a > 1)$, shrink $(0 < a < 1)$ by the factor a:	$y = a\sqrt{x}$	$y = a\sqrt[n]{x}$
Translation: Horizontal by h Vertical by k	$y = \sqrt{x - h} + k$	$y = \sqrt[n]{x - h} + k$

Problem 1 Translating a Square Root Function Vertically

Think

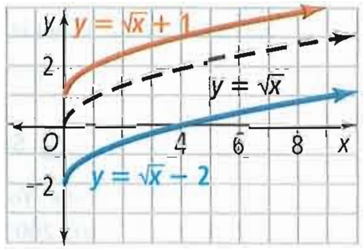
How is $y = \sqrt{x} + k$ related to the parent function $y = \sqrt{x}$?
It is related to the parent function in the same way that $y = f(x) + k$ is related to $y = f(x)$. It is a vertical translation of k units.

What are the graphs of $y = \sqrt{x} - 2$ and $y = \sqrt{x} + 1$?

The graph of $y = \sqrt{x} - 2$ is the graph of $y = \sqrt{x}$ shifted down 2 units.

The graph of $y = \sqrt{x} + 1$ is the graph of $y = \sqrt{x}$ shifted up 1 unit.

The domains of both functions are the set of nonnegative numbers, but their ranges differ.



Got It? **1.** What are the graphs of $y = \sqrt{x} + 2$ and $y = \sqrt{x} - 3$?

Problem 2 Translating a Square Root Function Horizontally

Think

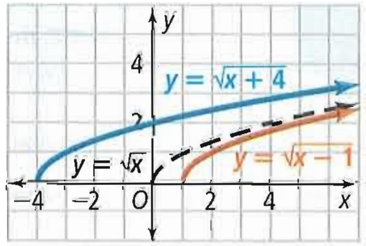
How is $y = \sqrt{x - h}$ related to the parent function $y = \sqrt{x}$?
It is a horizontal translation of h units.

What are the graphs of $y = \sqrt{x + 4}$ and $y = \sqrt{x - 1}$?

The graph of $y = \sqrt{x + 4}$ is the graph of $y = \sqrt{x}$ shifted left 4 units.

The graph of $y = \sqrt{x - 1}$ is the graph of $y = \sqrt{x}$ shifted right 1 unit.

The ranges of both functions are the set of nonnegative numbers, but their domains differ.



Got It? **2.** What are the graphs of $y = \sqrt{x - 3}$ and $y = \sqrt{x + 2}$?

Recall from Lesson 2-7 that for any transformation, $y = af(x - h) + k$ of the parent function $f(x)$, a indicates a vertical stretch or shrink.

Similarly, for the combined transformation $y = a\sqrt{x - h} + k$, a indicates a vertical stretch ($|a| > 1$) or shrink ($|a| < 1$). A negative value of a indicates a reflection in the x-axis.

Think

What would be good points to choose?
Points that have integer x- and y-coordinates.

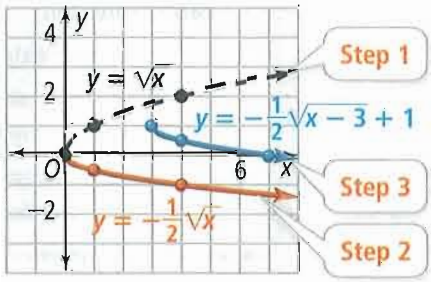
Problem 3 Graphing a Square Root Function

What is the graph of $y = -\frac{1}{2}\sqrt{x - 3} + 1$?

Step 1 Choose several points from the parent function $y = \sqrt{x}$.

Step 2 Multiply the y-coordinates by $a = -\frac{1}{2}$. This shrinks the parent graph vertically by the factor $\frac{1}{2}$ and reflects the result in the x-axis.

Step 3 The values of h and k give the horizontal and vertical translations. Translate the graph from Step 2 to the right 3 units and up 1 unit.



Got It? **3.** What is the graph of $y = 3\sqrt{x + 2} - 4$?

Think

How can you rewrite a radical function using an exponent?
You can write a radical function $y = \sqrt[n]{x}$ as $y = x^{\frac{1}{n}}$.

Problem 4 Solving a Radical Equation by Graphing

Multiple Choice **You can model the population P of Corpus Christi, Texas, between the years 1970 and 2005 by the radical function $P(x) = 75{,}000\sqrt[3]{x - 1950}$, where x is the year. Using this model, in what year was the population of Corpus Christi 250,000?**

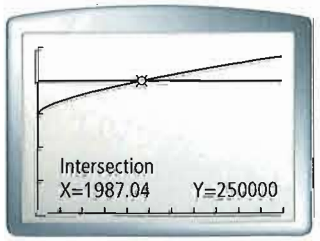
Ⓐ 1980 Ⓑ 1983 Ⓒ 1987 Ⓓ 1990

For $P = 250{,}000$, solve the equation $250{,}000 = 75{,}000\sqrt[3]{x - 1950}$.

Graph **Y1 = 75000(X − 1950)^(1/3)** and **Y2 = 250000**. Adjust the window to find where the graphs intersect.

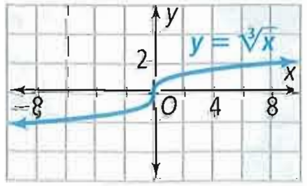
Use the **INTERSECT** feature to find the x-coordinate of the intersection.

In the year 1987, the population of Corpus Christi was 250,000. The correct answer is C.


Intersection
X=1987.04 Y=250000

Got It? **4.** In what year was the population of Corpus Christi 275,000?

Problem 4 uses a transformation of $y = \sqrt[3]{x}$. The function $f(x) = \sqrt[3]{x}$ is the inverse of $g(x) = x^3$. Unlike $y = \sqrt{x}$, the domain and range of $f(x) = \sqrt[3]{x}$ are all real numbers.



The patterns for graphing square root functions apply to other radical functions.

Plan

How is $y = a\sqrt[n]{x - h} + k$ related to its parent function?
a stretches or shrinks the parent function and h and k translate it horizontally and vertically.

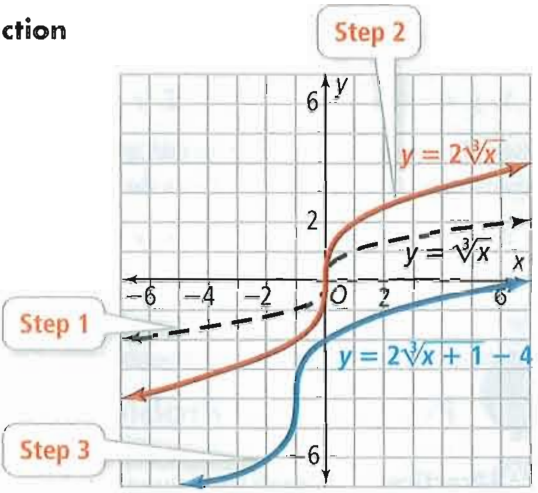
Problem 5 Graphing a Cube Root Function

What is the graph of $y = 2\sqrt[3]{x + 1} - 4$?

Step 1 Graph the parent function, $y = \sqrt[3]{x}$.

Step 2 Multiply the y-coordinates by 2. This stretches the graph vertically.

Step 3 Translate the graph from step 2, 1 unit to the left and 4 units down.



Got It? **5.** What is the graph of $y = 3 - \frac{1}{2}\sqrt[3]{x - 2}$?

You can graph functions of the form $y = \sqrt[n]{bx + c}$ using transformations, if you can simplify the radicand so that x has a coefficient of 1. This is also true for functions in the form $y = a\sqrt[n]{bx + c} + k$.

Problem 6 Rewriting a Radical Function

How can you rewrite $y = \sqrt{9x + 18}$ so you can graph it using transformations? Describe the graph.

Think

The form $y = a\sqrt{x}$ shows the stretch or shrink. Factor to get $x - h$ in the radicand.

Find the square root of 9. Now, you have the form $y = a\sqrt{x - h}$ that you can graph using transformations.

Write

$y = \sqrt{9x + 18}$
$y = \sqrt{9(x + 2)}$
$y = \sqrt{9(x - (-2))}$

$y = 3\sqrt{x - (-2)}$

The graph of $y = \sqrt{9x + 18}$ is the graph of $y = 3\sqrt{x}$ translated 2 units to the left.

Got It? **6. a.** How can you rewrite $y = \sqrt[3]{8x + 32} - 2$ so you can graph it using transformations? Describe the graph.

b. Reasoning Describe the graph of $y = |9x + 18|$ by rewriting it in the form $y = a|x - h|$. How is this similar to rewriting $y = \sqrt{9x + 18}$ in Problem 6?

Lesson Check

Do you know HOW?

Graph each function.

1. $y = -\sqrt{x + 3}$

2. $y = -\sqrt[3]{x} + 5$

Rewrite each function so you can graph it using transformations of its parent function. Describe the graph.

3. $y = \sqrt{4x - 4}$

4. $y = \sqrt[3]{8x + 16}$

Do you UNDERSTAND?

5. Writing Explain the effect that a has on the graph of $y = a\sqrt{x}$. How does this compare to its effect on other functions you have studied?

6. Error Analysis Your friend states that the graph of the function $g(x) = \sqrt{-x - 1}$ is a reflection of the graph of the function $f(x) = -\sqrt{x + 1}$ across the x-axis. Describe your friend's error.

Practice and Problem-Solving Exercises

A Practice

Graph each function. **See Problems 1 and 2.**

7. $y = \sqrt{x} + 1$

8. $y = \sqrt{x} - 2$

9. $y = \sqrt{x} - 4$

10. $y = \sqrt{x} + 5$

11. $y = \sqrt{x - 3}$

12. $y = \sqrt{x + 1}$

13. $y = \sqrt{x + 6}$

14. $y = \sqrt{x - 4}$

Graph each function. **See Problem 3.**

15. $y = 3\sqrt{x}$

16. $y = -\sqrt{x - 1}$

17. $y = -5\sqrt{x + 2}$

18. $y = -0.5\sqrt{x} + 3$

19. $y = \frac{1}{2}\sqrt{x + 2} - 1$

20. $y = 3\sqrt{x + 1} + 4$

Solve each square root equation by graphing. Round the answer to the nearest hundredth, if necessary. If there is no solution, explain why. **See Problem 4.**

21. $\sqrt{x - 3} = 12$

22. $\sqrt{2x - 3} = 4$

23. $\sqrt{2x + 5} = \sqrt{2 - x}$

24. Landscaping A sprinkler can water between 1 and 130 square yards of a lawn. The length L in inches of rotating pipe needed to water A square yards is given by the function $L = 117.75\sqrt{A}$.

a. Graph the equation on your calculator. Make a sketch of the graph.

b. How much area can be watered if the length of the pipe is 500, 800, or 1,300 inches long?

Graph each function. **See Problem 5.**

25. $y = \sqrt[3]{x + 5}$

26. $y = \sqrt[3]{x} - 4$

27. $y = \sqrt[3]{x + 2} - 7$

28. $y = -\sqrt[3]{x + 3} - 1$

29. $y = 2\sqrt[3]{x - 6} - 9$

30. $y = \frac{1}{2}\sqrt[3]{x - 1} + 3$

Rewrite each function to make it easy to graph using transformations of its parent function. Describe the graph. **See Problem 6.**

31. $y = \sqrt{9x - 9}$

32. $y = -\sqrt{16x + 32}$

33. $y = -2\sqrt{4x + 16}$

34. $y = \sqrt[3]{64x + 128}$

35. $y = \sqrt{25x + 125} - 3$

36. $y = \sqrt[3]{8x - 24} + 1$

B Apply

37. Think About a Plan The time t in seconds for a pendulum to complete one full cycle is given by the function $t = 1.11\sqrt{l}$, where l is the length of the pendulum in feet. How long is a pendulum that takes 4.5 seconds to complete one full cycle? 6 seconds to complete one full cycle? Round your answers to the nearest hundredth.

- How can you use a graph to approximate the length of a pendulum?
- How can you check your answers algebraically?

Graph each function. Find the domain and range.

38. $y = 4\sqrt[3]{x - 2} + 1$

39. $y = \frac{1}{2}\sqrt{x - 1} + 3$

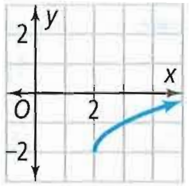
40. $y = 3\sqrt[3]{x - 6} + 2$

41. Suppose that a function pairs elements from set A with elements from set B. Recall that a function is called *onto* if every element in B is paired with at least one element in A.

a. The graph shows a transformation of $y = \sqrt{x}$. Write the function.

b. What are the domain and range of the function?

c. For the domain, is the function onto the set of nonnegative real numbers? Explain.



42. Open-Ended Write a radical function such that for its domain, the function is onto the set of real numbers such that $y \leq 3$.

Rewrite each function to make it easy to graph using transformations of its parent function. Describe the graph.

43. $y = \sqrt{25x - 100} - 1$

44. $y = \sqrt{36x + 108} + 4$

45. $y = -\sqrt[3]{8x - 2}$

46. $y = \sqrt{\frac{x - 1}{4}} - 2$

47. $y = 10 - \sqrt[3]{\frac{x + 3}{27}}$

48. $y = \sqrt{\frac{x}{9} + 1} + 5$

Graphing Calculator **Solve the following radical equations.**

49. $2\sqrt{x} = \sqrt{(x + 1)}$

50. $\sqrt{(x + 3)} = 4\sqrt{(x)} - 2$

51. $\sqrt[3]{x - 1} = \sqrt{x} - 1$

52. a. Solve $3 - \sqrt{(x - 3)} = x$ algebraically.

b. Solve the equation from part (a) graphically.

c. What do you notice about your answer to part (a) compared to your answer to part (b)?

53. Electronics The size of a computer monitor is given as the length of the screen's diagonal d in inches. The equation $d = \frac{5}{6}\sqrt{3A}$ models the length of a diagonal of a monitor screen with area A in square inches.

a. Graph the equation on your calculator.

b. Suppose you want to buy a new monitor with a screen that is twice the area of your old screen. Your old screen has a diagonal of 15 inches. What will be the diagonal of your new screen?

54. Physics You can model time t, in seconds, an object takes to reach the ground falling from height H, in meters, by $t(H) = \sqrt{\frac{2H}{g}}$. The value of g is 9.81 m/s^2. If an object takes 7 seconds to fall to the ground, what was its initial height?

Challenge **Rewrite each function to make it easy to graph using transformations of its parent function. Describe the graph. Find the domain and range of each function.**

55. $y = -\sqrt{2(4x - 3)}$ **56.** $y = \sqrt{3x - 5} + 6$ **57.** $y = -3 - \sqrt{12x + 18}$

58. a. Graph $y = \sqrt{-x}$, $y = \sqrt{1 - x}$, and $y = \sqrt{2 - x}$.

b. **Make a Conjecture** How does the graph of $y = \sqrt{h - x}$ differ from the graph of $y = \sqrt{x - h}$?

59. For what positive integers n are the domain and range of $y = \sqrt[n]{x}$ the set of real numbers? Assume that x is a real number.

Sunshine State Standards Practice

MA.912.A.2.10

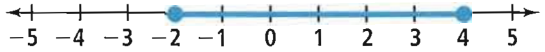
60. How is the graph of $y = \sqrt{x} - 5$ translated from the graph of $y = \sqrt{x}$?

(A) shifted 5 units left
(C) shifted 5 units up
(B) shifted 5 units right
(D) shifted 5 units down

MA.912.A.3.6

61. Which absolute value inequality has the graph shown here?

(F) $|x - 1| \leq 3$
(H) $|x + 1| \leq 3$
(G) $|x - 1| \geq 3$
(I) $|x + 1| \geq 3$



MA.912.A.4.3

62. Which polynomial cannot be factored in the real number system?

(A) $x^2 - 3x + 2$ (B) $x^2 + 4$ (C) $4x^2 - 1$ (D) $2x^2y - 2xy^2$

MA.912.A.2.6

63. **Short Response** How do the domains and ranges of the functions $f(x) = \sqrt{x - 1}$ and $g(x) = \sqrt{x} - 1$ compare?

Mixed Review

Find the inverse of each function. Is the inverse a function? **See Lesson 6-7.**

64. $f(x) = \frac{2}{3}x - 3$ **65.** $f(x) = \sqrt{x + 3} - 4$ **66.** $f(x) = (2x + 1)^2$

Rationalize the denominator of each expression. Assume that all variables are positive. **See Lesson 6-2.**

67. $\frac{\sqrt{36x^3}}{\sqrt{12y}}$ **68.** $\sqrt{\frac{3x}{2y}}$ **69.** $\frac{\sqrt[3]{x}}{\sqrt[3]{3y}}$ **70.** $\sqrt[5]{\frac{3x^3}{2y}}$

Solve using the Quadratic Formula. **See Lesson 4-7.**

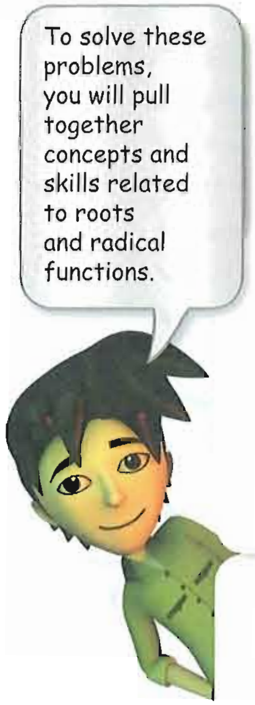
71. $x^2 - 9x + 15 = 0$ **72.** $3x^2 + 9x = 27$ **73.** $5x^2 + x = 3$

Get Ready! **To prepare for Lesson 7-1, do Exercises 74–76.**

Evaluate each expression for the given value of x. **See Lesson 1-3.**

74. 2^x for $x = 3$ **75.** 4^{x+1} for $x = 1$ **76.** 2^{3x+4} for $x = -1$

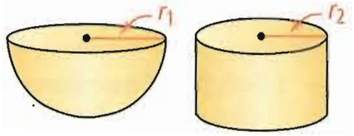
6 Pull It All Together



BIG idea Solving Equations and Inequalities

Solving an equation is the process of rewriting the equation to make what it says about its variables as simple as possible.

Task 1



An environmental equipment supplier sells hemispherical holding ponds for treatment of chemical waste. The volume of a pond is $V_1 = \frac{1}{2}\left(\frac{4}{3}\pi r_1^3\right)$, where r_1 is the radius in feet. The supplier also sells cylindrical collecting tanks. A collecting tank fills completely and then drains completely to fill the empty pond. The volume of the tank is $V_2 = 12\pi r_2^2$, where r_2 is the radius of the tank.

- **a.** Since $V_1 = V_2$, write an equation that shows r_1 as a function of r_2. Write an equation that shows r_2 as a function of r_1.
- **b.** You want to double the radius of the pond. How will the radius of the tank change?

BIG idea Solving Equations and Inequalities

The numbers and types of solutions vary based on the type of equation.

BIG idea Function

You can represent functions in a variety of ways (such as graphs, tables, equations, or words). Each representation is particularly useful in certain situations.

Task 2

Suppose $f(x) = \sqrt{x + 1}$.

- **a.** What are the domain and range of f?
- **b.** Find $f^{-1}(x)$. What are its domain and range? Be careful!
- **c.** Show that $(f \circ f^{-1})(a) = a = (f^{-1} \circ f)(a)$ for any a in the respective domains.
- **d.** Solve the equation $f(x) = f^{-1}(x)$. Remember to check for extraneous roots.
- **e.** Graph the functions f and f^{-1}. Be sure that you accurately represent the domains of each function. Interpret graphically the solution(s) you found to the equation in part (d).

6 Chapter Review

Connecting BIG ideas and Answering the Essential Questions

1 Equivalence
You can simplify the *n*th root of an expression that contains an *n*th power as a factor.

$$\sqrt[n]{x^n} = x^{\frac{n}{n}} = \begin{cases} x, & n \text{ odd} \\ |x|, & n \text{ even} \end{cases}$$

2 Solving Equations and Inequalities
When you square each side of an equation, the resulting equation may have more solutions than the original equation.

3 Function
If f and f^{-1} are inverse functions and if one maps *a* to *b*, then the other maps *b* to *a*, i.e.,

$$(f \circ f^{-1})(a) = (f^{-1} \circ f)(a) = a.$$

Radical Expressions and Rational Exponents (Lessons 6-1, 6-2 and 6-4)

$$\begin{aligned} \sqrt[3]{-8x^5}\,\sqrt[3]{x^2} &= \sqrt[3]{-8x^7} \\ &= \sqrt[3]{(-2)^3x^6 \cdot x} \\ &= -2x^2\sqrt[3]{x} \\ (-8x^5)^{\frac{1}{3}}(x^2)^{\frac{1}{3}} &= (-8x^7)^{\frac{1}{3}} \\ &= ((-2)^3 \cdot x^6 \cdot x)^{\frac{1}{3}} \\ &= -2x^2\,x^{\frac{1}{3}} \end{aligned}$$

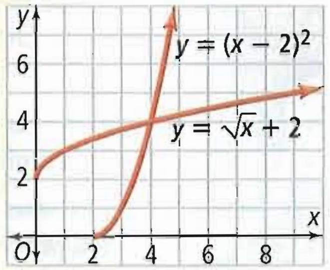
Inverse Relations and Functions (Lesson 6-7)

The inverse of $y = \sqrt{x} + 2, x \geq 0, y \geq 2$ is $x = \sqrt{y} + 2$, or $\sqrt{y} = x - 2$, or $y = (x - 2)^2, y \geq 0, x \geq 2$.

Solving Square Root Equations (Lesson 6-5)

$$\begin{aligned} x - 2 &= \sqrt{x} \\ x^2 - 4x + 4 &= x \\ x^2 - 5x + 4 &= 0 \\ (x - 4)(x - 1) &= 0 \\ x = 4 \text{ or } x &= 1 \\ 4 - 2 &= \sqrt{4} \checkmark \\ 1 - 2 &\neq \sqrt{1} \times \end{aligned}$$

Graphing Radical Functions (Lesson 6-8)



Chapter Vocabulary

- composite function (p. 399)
- index (p. 362)
- inverse function (p. 405)
- inverse relation (p. 405)
- like radicals (p. 374)
- *n*th root (p. 361)
- one-to-one function (p. 408)
- principal root (p. 361)
- radical equation (p. 390)
- radical function (p. 415)
- radicand (p. 362)
- rational exponent (p. 382)
- rationalize the denominator (p. 369)
- simplest form of a radical (p. 368)
- square root equation (p. 390)
- square root function (p. 415)

Choose the correct term to complete each sentence.

1. The number under a radical sign is called the (index/radicand).

2. (Radical functions/Inverse functions) are of the form $f(x) = \sqrt[n]{x}$.

3. A radical expression can always be rewritten using a(n) (rational exponent/inverse relation).

4. When two functions are combined so the range of one becomes the domain of the other, the resulting function is called a (square root function/composite function).

6-1 Roots and Radical Expressions

Quick Review

You can simplify a radical expression by finding the roots. The **principal root** of a number with two real roots is the positive root. The principal ***n*th root** of b is written as $\sqrt[n]{b}$, where b is the **radicand** and n is the **index** of the radical expression.

For any real number a, $\sqrt[n]{a^n} = \begin{cases} a \text{ if } n \text{ is odd} \\ |a| \text{ if } n \text{ is even} \end{cases}$.

Example

What is the simplified form of $\sqrt{36x^6}$?

$\sqrt{6^2x^6}$	Find the root of the integer.
$= \sqrt{6^2(x^3)^2}$	Find the root of the variable.
$= 6\lvert x^3\rvert$	Take the square root of each term. Since the index is even, include the absolute value symbol to ensure that the root is positive even when x^3 is negative.

Exercises

Find each real root.

5. $\sqrt{25}$

6. $\sqrt{0.49}$

7. $\sqrt[3]{-8}$

8. $-\sqrt[3]{8}$

Simplify each radical expression. Use absolute value symbols when needed.

9. $\sqrt{81x^2}$

10. $\sqrt[3]{64x^6}$

11. $\sqrt[4]{16x^{12}}$

12. $\sqrt[5]{0.00032x^5}$

13. $\sqrt{\frac{9x^4}{36}}$

14. $\sqrt[3]{125x^6y^9}$

6-2 Multiplying and Dividing Radical Expressions

Quick Review

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $(\sqrt[n]{a})(\sqrt[n]{b}) = \sqrt[n]{ab}$, and, if $b \neq 0$, then $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.

To **rationalize the denominator** of an expression, rewrite it so that the denominator contains no radical expressions.

Example

What is the simplest form of $\sqrt{32x^2y} \cdot \sqrt{18xy^3}$?

$\sqrt{(32x^2y)(18xy^3)}$	Combine terms.
$= \sqrt{(4^2 \cdot 2x^2y)(3^2 \cdot 2xy^3)}$	Factor.
$= \sqrt{4^2 \cdot 3^2 \cdot 2^2x^3y^4}$	Consolidate like terms.
$= \sqrt{4^2 \cdot 3^2 \cdot 2^2(x^2x)(y^2)^2}$	Identify perfect squares.
$= 4 \cdot 3 \cdot 2xy^2\sqrt{x} = 24xy^2\sqrt{x}$	Extract perfect squares.

Exercises

Multiply if possible. Then simplify.

15. $\sqrt[3]{9} \cdot \sqrt[3]{3}$

16. $\sqrt[3]{-7} \cdot \sqrt[3]{49}$

17. $\sqrt{2} \cdot \sqrt{8}$

Multiply and simplify.

18. $\sqrt{8x^2} \cdot \sqrt{2x^2}$

19. $5\sqrt[3]{9y^2} \cdot \sqrt[3]{24y}$

Divide and simplify.

20. $\sqrt{\frac{128}{8}}$

21. $\frac{\sqrt[3]{81x^5y^3}}{\sqrt[3]{3x^2}}$

22. $\frac{\sqrt[4]{162x^4}}{\sqrt[4]{2y^8}}$

Divide. Rationalize all denominators.

23. $\frac{\sqrt{8}}{\sqrt{6}}$

24. $\frac{\sqrt{3x^5}}{8x^2}$

25. $\frac{\sqrt[3]{6x^2y^4}}{2\sqrt[3]{5x^7y}}$

6-3 Binomial Radical Expressions

Quick Review

Like radicals have the same index and the same radicand. Use the distributive property to add and subtract them. Use the FOIL method to multiply binomial radical expressions. To rationalize a denominator that is a square root binomial, multiply the numerator and denominator by the conjugate of the denominator.

Example

What is the simplified form of $\sqrt{18} + \sqrt{50} - \sqrt{8}$?

$$\begin{aligned} &\sqrt{18} + \sqrt{50} - \sqrt{8} && \\ &= \sqrt{3^2 \cdot 2} + \sqrt{5^2 \cdot 2} - \sqrt{2^2 \cdot 2} && \text{Factor.} \\ &= 3\sqrt{2} + 5\sqrt{2} - 2\sqrt{2} && \text{Simplify each radical.} \\ &= (3 + 5 - 2)\sqrt{2} && \text{Combine like terms.} \\ &= 6\sqrt{2} && \text{Simplify.} \end{aligned}$$

Exercises

Add or subtract if possible.

26. $10\sqrt{27} - 4\sqrt{12}$

27. $3\sqrt{20x} + 8\sqrt{45x} = 4\sqrt{5x}$

28. $\sqrt[3]{54x^3} - \sqrt[3]{16x^3}$

Multiply.

29. $(3 + \sqrt{2})(4 + \sqrt{2})$

30. $(\sqrt{5} + \sqrt{11})(\sqrt{5} - \sqrt{11})$

31. $(10 + \sqrt{6})(10 - \sqrt{3})$

Divide. Rationalize all denominators.

32. $\dfrac{2 + \sqrt{5}}{\sqrt{5}}$

33. $\dfrac{3 + \sqrt{18}}{1 + \sqrt{8}}$

6-4 Rational Exponents

Quick Review

You can rewrite a radical expression with a rational exponent. By definition, if the nth root of a is a real number and m is an integer, then $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$; if m is negative then $a \neq 0$. Rational exponents can be used to simplify radical expressions.

Example

Multiply and simplify $\sqrt{x}\left(\sqrt[4]{x^3}\right)$.

$$\begin{aligned} \sqrt{x}\left(\sqrt[4]{x^3}\right) &= x^{\frac{1}{2}} \cdot x^{\frac{3}{4}} && \text{Rewrite with rational exponents.} \\ &= x^{\frac{5}{4}} && \text{Combine exponents.} \\ &= \sqrt[4]{x^5} && \text{Rewrite as a radical expression.} \end{aligned}$$

Exercises

Simplify each expression.

34. $25^{\frac{1}{2}}$

35. $81^{\frac{1}{4}}$

36. $16^{\frac{1}{3}} \cdot 4^{\frac{1}{3}}$

37. $5^{\frac{3}{2}} \cdot 5^{\frac{1}{2}}$

Write each expression in simplest form.

38. $\left(x^{\frac{1}{4}}\right)^4$

39. $\left(-8y^9\right)^{\frac{1}{3}}$

40. $\left(\sqrt{9xy^2}\right)^4$

41. $\left(x^{\frac{1}{6}} y^{\frac{1}{3}}\right)^{-18}$

42. $\left(\dfrac{x^4}{x^{-1}}\right)^{-\frac{1}{5}}$

43. $\left(\dfrac{x^{\frac{1}{3}}}{y^{-\frac{2}{3}}}\right)^9$

6-5 Solving Square Root and Other Radical Equations

Quick Review

To solve a **radical equation**, you must isolate a radical expression on one side of the equation. You can then rewrite the radical expression using a rational exponent and use the reciprocal of the exponent to solve the equation.

For example, to solve a square root equation, you square each side of the equation. Check all possible solutions in the original equation to eliminate extraneous solutions.

Example

What is the solution of $4(x-2)^{\frac{2}{3}} = 16$?

$(x-2)^{\frac{2}{3}} = 4$ Isolate the radical.

$((x-2)^{\frac{2}{3}})^{\frac{3}{2}} = 4^{\frac{3}{2}}$ Raise both sides to the $\frac{3}{2}$ power.

$(x-2)^{\frac{6}{6}} = 4^{\frac{3}{2}}$ Law of exponents.

$|x-2| = 8$ Simplify.

$x = 10$ or $x = -6$ Solve for x.

Exercises

Solve each equation. Check for extraneous solutions.

44. $2 + \sqrt{x+5} = 4$

45. $3\sqrt{2x+6} = 18$

46. $5(3x+1)^{\frac{1}{4}} = 10$

47. $4(3x-3)^{\frac{2}{3}} = 36$

48. $\sqrt{3x+3} - 1 = x$

49. $\sqrt{x+6} + 2 = x + 6$

50. $\sqrt{5x+1} - 2\sqrt{x} = 1$

51. $\sqrt{2x+9} - \sqrt{x} = 3$

52. Electricity The power P, in watts, that a circular solar cell produces and the radius of the cell r in centimeters are related by the square root equation $r = \sqrt{\frac{P}{0.02\pi}}$. About how much power is produced by a cell with a radius of 12 cm?

6-6 Function Operations

Quick Review

When performing function operations, you can use the same rules you used for real numbers, but you must take into consideration the domain and range of each function. The composition of function g with function f is defined as $(g \circ f)(x) = g(f(x))$.

Example

Let $f(x) = x + 3$ and $g(x) = x^2 - 2$. What is $(g \circ f)(-2)$?

$g(f(-2)) = g((-2) + 3)$ Evaluate $f(-2)$.

$= g(1)$ Simplify.

$= (1)^2 - 2$ Evaluate $g(f(-2))$.

$= -1$ Simplify.

Therefore, $(g \circ f)(-2) = -1$

Exercises

Let $f(x) = x - 4$ and $g(x) = x^2 - 16$. Perform each function operation and then find the domain.

53. $f(x) + g(x)$

54. $g(x) - f(x)$

55. $f(x) \cdot g(x)$

56. $\frac{g(x)}{f(x)}$

Let $g(x) = 5x - 2$ and $h(x) = x^2 + 1$. Find the value of each expression.

57. $(h \circ g)(-1)$

58. $(h \circ g)(0)$

59. $(g \circ h)(2)$

60. $(g \circ h)(a)$

61. Discounts A grocery store is offering a 50% discount off a \$4.00 box of cereal. You also have a \$1.00 off coupon for the same cereal. Use a composite function to show whether it is better to use the coupon before or after the store discount.

6-7 Inverse Relations and Functions

Quick Review

If a relation or a function is described by an equation in x and y, you can interchange x and y to get the inverse. The domain of a function becomes the range of its inverse, and the range of a function becomes the domain of its inverse.

Example

What is the inverse of $f(x) = \sqrt{x - 10}$?

$y = \sqrt{x - 10}$ Rewrite using y.

$x = \sqrt{y - 10}$ Interchange the x and y values.

$x^2 = y - 10$ Square each side.

$y = x^2 + 10$ Solve for y.

$f^{-1}(x) = x^2 + 10$ Write the inverse function.

The domain of $f(x)$ is $x \geq 10$, which means the range of $f^{-1}(x)$ is $y \geq 10$. Also, since the range of $f(x)$ is $y \geq 0$, the domain of $f^{-1}(x)$ is $x \geq 0$.

Exercises

Find the inverse of each function. Determine whether each inverse is a function.

62. $f(x) = 2x^2 - 8$

63. $f(x) = 15 - 3x$

64. $f(x) = \sqrt{x + 6}$

65. $f(x) = (2x - 3)^2$

Graph each function and its inverse. Describe the domain and range of each.

66. $f(x) = 4x - 1$

67. $f(x) = (x + 3)^2$

68. $f(x) = \sqrt{x - 3}$

69. $f(x) = 6 - 5x^2$

70. **Geometry** The volume of cube is determined by the formula $V = s^3$, where s is the length of one side. Find the inverse formula. Use it to find the side length of a cube with a volume of 64 ft^3.

6-8 Graphing Radical Functions

Quick Review

The function $f(x) = \sqrt{x}$ is the parent function of the **square root function** $f(x) = a\sqrt{x - h} + k$. The graph of $f(x) = a\sqrt{x}$ is a stretch $(a > 1)$ or a shrink $(0 < a < 1)$ of the parent function. The graph of $f(x) = a\sqrt{x - h} + k$ is a translation h units horizontally and k units vertically of $y = a\sqrt{x}$. The graph of $f(x) = \sqrt[n]{x}$ is transformed by a, h, and k in the same way as the graph of $f(x) = \sqrt{x}$.

Example

Describe the graph of $y = \sqrt{4x + 12}$.

$y = \sqrt{4x + 12}$

$y = \sqrt{4(x + 3)}$ Factor the polynomial.

$y = 2\sqrt{x + 3}$ Simplify the radical.

The graph of $y = \sqrt{4x + 12}$ is the graph of $y = 2\sqrt{x}$ translated 3 units to the left.

Exercises

Graph each function. Find the domain and range.

71. $y = \sqrt{x} - 5$

72. $y = \sqrt{x + 8}$

73. $y = 5\sqrt{x} + 9$

74. $y = -\sqrt{x - 4}$

75. $y = \sqrt[3]{x + 10}$

76. $y = -\sqrt[3]{x - 2} + 5$

Rewrite each function to make it easy to graph using transformations. Describe each graph.

77. $y = \sqrt{9x - 27} + 4$

78. $y = -3\sqrt{4x - 16}$

79. $y = \sqrt[3]{8x + 24}$

80. $y = \sqrt{\frac{x - 4}{4}} + 6$

Solve each equation by graphing.

81. $5 = -\sqrt{x - 3}$

82. $\sqrt{8x - 16} = 2\sqrt{x + 2}$

6 Chapter Test

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

Simplify each radical expression. Use absolute value symbols when needed.

1. $\sqrt{54x^3y^5}$

2. $\sqrt[3]{-0.027}$

3. $\sqrt[5]{-64x^{14}y^{20}}$

Simplify each expression. Rationalize all denominators.

4. $\sqrt{7x^3} \cdot \sqrt{14x}$

5. $\frac{1 - \sqrt{3x}}{\sqrt{6x}}$

6. $\sqrt{48} + 2\sqrt{27} + 5\sqrt{12}$

7. $(3 + 2\sqrt{5})(1 - \sqrt{20})$

8. $4\sqrt{7xz} + 2\sqrt{7xz}$

9. $\frac{5\sqrt{2}}{\sqrt{7} - \sqrt{2}}$

Simplify each expression.

10. $(125)^{-\frac{2}{3}}$

11. $x^{\frac{1}{6}} \cdot x^{\frac{1}{3}}$

12. $\left(\frac{8x^9y^3}{27x^2y^{12}}\right)^{\frac{2}{3}}$

13. $\sqrt{8x^5} - \sqrt{18x^5}$

Solve each equation. Check for extraneous solutions.

14. $\sqrt{x - 3} = x - 5$

15. $\sqrt{x + 4} = \sqrt{3x}$

16. $2(x - 1)^{\frac{3}{4}} = 16$

17. $\sqrt{x + 3} - 1 = x$

Let $f(x) = x - 2$ and $g(x) = x^2 - 3x + 2$. Perform each function operation and then find the domain.

18. $-2g(x) + f(x)$

19. $-f(x) \cdot g(x)$

20. $\frac{g(x)}{f(x)}$

Find each product or quotient.

21. $\sqrt{5}(\sqrt[4]{5})$

22. $\frac{\sqrt{x^3}}{\sqrt[5]{x^2}}$

For each pair of functions, find $(g \circ f)(x)$ and $(f \circ g)(x)$.

23. $f(x) = x^2 - 2$, $g(x) = 4x + 1$

24. $f(x) = 2x^2 + x - 7$, $g(x) = -3x - 1$

Find the inverse of each function. Is the inverse a function?

25. $f(x) = (x + 3)^2 + 1$

26. $f(x) = \sqrt{2x + 1}$

27. $g(x) = 3x^3 - 4$

28. $f(x) = \frac{1}{4}x$

Rewrite each function to make it easy to graph using transformations. Describe the graph.

29. $y = \sqrt{16x + 80} - 1$

30. $y = \sqrt{9x + 3}$

Graph. Find the domain and range of each function.

31. $y = 2\sqrt{x} + 3$

32. $y = -\sqrt{2x + 3}$

33. $y = \sqrt{x + 3} - 4$

Do you UNDERSTAND?

34. **Writing** Explain why −108 has no real 6th roots.

35. **Open-Ended** Write a relation that is not a function, but whose inverse is a function.

36. **Measurement** The time t in seconds for a swinging pendulum to complete one full cycle is given by the function $t = 0.2\sqrt{l}$, where l is the length of the pendulum in centimeters. To the nearest tenth, how long is a full cycle if the pendulum is 10 cm long? 20 cm long? How long, in centimeters, is a pendulum that takes 2 seconds for one full cycle?

6

Sunshine State Standards Practice

TIPS FOR SUCCESS

Some problems require you to find the inverse of a function.

What is the inverse of the function $y = x^2 + 3$?

(A) $y = x - 3$

(B) $y = \pm\sqrt{x - 3}$

(C) $y = \pm\sqrt{x^2 + 3}$

(D) $y = (x - 3)^2$

TIP 1

To find the inverse of a function, interchange x and y.

TIP 2

After you interchange x and y, solve for y.

Think It Through

$y = x^2 + 3$

$x = y^2 + 3$

$x - 3 = y^2$

$\pm\sqrt{x - 3} = y$

$y = \pm\sqrt{x - 3}$

The correct answer is B.

Vocabulary Review

As you solve test items, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. radicand

B. index

C. composite function

D. inverse functions

E. radical function

I. the combination of two functions such that the output from the first becomes the input for the second

II. the degree of a root in a radical expression

III. the number under the radical sign in a radical expression

IV. a function that can be written in the form $f(x) = a\sqrt[n]{x - h} + k$

V. the range of one function is the domain of the other and vice versa

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

1. Find all the roots of $2x^4 + x^3 - 8x^2 - 4x = 0$.

(A) $x = -2, x = -0.5, x = 0, x = 2$

(B) $x = -2, x = -0.5, x = 2$

(C) $x = -2, x = 0.5, x = 0, x = 2$

(D) $x = -2, x = 0.5, x = 2$

2. Solve the equation $ax^2 + bx + c = 0$ for b.

(F) $b = -cx - ax^2$

(H) $b = -(cx - ax^2)$

(G) $b = \frac{-c - ax^2}{x}$

(I) $b = \frac{-(c - ax^2)}{x}$

3. Use the sum of cubes formula to factor $x^3 + 64$.

(A) $(x + 4)(x^2 - 4x + 4)$

(B) $(x + 4)(x^2 + 4x + 4)$

(C) $(x + 4)(x^2 - 4x + 16)$

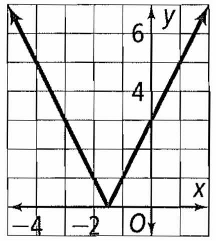
(D) $(x + 4)(x^2 + 4x + 16)$

4. The time it takes to copy pages varies directly with the number of pages being copied. The copier at your office can copy 21 color pages per minute and 40 black and white pages per minute. Approximately how long will it take to copy 60 color pages and 35 black and white pages?

Ⓕ 0.9 minute　　Ⓗ 2.9 minutes

Ⓖ 2.5 minutes　　Ⓘ 3.7 minutes

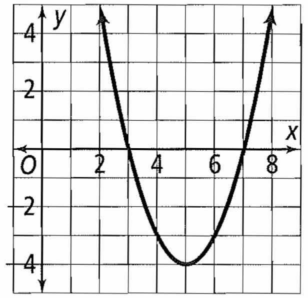
5. Which equation is modeled by the graph?



Ⓐ $y = |2x - 3|$　　Ⓒ $y = |2x + 3|$

Ⓑ $y = 2|x - 3|$　　Ⓓ $y = 2|x + 3|$

6. What are the vertex and axis of symmetry for the given parabola?



Ⓕ $(-4, 5), y = 5$　　Ⓗ $(5, -4), y = -4$

Ⓖ $(-4, 5), x = 5$　　Ⓘ $(5, -4), x = 5$

7. What is the product of $\sqrt[3]{3}$ and $\sqrt[5]{3}$?

Ⓐ $\sqrt[8]{3}$　　Ⓒ $\sqrt[15]{3^8}$

Ⓑ $\sqrt[8]{9}$　　Ⓓ $\sqrt[8]{3^{15}}$

8. Solve the equation $x^2 - 6x = 3$ by completing the square.

Ⓕ $-3 \pm 2\sqrt{3}$　　Ⓗ $-3 \pm 3\sqrt{2}$

Ⓖ $3 \pm 2\sqrt{3}$　　Ⓘ $3 \pm 3\sqrt{2}$

9. Solve $7x^2 + 196 = 0$ for x.

Ⓐ $\pm 4i\sqrt{7}$　　Ⓒ $\pm 2i\sqrt{7}$

Ⓑ $\pm 4\sqrt{7}$　　Ⓓ $\pm 2\sqrt{7}$

10. Which inequality is modeled by the graph?

Ⓕ $k + 1 \leq 7$　　Ⓗ $k - 4 \leq 3$

Ⓖ $|k + 1| \leq 7$　　Ⓘ $|k - 4| \leq 3$

11. Which equation represents a line that contains the point (3, 2) and is parallel to $y = 3x - 12$?

Ⓐ $y = \frac{1}{3}x - 7$　　Ⓒ $y = 3x - 3$

Ⓑ $y = 3x - 7$　　Ⓓ $y = \frac{1}{3}x - 3$

12. Which is the first *incorrect* step in simplifying $\sqrt[3]{x^9y^6z}$?

Step 1: $\sqrt[3]{x^9y^6z} = \sqrt[3]{x^9} \cdot \sqrt[3]{y^6} \cdot \sqrt[3]{z}$

Step 2: $= x^3 \cdot \sqrt[3]{y^6} \cdot \sqrt[3]{z}$

Step 3: $= x^3 \cdot y^3 \cdot \sqrt[3]{z}$

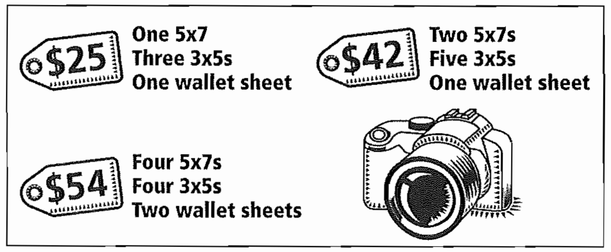
Ⓕ Step 1

Ⓖ Step 2

Ⓗ Step 3

Ⓘ Each step is correct.

13. A photographer is promoting three photo specials. How much does it cost for each type of print?



$25
One 5x7
Three 3x5s
One wallet sheet

$42
Two 5x7s
Five 3x5s
One wallet sheet

$54
Four 5x7s
Four 3x5s
Two wallet sheets

Ⓐ 5 × 7 costs $7, 3 × 5 costs $5, Wallet costs $3

Ⓑ 5 × 7 costs $11, 3 × 5 costs $7, Wallet costs $3

Ⓒ 5 × 7 costs $12, 3 × 5 costs $11, Wallet costs $7

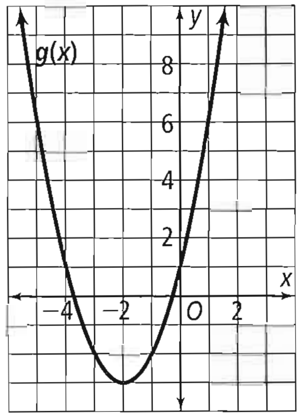
Ⓓ 5 × 7 costs $7, 3 × 5 costs $5, Wallet costs $5

14. What is an equivalent form of $\frac{5}{2 + 2i}$?

Ⓕ $\frac{5}{4i}$　　Ⓗ $\frac{5 + 5i}{4}$

Ⓖ $\frac{10 - 10i}{4 - 4i}$　　Ⓘ $\frac{5 - 5i}{4}$

15. The graph shows a transformation of $f(x) = x^2$. What is an equation of the graph?



Ⓐ $g(x) = (x - 2)^2 - 3$ Ⓒ $g(x) = (x + 2)^2 - 3$

Ⓑ $g(x) = (x - 2)^2 + 3$ Ⓓ $g(x) = (x + 2)^2 + 3$

16. The total cost of $x + 2$ markers is $x^3 + 5x^2 + 2x - 8$. What is the cost of each marker?

Ⓕ $x^2 + 3x - 4$ Ⓗ $x^2 - 3x - 4$

Ⓖ $x^2 - 3x + 4$ Ⓘ $x^2 + 5x + 2$

17. Which description best fits the graph of $f(x) = -x^5 + 4x^3 + 8x^2 - 32$?

Ⓐ a zero at −2, a zero of multiplicity 2 at 2, and with end behavior up and down

Ⓑ a zero at −2, a zero at 2, and with end behavior up and down

Ⓒ a zero at −2, a zero of multiplicity 2 at 2, and with end behavior down and up

Ⓓ a zero at −2, a zero at 2, and with end behavior down and up

18. What can you correctly say about the following statement?

If n is a real number, then $0^n = 0$.

Ⓕ The statement is always true.

Ⓖ The statement is sometimes true.

Ⓗ The statement is never true.

Ⓘ The statement cannot be labeled *true* or *false*.

19. If $f(x) = x^2$ and $g(x) = \frac{2}{x}$, what is $f(g(x)) - g(f(x))$?

Ⓐ $f(g(x))$ Ⓒ $f(f(x))$

Ⓑ $g(f(x))$ Ⓓ $g(g(x))$

GRIDDED RESPONSE

20. Let $g(x) = x - 3$ and $h(x) = x^2 + 6$. What is $(h \circ g)(1)$?

21. A laptop comes without any programs installed on it. Each program costs \$20 and the laptop you want costs \$319. What is the greatest number of programs you can buy if you want to spend at most \$500 for the laptop?

22. You are building an entertainment center with shelves that are x in. deep by x in. long. The height of the unit will be twice the depth. If the volume of the unit will be 8,192 in^3, what is the height, in inches, of the entertainment center?

23. Use Descartes' Rule of Signs to find the maximum number of positive real roots of $P(x) = 5x^4 + x^3 - 4x^2 + 3x + 1 = 0$.

24. Using the discriminant, determine the number of real roots of the equation $2x^2 + 3x = 4$.

25. What is the quotient $\frac{\sqrt[3]{8x^6y^{12}}}{\sqrt{4x^4y^8}}$?

26. What is the solution of $4 + \sqrt{3x + 5} = 7$? Express your answer as a fraction.

27. Simplify the expression $256^{-\frac{3}{4}}$. Express your answer as a fraction.

28. All 385 tickets for a high-school play sold in 10 days. The ticket receipts totaled \$1960. If the cost of a child's ticket was \$4 and the cost of an adult's ticket was \$6, how many adult tickets were sold?

29. What is the x-value of the x-intercept of the graph of $f(x) = x^2 + 4x + 4$?

30. You are given that $f(x) = x^2 - 4$ and $g(x) = 3x + 1$. What is the value of $f(x)$ at the one value of x where $f(x) + g(x) = f(x) - g(x)$? Express your answer as a fraction.

31. Two friends went shopping together. One friend bought 2 hats and 1 shirt and spent \$70, while the other friend bought 1 hat and 3 shirts and spent \$85. All the shirts were one price, and all the hats were another price. How many dollars did they pay for all three hats?

32. A student found that a third-degree cubic function with real coefficient has zeros 16 and $1 - 2i$. It has a leading coefficient of 3. What is the constant term of this polynomial function?

CHAPTER 7

Get Ready!

Lesson 1-3

Evaluating Expressions

Evaluate each expression for $x = -2, 0,$ and 2.

1. 10^{x+1}

2. $\left(\frac{3}{2}\right)^x$

3. -5^{x-2}

4. $-(3)^{0.5x}$

Lesson 2-5

Using Linear Models

Draw a scatter plot and find the line of best fit for each set of data.

5. (0, 2), (1, 4), (2, 6.5), (3, 8.5), (4, 10), (5, 12), (6, 14)

6. (3, 100), (5, 150), (7, 195), (9, 244), (11, 296), (13, 346), (15, 396)

Lessons 4-1 and 5-9

Graphing Transformations

Identify the parent function of each equation. Graph each equation as a transformation of its parent function.

7. $y = (x + 5)^2 - 3$

8. $y = -2(x - 6)^3$

Lesson 6-4

Simplifying Rational Exponents

Simplify each expression.

9. $\left(x^{\frac{1}{5}}\right)^{10}$

10. $\left(-8x^3\right)^{\frac{4}{3}}$

Lesson 6-7

Finding Inverses

Find the inverse of each function. Is the inverse a function?

11. $y = 10 - 2x^2$

12. $y = (x + 4)^3 - 1$

Looking Ahead Vocabulary

13. In advertising, the *decay factor* describes how an advertisement loses its effectiveness over time. In math, would you expect a decay factor to increase or decrease the value of y as x increases?

14. There are many different kinds of growth patterns. Patterns that increase by a constant rate are linear. Patterns that grow *exponentially* increase by an ever-increasing rate. If your allowance doubles each week, does that represent linear growth or exponential growth?

15. The word *asymptote* comes from a Greek word meaning "not falling together." When looking at the end behavior of a function, do you expect the graph to intersect its asymptote?

CHAPTER

7

Exponential and Logarithmic Functions

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VOCABULARY

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Logarithms provide a way to work with the inverses of exponential functions. Exponential functions model what some might call "explosive" growth, but logarithmic values grow very slowly. Decibels are logarithms that measure sound, and when sound energy increases dramatically, the decibel values creep upward. A few extra decibels can bust your eardrums!

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
asymptote, *p. 435*	asíntota
Change of Base Formula, *p. 464*	fórmula de cambio de base
common logarithm, *p. 453*	logaritmo común
exponential equation, *p. 469*	ecuación exponencial
exponential function, *p. 434*	función exponencial
exponential growth, *p. 435*	incremento exponencial
logarithm, *p. 451*	logaritmo
logarithmic equation, *p. 471*	ecuación logarítmica
logarithmic function, *p. 454*	función logarítmica
natural logarithmic function, *p. 478*	función logarítmica natural

BIG ideas

1 **Modeling**

Essential Question How do you model a quantity that changes regularly over time by the same percentage?

2 **Equivalence**

Essential Question How are exponents and logarithms related?

3 **Function**

Essential Question How are exponential functions and logarithmic functions related?

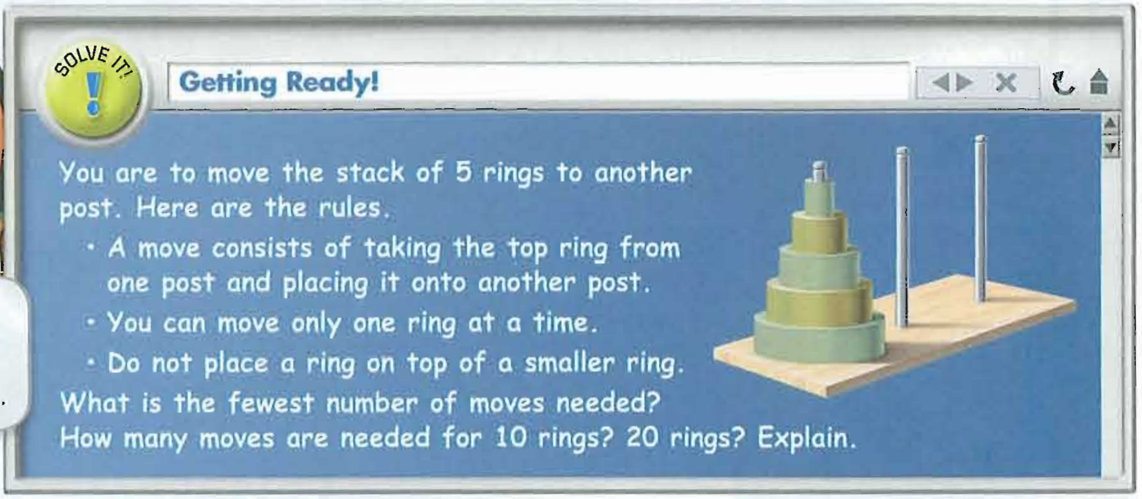
Chapter Preview

7-1 Exploring Exponential Models

Sunshine State Standards

MA.912.A.8.1 Define exponential functions.
MA.912.A.8.3 Graph exponential functions.
MA.912.A.8.7 Solve applications of exponential growth and decay.

Objective To model exponential growth and decay



Getting Ready!

You are to move the stack of 5 rings to another post. Here are the rules.

- A move consists of taking the top ring from one post and placing it onto another post.
- You can move only one ring at a time.
- Do not place a ring on top of a smaller ring.

What is the fewest number of moves needed? How many moves are needed for 10 rings? 20 rings? Explain.

Lesson Vocabulary

- exponential function
- exponential growth
- exponential decay
- asymptote
- growth factor
- decay factor

The number of moves needed for additional rings in the Solve It suggests a pattern that approximates repeated multiplication.

Essential Understanding You can represent repeated multiplication with a function of the form $y = ab^x$ where b is a positive number other than 1.

An **exponential function** is a function with the general form $y = ab^x$, $a \neq 0$, with $b > 0$, and $b \neq 1$. In an exponential function, the base b is a constant. The exponent x is the independent variable with domain the set of real numbers.

Problem 1 Graphing an Exponential Function

Plan

How does making a table help you sketch the graph?
The table shows coordinates of several points on the graph.

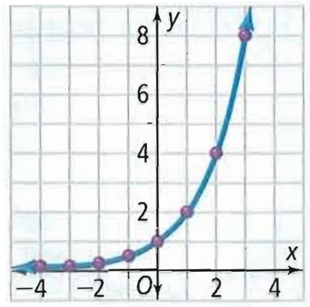
What is the graph of $y = 2^x$?

Step 1 Make a table of values.

x	2^x	y
-4	2^{-4}	$\frac{1}{16} = 0.0625$
-3	2^{-3}	$\frac{1}{8} = 0.125$
-2	2^{-2}	$\frac{1}{4} = 0.25$
-1	2^{-1}	$\frac{1}{2} = 0.5$

x	2^x	y
0	2^0	1
1	2^1	2
2	2^2	4
3	2^3	8

Step 2 Plot and connect the points.



Got It? **1.** What is the graph of each function?

a. $y = 4^x$ **b.** $y = \left(\frac{1}{3}\right)^x$ **c.** $y = 2(3)^x$

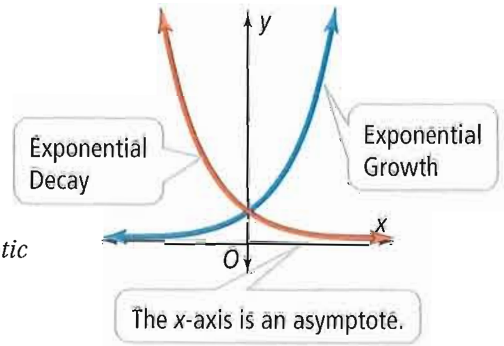
d. **Reasoning** What generalizations can you make about the domain, range, and y-intercepts of these functions?

Dynamic Activity
Exponential
Growth and Decay

Two types of exponential behavior are *exponential growth* and *exponential decay*.

For **exponential growth**, as the value of x increases, the value of y increases. For **exponential decay**, as the value of x increases, the value of y decreases, approaching zero.

The exponential functions shown here are *asymptotic* to the x-axis. An **asymptote** is a line that a graph approaches as x or y increases in absolute value.



take note

Concept Summary Exponential Functions

For the function $y = ab^x$,

- if $a > 0$ and $b > 1$, the function represents exponential growth.
- if $a > 0$ and $0 < b < 1$, the function represents exponential decay.

In either case, the y-intercept is $(0, a)$, the domain is all real numbers, the asymptote is $y = 0$, and the range is $y > 0$.

Problem 2 Identifying Exponential Growth and Decay

Identify each function or situation as an example of exponential growth or decay. What is the y-intercept?

A $y = 12(0.95)^x$

Since $0 < b < 1$, the function represents exponential decay.
The y-intercept is $(0, a) = (0, 12)$.

B $y = 0.25(2)^x$

Since $b > 1$, the function represents exponential growth.
The y-intercept is $(0, a) = (0, 0.25)$.

Think

What quantity does the y-intercept represent?
The y-intercept is the amount of money at $t = 0$, which is the initial investment.

C **You put \$1000 into a college savings account for four years. The account pays 5% interest annually.**

The amount of money in the bank grows by 5% annually. It represents exponential growth. The y-intercept is 1000, which is the dollar value of the initial investment.

Got It? **2.** Identify each function or situation as an example of exponential growth or decay. What is the y-intercept?

a. $y = 3(4^x)$ **b.** $y = 11(0.75^x)$

c. You put \$2000 into a college savings account for four years. The account pays 6% interest annually.

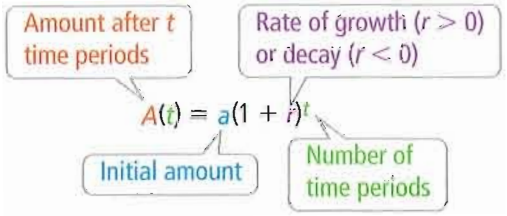
For exponential growth $y = ab^x$, with $b > 1$, the value b is the **growth factor**. A quantity that exhibits exponential growth increases by a constant percentage each time period. The percentage increase r, written as a decimal, is the *rate of increase* or *growth rate*. For exponential growth, $b = 1 + r$.

For exponential decay, $0 < b < 1$ and b is the **decay factor**. The quantity decreases by a constant percentage each time period. The percentage decrease, r, is the *rate of decay*. Usually a rate of decay is expressed as a negative quantity, so $b = 1 + r$.

take note

Key Concept Exponential Growth and Decay

You can model exponential growth or decay with this function.



$$A(t) = a(1 + r)^t$$

For growth or decay to be exponential, a quantity changes by a fixed percentage each time period.

Problem 3 Modeling Exponential Growth

You invested \$1000 in a savings account at the end of 6th grade. The account pays 5% annual interest. How much money will be in the account after six years?

Step 1 Determine if an exponential function is a reasonable model.

The money grows at a fixed rate of 5% per year. An exponential model is appropriate.

Step 2 Define the variables and determine the model.

Let t = the number of years since the money was invested.
Let $A(t)$ = the amount in the account after each year.

A reasonable model is $A(t) = a(1 + r)^t$.

Step 3 Use the model to solve the problem.

$$\begin{aligned} A(6) &= 1000(1 + 0.05)^6 && \text{Substitute } a = 1000,\ r = 0.05, \text{ and } t = 6. \\ &= 1000(1.05)^6 && \text{Simplify.} \\ &\approx \$1340.10 \end{aligned}$$

The account contains \$1340.10 after six years.

Think

What is the growth rate r?
It is the annual interest rate, written as a decimal: 5% = 0.05.

Got It? **3.** Suppose you invest \$500 in a savings account that pays 3.5% annual interest. How much will be in the account after five years?

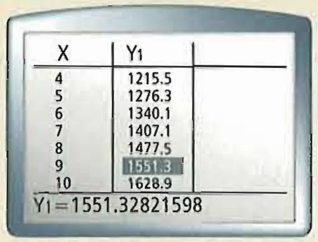
Problem 4 Using Exponential Growth

Suppose you invest $1000 in a savings account that pays 5% annual interest. If you make no additional deposits or withdrawals, how many years will it take for the account to grow to at least $1500?

Plan

How can you make a table to solve this problem? Define the variables, write an equation and enter it into a graphing calculator. Then you can inspect a table to find the solution.

Think	Write
Define the variables.	Let t = the number of years. Let A(t) = the amount in the account after t years.
Determine the model.	$A(t) = 1000(1 + 0.05)^t$ $= 1000(1.05)^t$
Make a table using the table feature on a graphing calculator. Find the input when the output is 1500.	see screen below
The account pays interest only once a year. The balance after the 8th year is not yet $1500.	The account will not contain $1500 until the ninth year. After nine years, the balance will be $1551.33.



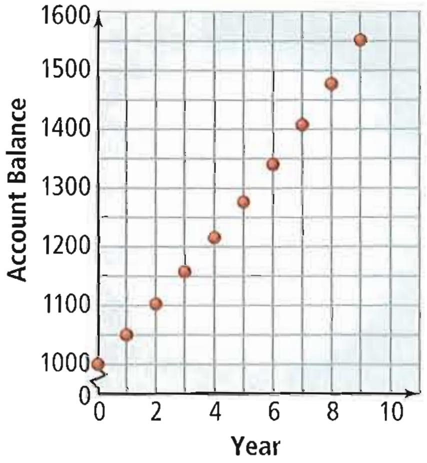
X	Y1
4	1215.5
5	1276.3
6	1340.1
7	1407.1
8	1477.5
9	1551.3
10	1628.9

Y1=1551.32821598

Got It? 4. a. Suppose you invest $500 in a savings account that pays 3.5% annual interest. When will the account contain at least $650?

b. Reasoning Use the table in Problem 4 to determine when that account will contain at least $1650. Explain.

Exponential functions are often discrete. In Problem 4, interest is paid only once a year. So the graph consists of individual points corresponding to $t = 1, 2, 3,$ and so on. It is not continuous. Both the table and the graph show that there is never *exactly* $1500 in the account and that the account will not contain more than $1500 until the ninth year.



To model a discrete situation using an exponential function of the form $y = ab^x$, you need to find the growth or decay factor b. If you know y-values for two consecutive x-values, you can find the rate of change r, and then find b using $r = \frac{(y_2 - y_1)}{y_1}$ and $b = 1 + r$.

B Apply

30. Think About a Plan Your friend invested \$1000 in an account that pays 6% annual interest. How much interest will your friend have after her college graduation in 4 years?
- Is an exponential model reasonable for this situation?
- What equation should you use to model this situation?
- Is the solution of the equation the final answer to the problem?

31. Oceanography The function $y = 20(0.975)^x$ models the intensity of sunlight beneath the surface of the ocean. The output y represents the percent of surface sunlight intensity that reaches a depth of x feet. The model is accurate from about 20 feet to about 600 feet beneath the surface.

a. Find the percent of sunlight 50 feet beneath the surface of the ocean.

b. Find the percent of sunlight at a depth of 370 feet.

32. Population The population of a certain animal species decreases at a rate of 3.5% per year. You have counted 80 of the animals in the habitat you are studying.

a. Write a function that models the change in the animal population.

b. Graphing Calculator Graph the function. Estimate the number of years until the population first drops below 15 animals.

33. Sports While you are waiting for your tennis partner to show up, you drop your tennis ball from 5 feet. Its rebound was approximately 35 inches on the first bounce and 21.5 inches on the second. What exponential function would be a good model for the bouncing ball?

For each annual rate of change, find the corresponding growth or decay factor.

34. +70% **35.** +500% **36.** −75% **37.** −55%

38. +12.5% **39.** −0.1% **40.** +0.1% **41.** +100%

C Challenge

42. Manufacturing The value of an industrial machine has a decay factor of 0.75 per year. After six years, the machine is worth \$7500. What was the original value of the machine?

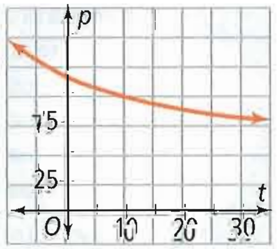
43. Zoology Determine which situation best matches the graph.

(A) A population of 120 cougars decreases 98.75% yearly.

(B) A population of 120 cougars increases 1.25% yearly.

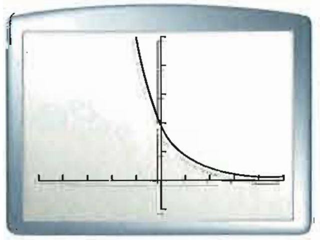
(C) A population of 115 cougars decreases 1.25% yearly.

(D) A population of 115 cougars decreases 50% yearly.



44. Open-Ended Write a problem that could be modeled with $y = 20(1.1)^x$.

45. Reasoning Which function does the graph represent? Explain. (Each interval represents one unit.)



(A) $y = \left(\frac{1}{3}\right)2^x$

(B) $y = 2\left(\frac{1}{3}\right)^x$

(C) $y = -2\left(\frac{1}{3}\right)^x$

Sunshine State Standards Practice

MA.912.A.2.6

46. Which function represents the value after x years of a new delivery van that costs $25,000 and depreciates 15% each year?

Ⓐ $y = -15(25{,}000)^x$

Ⓑ $y = 25{,}000(0.15)^x$

Ⓒ $y = 25{,}000(0.85)^x$

Ⓓ $y = 25{,}000(1.15)^x$

MA.912.A.6.3

47. What is $f(x) = 3x^{\frac{1}{3}}$ for $x = \frac{1}{125}$?

Ⓕ 15

Ⓖ $\frac{3}{5}$

Ⓗ $\frac{\sqrt[3]{3}}{5}$

Ⓘ $5\sqrt[3]{3}$

MA.912.A.1.6

48. What is the simplified form of $\frac{2 + i}{2 - i}$?

Ⓐ -1

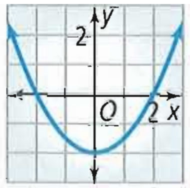
Ⓑ $\frac{3 + 4i}{3}$

Ⓒ $\frac{5 + 4i}{5}$

Ⓓ $\frac{3 + 4i}{5}$

MA.912.A.4.5

49. Which graph represents the equation $y = x^2 - x - 2$?

Ⓕ


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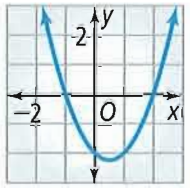

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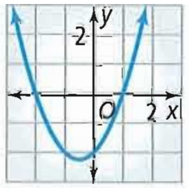

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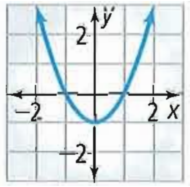


MA.912.A.2.11

50. Extended Response You are driving a car when a deer suddenly darts across the road in front of you. Your brain registers the emergency and sends a signal to your foot to hit the brake. The car travels a reaction distance D, in feet, during this time, where D is a function of the speed r, in miles per hour, that the car is traveling when you see the deer, given by $D(r) = \frac{11r + 5}{10}$. Find the inverse and explain what it represents. Is the inverse a function?

Mixed Review

Graph each function. See Lesson 6-8.

51. $y = 3 - 2\sqrt{x + 2}$

52. $y = 3\sqrt[3]{2x - 1}$

53. $y = -2 + \sqrt{x}$

Factor the expression. See Lesson 4-4.

54. $8 + 27x^3$

55. $3x^2 + 11x - 4$

56. $25 - 40x + 16x^2$

Solve the system of equations using a matrix. See Lesson 3-6.

57. $\begin{cases} x + 5y = -4 \\ x + 6y = -5 \end{cases}$

58. $\begin{cases} 3a + 5b = 0 \\ a + b = 0 \end{cases}$

59. $\begin{cases} -x + 2y + z = 0 \\ y = -2x + 3 \\ z = 3x \end{cases}$

Get Ready! **To prepare for Lesson 7-2, do Exercises 60–63.**

Graph each function. See Lesson 7-1.

60. $y = 3^x$

61. $y = 4(2)^x$

62. $y = 0.75^x$

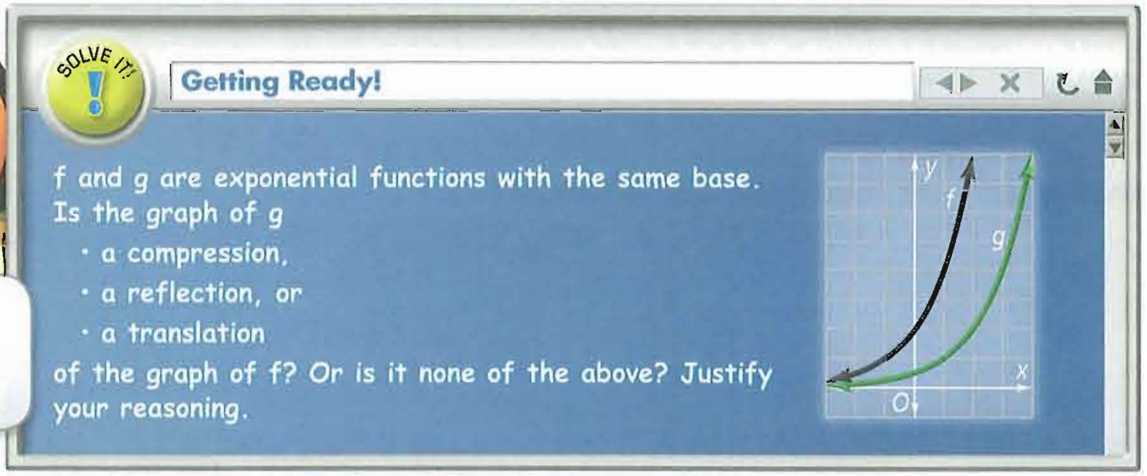
63. $y = 0.5(4)^x$

7-2

Properties of Exponential Functions

Sunshine State Standards
MA.912.A.8.1 Define exponential functions.
MA.912.A.8.3 Graph exponential functions.
MA.912.A.2.10 Describe and graph transformations of functions.

Objectives To explore the properties of functions of the form $y = ab^x$
To graph exponential functions that have base e



Getting Ready!

f and g are exponential functions with the same base. Is the graph of g

- a compression,
- a reflection, or
- a translation

of the graph of f? Or is it none of the above? Justify your reasoning.

Dynamic Activity
Exponential Functions

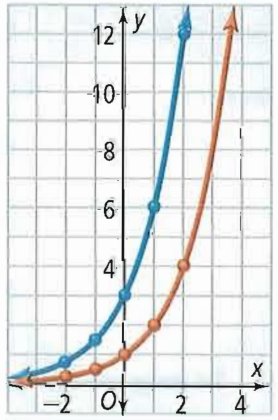
Lesson Vocabulary
- natural base exponential function
- continuously compounded interest

You can apply the four types of transformations—stretches, compressions, reflections, and translations—to exponential functions.

Essential Understanding The factor a in $y = ab^x$ can stretch or compress, and possibly reflect the graph of the parent function $y = b^x$.

The graphs of $y = 2^x$ (in red) and $y = 3 \cdot 2^x$ (in blue) are shown. Each y-value of $y = 3 \cdot 2^x$ is 3 times the corresponding y-value of the parent function $y = 2^x$.

x	$y = 2^x$	$y = 3 \cdot 2^x$
-2	$\frac{1}{4}$	$\frac{3}{4}$
-1	$\frac{1}{2}$	$\frac{3}{2}$
0	1	3
1	2	6
2	4	12



$y = 3 \cdot 2^x$ stretches the graph of the parent function $y = 2^x$ by the factor 3.

Problem 1 Graphing $y = ab^x$

Think

Which x-values should you use to make a table?

Use $x = 0$ and then choose both positive and negative values.

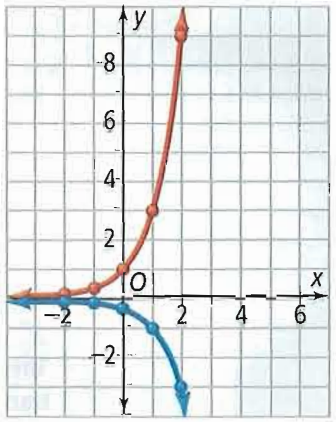
How does the graph of $y = -\frac{1}{3} \cdot 3^x$ compare to the graph of the parent function?

Step 1 Make a table of values.

x	$y = 3^x$	$y = -\frac{1}{3} \cdot 3^x$
-2	$\frac{1}{9}$	$-\frac{1}{27}$
-1	$\frac{1}{3}$	$-\frac{1}{9}$
0	1	$-\frac{1}{3}$
1	3	-1
2	9	-3

Each value is $-\frac{1}{3}$ times the corresponding value of the parent function.

Step 2 Graph the function.



The $-\frac{1}{3}$ in $y = -\frac{1}{3} \cdot 3^x$ reflects the graph of the parent function $y = 3^x$ across the x-axis and compresses it by the factor $\frac{1}{3}$. The domain and asymptote remain unchanged. The y-intercept becomes $-\frac{1}{3}$ and the range becomes $y < 0$.

Got It? **1.** How does the graph of $y = -0.5 \cdot 5^x$ compare to the graph of the parent function?

A horizontal shift $y = ab^{(x-h)}$ is the same as the vertical stretch or compression $y = (ab^{-h})b^x$. A vertical shift $y = ab^x + k$ also shifts the horizontal asymptote from $y = 0$ to $y = k$.

Problem 2 Translating the Parent Function $y = b^x$

Think

How is the graph of $y = 2^{(x-4)}$ different from the graph of $y = 2^x$?

The graph of $y = 2^{(x-4)}$ is a horizontal translation of $y = 2^x$ to the right 4 units.

How does the graph of each function compare to the graph of the parent function?

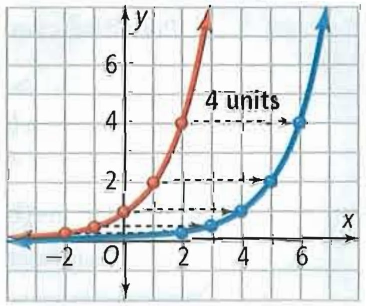
A $y = 2^{(x-4)}$

Step 1 Make a table of values of the parent function $y = 2^x$.

x	$y = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1

x	$y = 2^x$
1	2
2	4
3	8

Step 2 Graph $y = 2^x$ then translate 4 units to the right.



The $(x - 4)$ in $y = 2^{(x-4)}$ translates the graph of $y = 2^x$ to the right 4 units. The asymptote remains $y = 0$. The y-intercept becomes $\frac{1}{16}$.

Think

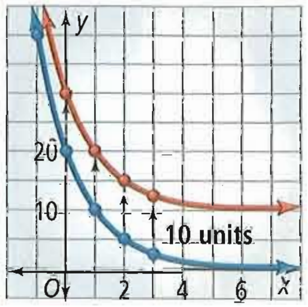
Where have you seen this situation before?
The graph of a function like $y = 20\left(\frac{1}{2}\right)^x + 10$ is both a stretch and a vertical translation of its parent function.

B $y = 20\left(\frac{1}{2}\right)^x + 10$

Step 1 Make a table of values for $y = 20\left(\frac{1}{2}\right)^x$.

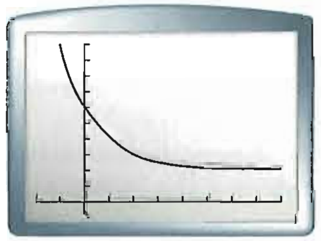
x	$y = 20 \cdot \left(\frac{1}{2}\right)^x$
−1	40
0	20
1	10
2	5
3	2.5

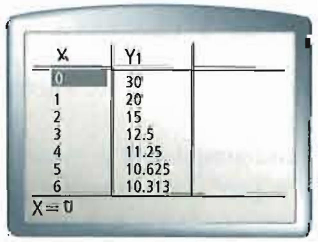
Step 2 Graph $y = 20\left(\frac{1}{2}\right)^x$, then translate 10 units up.



The "+ 10" in $y = 20\left(\frac{1}{2}\right)^x + 10$ translates the graph of $y = 20\left(\frac{1}{2}\right)^x$ up 10 units. It also translates the asymptote, the y-intercept, and the range 10 units up. The asymptote becomes $y = 10$, the y-intercept becomes 30, and the range becomes $y > 10$. The domain is unchanged.

Check Use a graphing calculator to graph $y = 20\left(\frac{1}{2}\right)^x + 10$.





```
X     Y1
0     30
1     20
2     15
3     12.5
4     11.25
5     10.625
6     10.313
X=0
```

Got It? **2.** How does the graph of each function compare to the graph of the parent function?

a. $y = 4^{(x+2)}$

b. $y = 5 \cdot 0.25^x + 5$

take note

Concept Summary Families of Exponential Functions

Parent function	$y = b^x$
Stretch ($\lvert a \rvert > 1$) Compression (Shrink) ($0 < \lvert a \rvert < 1$) Reflection ($a < 0$) in x-axis	$y = ab^x$
Translations (horizontal by h; vertical by k)	$y = b^{(x-h)} + k$
All transformations combined	$y = ab^{(x-h)} + k$

Problem 3 Using an Exponential Model

Physics **The best temperature to brew coffee is between 195°F and 205°F. Coffee is cool enough to drink at 185°F. The table shows temperature readings from a sample cup of coffee. How long does it take for a cup of coffee to be cool enough to drink? Use an exponential model.**

Time (min)	Temp (°F)
0	203
5	177
10	153
15	137
20	121
25	111
30	104

Know
- Set of values
- Best serving temperature

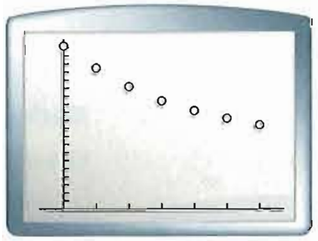
Need
Time it takes for a cup of coffee to become cool enough to drink

Plan
Use an exponential model to find the time it takes for coffee to reach 185°F.

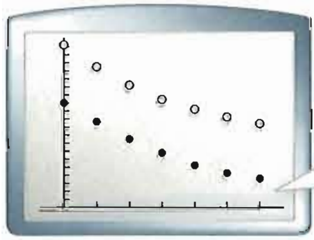
Think

Why does it make sense that a graph of this data would have an asymptote?
The temperature of the hot coffee will get closer and closer to room temperature as it cools, but it cannot cool below room temperature.

Step 1
Plot the data to determine if an exponential model is realistic.

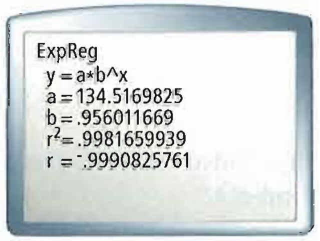


Step 2
The graphing calculator exponential model assumes the asymptote is $y = 0$. Since room temperature is about 68°F, subtract 68 from each temperature value. Calculate the third list by letting **L3 = L2 − 68**.



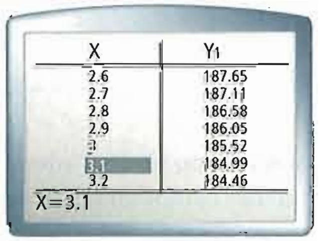
The graphing calculator exponential model assumes the asymptote is $y = 0$.

Step 3
Use the **ExpReg L1**, **L3** function on the transformed data to find an exponential model.



```
ExpReg
 y=a*b^x
 a=134.5169825
 b=.956011669
 r²=.9981659939
 r =⁻.9990825761
```

Step 4
Translate $y = 134.5(0.956)^x$ vertically by 68 units to model the original data. Use the model $y = 134.5 \cdot 0.956^x + 68$ to find how long it takes the coffee to cool to 185°F.



```
   X     |    Y1
   2.6   |  187.65
   2.7   |  187.11
   2.8   |  186.58
   2.9   |  186.05
   3     |  185.52
   3.1   |  184.99
   3.2   |  184.46
X=3.1
```

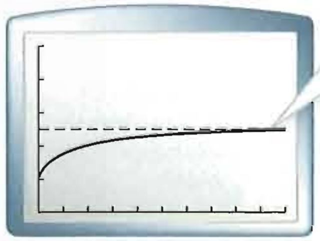
The coffee takes about 3.1 min to cool to 185°F.

Got It? **3. a.** Use the exponential model. How long does it take for the coffee to reach a temperature of 100 degrees?

b. **Reasoning** In Problem 3, would the model of the exponential data be useful if you did not translate the data by 68 units? Explain.

Up to this point you have worked with rational bases. However, exponential functions can have irrational bases as well. One important irrational base is the number e. The graph of $y = \left(1 + \frac{1}{x}\right)^x$ has an asymptote at $y = e$ or $y \approx 2.71828$.

x	$y = \left(1 + \frac{1}{x}\right)^x$
1	$y = 2$
10	$y \approx 2.594$
100	$y \approx 2.70$
1000	$y \approx 2.717$



As x approaches infinity the graph approaches the value of e.

Natural base exponential functions are exponential functions with base e. These functions are useful for describing continuous growth or decay. Exponential functions with base e have the same properties as other exponential functions.

Problem 4 Evaluating e^x

How can you use a graphing calculator to evaluate e^3?

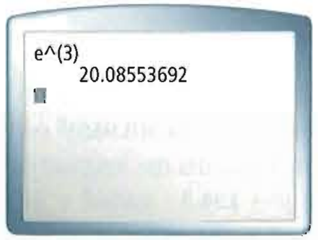
Think

After you press the e^x key, what keys should you press?

Press 3,), and enter.

Method 1

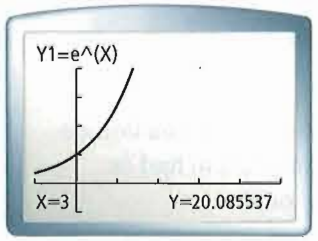
Use the e^x key.



```
e^(3)
    20.08553692
```

Method 2

Use the graph of $y = e^x$.

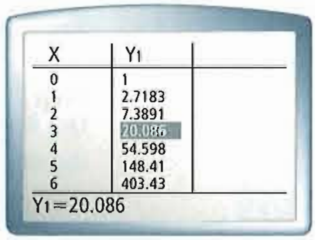


```
Y1=e^(X)

X=3            Y=20.085537
```

Method 3

Use a table of values for $y = e^x$.



```
X    | Y1     |
0    | 1
1    | 2.7183
2    | 7.3891
3    | 20.086
4    | 54.598
5    | 148.41
6    | 403.43
Y1=20.086
```

$e^3 \approx 20.086$

Got It? **4.** How can you use a graphing calculator to calculate e^8?

In Lesson 7-1 you studied interest that was compounded annually. The formula for continuously compounded interest uses the number e.

take note

Key Concept Continuously Compounded Interest

amount in account at time t

interest rate (annual)

$$A(t) = P \cdot e^{rt}$$

Principal

time in years

Problem 5 Continuously Compounded Interest

GRIDDED RESPONSE

Plan

What is the unknown?
The amount A in the account after 4 years.

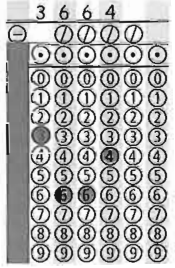
Scholarships Suppose you won a contest at the start of 5th grade that deposited $3000 in an account that pays 5% annual interest compounded continuously. How much will you have in the account when you enter high school 4 years later? Express the answer to the nearest dollar.

$$A = P \cdot e^{rt}$$

$= 3000e^{(0.05)(4)}$ Substitute values for P, r, and t.

$= 3000e^{0.2}$ Simplify.

≈ 3664 Use a calculator. Round to the nearest dollar.



The amount in the account, to the nearest dollar, is $3664.
Write your answer, 3664 in the grid.

Got It? **5.** About how much will be in the account after 4 years of high school?

Lesson Check

Do you know HOW?

For each function, identify the transformation from the parent function $y = b^x$.

1. $y = -2 \cdot 3^x$

2. $y = \frac{1}{2}(9)^x$

3. $y = 7^{(x-5)}$

4. $y = 5^x + 3$

Do you UNDERSTAND?

5. Vocabulary Is $y = e^{(x+7)}$ a natural base exponential function?

6. Reasoning Is investing $2000 in an account that pays 5% annual interest compounded continuously the same as investing $1000 at 4% and $1000 at 6%, each compounded continuously? Explain.

Practice and Problem-Solving Exercises

A Practice

Graph each function.

See Problem 1.

7. $y = -5^x$

8. $y = \left(\frac{1}{2}\right)^x$

9. $y = 2(4)^x$

10. $y = -9(3)^x$

11. $y = 3(2)^x$

12. $y = 24\left(\frac{1}{2}\right)^x$

13. $y = -4^x$

14. $y = -\left(\frac{1}{3}\right)^x$

15. $y = 2\left(\frac{3}{2}\right)^x$

Graph each function as a transformation of its parent function.

See Problem 2.

16. $y = 2^x + 5$

17. $y = 5\left(\frac{1}{3}\right)^x - 8$

18. $y = -(0.3)^{x-2}$

19. $y = -2(5)^{x+3}$

20. $y = 3(2)^{x-1} + 4$

21. $y = -2(3)^{x+1} - 5$

See Problem 3.

22. **Baking** A cake recipe says to bake the cake until the center is 180°F, then let the cake cool to 120°F. The table shows temperature readings for the cake.

a. Given a room temperature of 70°F, what is an exponential model for this data set?

b. How long does it take the cake to cool to the desired temperature?

Time (min)	Temp (°F)
0	180
5	126
10	94
15	80
20	73

See Problem 4.

Graphing Calculator **Use the graph of $y = e^x$ to evaluate each expression to four decimal places.**

23. e^6

24. e^{-2}

25. e^0

26. $e^{\frac{5}{2}}$

27. e^e

See Problem 5.

Find the amount in a continuously compounded account for the given conditions.

28. principal: \$2000
annual interest rate: 5.1%
time: 3 years

29. principal: \$400
annual interest rate: 7.6%
time: 1.5 years

30. principal: \$950
annual interest rate: 6.5%
time: 10 years

B Apply

31. **Think About a Plan** A student wants to save \$8000 for college in five years. How much should be put into an account that pays 5.2% annual interest compounded continuously?

- What formula should you use?
- What information do you know?
- What do you need to find?

32. **Investment** How long would it take to double your principal in an account that pays 6.5% annual interest compounded continuously?

33. **Error Analysis** A student says that the graph of $f(x) = \left(\frac{1}{3}\right)^{x+2} + 1$ is a shift of the parent function 2 units up and 1 unit to the left. Describe and correct the student's error.

34. Assume that a is positive and $b \geq 1$. Describe the effects of $c > 0$, $c = 0$, and $c < 0$ on the graph of the function $y = ab^{cx}$.

35. **Graphing Calculator** Using a graphing calculator, graph each of the functions below on the same coordinate grid. What do you notice? Explain why the definition of exponential functions has the constraint that $b \neq 1$.

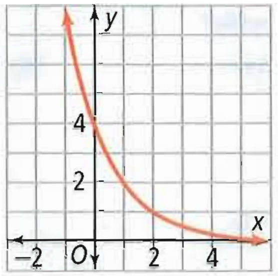
$y = \left(\frac{1}{2}\right)^x$ $y = \left(\frac{8}{10}\right)^x$ $y = \left(\frac{9}{10}\right)^x$ $y = \left(\frac{99}{100}\right)^x$

36. **Botany** The half-life of a radioactive substance is the time it takes for half of the material to decay. Phosphorus-32 is used to study a plant's use of fertilizer. It has a half-life of 14.3 days. Write the exponential decay function for a 50-mg sample. Find the amount of phosphorus-32 remaining after 84 days.

37. **Archaeology** Archaeologists use carbon-14, which has a half-life of 5730 years, to determine the age of artifacts in carbon dating. Write the exponential decay function for a 24-mg sample. How much carbon-14 remains after 30 millennia? (*Hint:* 1 millennium = 1000 years)

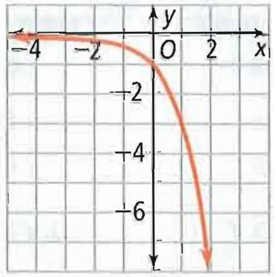
The parent function for each graph below is of the form $y = ab^x$. Write the parent function. Then write a function for the translation indicated.

38.



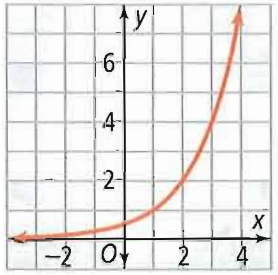
translation: left 4 units, up 3 units

39.



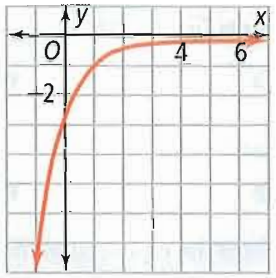
translation: right 8 units, up 2 units

40.



translation: right 6 units, down 7 units

41.



translation: left 15 units, down 1 unit

42. Physics At a constant temperature, the atmospheric pressure p in pascals is given by the formula $p = 101.3e^{-0.001h}$, where h is the altitude in meters. What is p at an altitude of 500 m?

Challenge

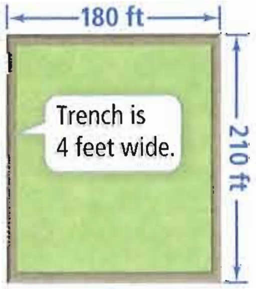
43. Psychology Psychologists use an exponential model of the learning process, $f(t) = c(1 - e^{-kt})$, where c is the total number of tasks to be learned, k is the rate of learning, t is time, and $f(t)$ is the number of tasks learned.

a. Suppose you move to a new school, and you want to learn the names of 25 classmates in your homeroom. If your learning rate for new tasks is 20% per day, how many complete names will you know after 2 days? After 8 days?

b. Graphing Calculator Graph the function on your graphing calculator. How many days will it take to learn everyone's name? Explain.

c. Open-Ended Does this function seem to describe your own learning rate? If not, how could you adapt it to reflect your learning rate?

44. Landscaping A homeowner is planting hedges and begins to dig a 3-ft-deep trench around the perimeter of his property. After the first weekend, the homeowner recruits a friend to help. After every succeeding weekend, each digger recruits another friend. One person can dig 405 ft^3 of dirt per weekend. The figure at the right shows the dimensions of the property and the width of the trench.



a. Geometry Determine the volume of dirt that must be removed for the trench.

b. Write an exponential function to model the volume of dirt remaining to be shoveled after x weekends. Then, use the model to determine how many weekends it will take to complete the trench.

Sunshine State Standards Practice

MA.912.A.8.7

45. A savings account earns 4.62% annual interest, compounded continuously. After approximately how many years will a principal of $500 double?

Ⓐ 2 years　Ⓑ 10 years　Ⓒ 15 years　Ⓓ 44 years

MA.912.A.2.11

46. What is the inverse of the function $f(x) = \sqrt{x - 4}$?

Ⓕ $f^{-1}(x) = x^2 - 4, x \geq 0$　Ⓗ $f^{-1}(x) = \sqrt{x + 4}$

Ⓖ $f^{-1}(x) = x^2 + 4, x \geq 0$　Ⓘ $f^{-1}(x) = \frac{\sqrt{x - 4}}{x - 4}$

MA.912.A.2.8

In Exercises 47 and 48, let $f(x) = x^2 - 4$ and $g(x) = \frac{1}{x + 4}$.

47. What is $(g \circ f)(x)$?

Ⓐ $\frac{1}{x^2}$　Ⓑ $\frac{1}{x^2 - 8x + 16} - 4$　Ⓒ $\frac{x^2 - 4}{x + 4}$　Ⓓ $x - 4$

48. What is $(f \circ f)(3)$?

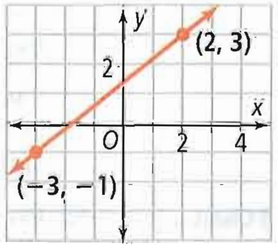
Ⓕ 1　Ⓖ 5　Ⓗ 21　Ⓘ 77

MA.912.A.3.10

49. What is the equation of the line shown at the right?

Ⓐ $y = -\frac{4}{5}x + 2$　Ⓒ $-4x + 5y = 7$

Ⓑ $y = \frac{5}{4}x - 2$　Ⓓ $4x - 5y = 15$



MA.912.A.8.7

50. Short Response How much should you invest in an account that pays 6% annual interest compounded continuously if you want exactly $8000 after four years? Show your work.

Mixed Review

Without graphing, determine whether the function represents exponential growth or exponential decay. Then find the *y*-intercept.　See Lesson 7-1.

51. $y = 23(3.03)^x$　**52.** $f(x) = 3(5)^x$

53. $y = 2\left(\frac{3}{4}\right)^x$　**54.** $y = 5\left(\frac{8}{3}\right)^x$

Simplify.　See Lesson 6-3.

55. $5\sqrt{5} + \sqrt{5}$　**56.** $\sqrt[3]{4} - 2\sqrt[3]{4}$　**57.** $\sqrt{75} + \sqrt{125}$

58. $\sqrt[4]{32} + \sqrt[4]{128}$　**59.** $5\sqrt{3} - 2\sqrt{12}$　**60.** $3\sqrt{63} + \sqrt{28}$

Get Ready! **To prepare for Lesson 7-3, do Exercises 61–63.**

Find the inverse of each function. Is the inverse a function?　See Lesson 6-7.

61. $f(x) = 4x - 1$　**62.** $f(x) = x^7$　**63.** $f(x) = 5x^3 + 1$

7-3

Logarithmic Functions as Inverses

Sunshine State Standards

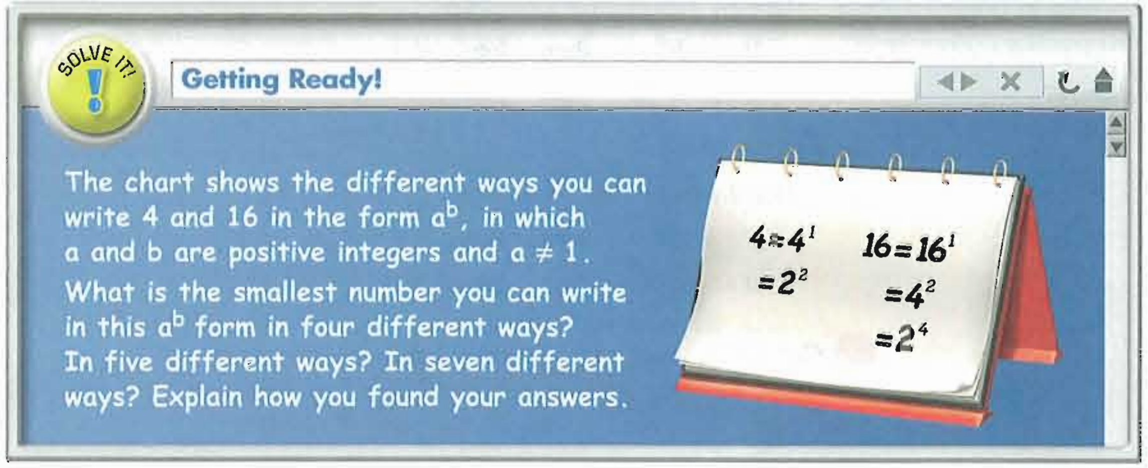
MA.912.A.8.1 Define exponential and logarithmic functions and determine their relationship.

MA.912.A.8.3 Graph logarithmic functions.

MA.912.A.2.11 Solve problems involving functions and their inverses.

Objectives To write and evaluate logarithmic expressions

To graph logarithmic functions



The chart shows the different ways you can write 4 and 16 in the form a^b, in which a and b are positive integers and $a \neq 1$. What is the smallest number you can write in this a^b form in four different ways? In five different ways? In seven different ways? Explain how you found your answers.

Dynamic Activity
Logarithmic Functions

Lesson Vocabulary
- logarithm
- logarithmic function
- common logarithm
- logarithmic scale

Many even numbers can be written as power functions with base 2. In this lesson you will find ways to express all numbers as powers of a common base.

Essential Understanding The exponential function $y = b^x$ is one-to-one, so its inverse $x = b^y$ is a function. To express "y as a function of x" for the inverse, write $y = \log_b x$.

take note

Key Concept Logarithm

A **logarithm** base b of a positive number x satisfies the following definition.

For $b > 0$, $b \neq 1$, $\log_b x = y$ if and only if $b^y = x$.

You can read $\log_b x$ as "log base b of x." In other words, the logarithm y is the exponent to which b must be raised to get x.

The exponent y in the expression b^y is the logarithm in the equation $\log_b x = y$. The base b in b^y and the base b in $\log_b x$ are the same. In both, $b \neq 1$ and $b > 0$.

Since $b \neq 1$ and $b > 0$, it follows that $b^y > 0$. Since $b^y = x$ then $x > 0$, so $\log_b x$ is defined only for $x > 0$.

Because $y = b^x$ and $y = \log_b x$ are inverse functions, their compositions map a number a to itself. In other words, $b^{\log_b a} = a$ for $a > 0$ and $\log_b b^a = a$ for all a.

You can use the definition of a logarithm to write exponential equations in logarithmic form.

Problem 1 Writing Exponential Equations in Logarithmic Form

Think

To what power do you raise 10 to get 100?
10 raised to the 2nd power equals 100.

What is the logarithmic form of each equation?

A $100 = 10^2$

Use the definition of logarithm.

If $x = b^y$ then $\log_b x = y$

If $100 = 10^2$ then $\log_{10} 100 = 2$

B $81 = 3^4$

Use the definition of logarithm.

If $x = b^y$ then $\log_b x = y$

If $81 = 3^4$ then $\log_3 81 = 4$

Got It? **1.** What is the logarithmic form of each equation?

a. $36 = 6^2$ **b.** $\frac{8}{27} = \left(\frac{2}{3}\right)^3$ **c.** $1 = 3^0$

You can use the exponential form to help you evaluate logarithms.

Problem 2 Evaluating a Logarithm

Plan

How can you use the definition of logarithm to help you find the value of $\log_8 32$?
If $\log_b x = y$ then $x = b^y$, so to what power must you raise 8 to get 32?

Multiple Choice **What is the value of $\log_8 32$?**

Ⓐ $\frac{3}{5}$ Ⓑ $\frac{5}{3}$ Ⓒ 3 Ⓓ 5

$\log_8 32 = x$	Write a logarithmic equation.
$32 = 8^x$	Use the definition of a logarithm to write an exponential equation.
$2^5 = (2^3)^x$	Write each side using base 2.
$2^5 = 2^{3x}$	Power Property of Exponents
$5 = 3x$	Since the bases are the same, the exponents must be equal.
$\frac{5}{3} = x$	Solve for x.

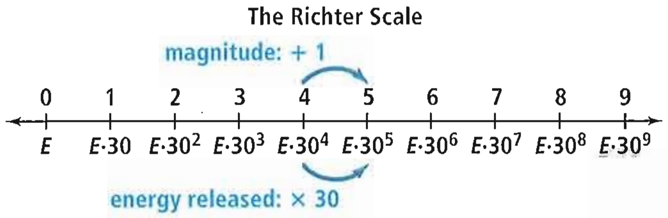
Since $8^{\frac{5}{3}} = 32$, then $\log_8 32 = \frac{5}{3}$.
The correct answer is B.

Got It? **2.** What is the value of each logarithm?

a. $\log_5 125$ **b.** $\log_4 32$ **c.** $\log_{64} \frac{1}{32}$

A **common logarithm** is a logarithm with base 10. You can write a common logarithm $\log_{10}x$ simply as $\log x$, without showing the 10.

Many measurements of physical phenomena have such a wide range of values that the reported measurements are logarithms (exponents) of the values, not the values themselves. When you use the logarithm of a quantity instead of the quantity, you are using a **logarithmic scale**. The Richter scale is a logarithmic scale. It gives logarithmic measurements of earthquake magnitude.



Problem 3 Using a Logarithmic Scale

In December 2004, an earthquake with magnitude 9.3 on the Richter scale hit off the northwest coast of Sumatra. The diagram shows the magnitude of an earthquake that hit Sumatra in March 2005. The formula $\log\frac{I_1}{I_2} = M_1 - M_2$ compares the intensity levels of earthquakes where I is the intensity level determined by a seismograph, and M is the magnitude on a Richter scale. How many times more intense was the December earthquake than the March earthquake?

Think

What is the base of this logarithm?
This is the common logarithm. It has base 10.

$\log\frac{I_1}{I_2} = M_1 - M_2$ Use the formula.

$\log\frac{I_1}{I_2} = 9.3 - 8.7$ Substitute $M_1 = 9.3$ and $M_2 = 8.7$.

$\log\frac{I_1}{I_2} = 0.6$ Simplify.

$\frac{I_1}{I_2} = 10^{0.6}$ Apply the definition of common logarithm.

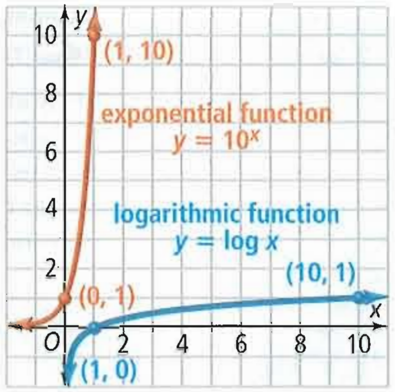
≈ 4 Use a calculator.

The December earthquake was about 4 times as strong as the one in March.

Got It? **3.** In 1995, an earthquake in Mexico registered 8.0 on the Richter scale. In 2001, an earthquake of magnitude 6.8 shook Washington state. How many times more intense was the 1995 earthquake than the 2001 earthquake?

A **logarithmic function** is the inverse of an exponential function. The graph shows $y = 10^x$ and its inverse $y = \log x$. Note that (0, 1) and (1, 10) are on the graph of $y = 10^x$, and that (1, 0) and (10, 1) are on the graph of $y = \log x$.

Recall that the graphs of inverse functions are reflections of each other across the line $y = x$. You can graph $y = \log_b x$ as the inverse of $y = b^x$.

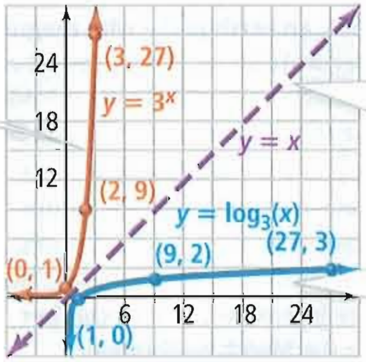


Problem 4 Graphing a Logarithmic Function

What is the graph of $y = \log_3 x$? Describe the domain and range and identify the y-intercept and the asymptote.

$y = \log_3 x$ is the inverse of $y = 3^x$.

Step 1 Graph $y = 3^x$.



Step 2 Reflecting across the line $y = x$ produces the inverse of $y = 3^x$.

Step 3 Choose a few points on $y = 3^x$ and reverse their coordinates. Plot these new points and graph $y = \log_3 x$.

Think

How are the domain and range of $y = 3^x$ and $y = \log_3 x$ related?
Since they are inverse functions, the domain and range of $y = \log_3 x$ are the same as the *range* and *domain* of $y = 3^x$.

The domain is $x > 0$. The range is all real numbers. There is no y-intercept. The vertical asymptote is $x = 0$.

Got It? **4. a.** What is the graph of $y = \log_4 x$? Describe the domain, range, y-intercept and asymptotes.

b. Reasoning Suppose you use the following table to help you graph $y = \log_2 x$. (Recall that if $y = \log_2 x$, then $2^y = x$.) Copy and complete the table. Explain your answers.

x	$2^y = x$	y
−1	$2^y = -1$	
0	$2^y = 0$	
1	$2^y = 1$	
2	$2^y = 2$	

The function $y = \log_b x$ is the parent for a function family. You can graph $y = \log_b (x - h) + k$ by translating the graph of the parent function, $y = \log_b x$, horizontally by h units and vertically by k units. The a in $y = a \log_b x$ indicates a stretch, a compression, and possibly a reflection.

take note Concept Summary Families of Logarithmic Functions

Parent functions:	$y = \log_b x, b > 0, b \neq 1$
Stretch ($\lvert a \rvert > 1$) Compression (Shrink) ($0 < \lvert a \rvert < 1$) Reflection ($a < 0$) in x-axis	$y = a \log_b x$
Translations (horizontal by h; vertical by k)	$y = \log_b (x - h) + k$
All transformations together	$y = a \log_b (x - h) + k$

Problem 5 Translating $y = \log_b x$

How does the graph of $y = \log_4 (x - 3) + 4$ compare to the graph of the parent function?

Think

How is the function $y = \log_4(x - 3) + 4$ similar to other functions you have seen?
Recall that the graph of $y = f(x - h) + k$ is a vertical and horizontal translation of the parent function, $y = f(x)$.

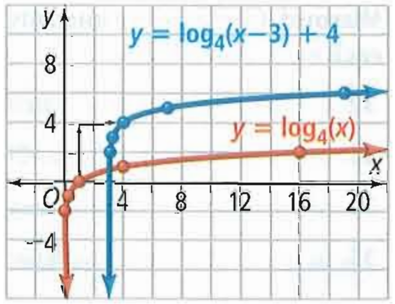
Step 1

Make a table of values for the parent function. Use the definition of logarithm.

x	$\log_4 x = y \longrightarrow 4^y = x$	y
$\frac{1}{16}$	$4^{-2} = \frac{1}{16}$	-2
$\frac{1}{4}$	$4^{-1} = \frac{1}{4}$	-1
1	$4^0 = 1$	0
4	$4^1 = 4$	1
16	$4^2 = 16$	2

Step 2

Graph the parent function. Shift the graph to the right 3 units and up 4 units to graph $y = \log_4 (x - 3) + 4$.



Because $y = \log_4 (x - 3) + 4$ translates the graph of the parent function 3 units to the right, the asymptote changes from $x = 0$ to $x = 3$. The domain changes from $x > 0$ to $x > 3$. The range remains all real numbers.

Got It? **5.** How does the graph of each function compare to the graph of the parent function?

a. $y = \log_2(x - 3) + 4$ **b.** $y = 5 \log_2 x$

Lesson Check

Do you know HOW?

Write each equation in logarithmic form.

1. $25 = 5^2$

2. $64 = 4^3$

3. $243 = 3^5$

4. $25 = 5^2$

Evaluate each logarithm.

5. $\log_2 8$

6. $\log_9 9$

7. $\log_7 49$

8. $\log_2 \frac{1}{4}$

Do you UNDERSTAND?

9. Vocabulary Determine whether each logarithm is a common logarithm.

a. $\log_2 4$ **b.** $\log 64$ **c.** $\log_{10} 100$ **d.** $\log_5 5$

10. Reasoning Explain how you could use an inverse function to graph the logarithmic function $y = \log_6 x$.

11. Compare and Contrast Compare the graph of $y = \log_2 (x + 4)$ to the graph of $y = \log_2 x$. How are the graphs alike? How are they different?

Practice and Problem-Solving Exercises

A Practice

Write each equation in logarithmic form.

See Problem 1.

12. $49 = 7^2$

13. $10^3 = 1000$

14. $625 = 5^4$

15. $\frac{1}{10} = 10^{-1}$

16. $8^2 = 64$

17. $4 = \left(\frac{1}{2}\right)^{-2}$

18. $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

19. $10^{-2} = 0.01$

Evaluate each logarithm.

See Problem 2.

20. $\log_2 16$

21. $\log_4 2$

22. $\log_8 8$

23. $\log_4 8$

24. $\log_2 8$

25. $\log_{49} 7$

26. $\log_5 (-25)$

27. $\log_3 9$

28. $\log_2 2^5$

29. $\log_{\frac{1}{2}} \frac{1}{2}$

30. $\log 10{,}000$

31. $\log_5 125$

Seismology **In 1812, an earthquake of magnitude 7.9 shook New Madrid, Missouri. Compare the intensity level of that earthquake to the intensity level of each earthquake below.**

See Problem 3.

32. magnitude 7.7 in San Francisco, California, in 1906

33. magnitude 9.5 in Valdivia, Chile, in 1960

34. magnitude 3.2 in Charlottesville, Virginia, in 2001

35. magnitude 6.9 in Kobe, Japan, in 1995

Graph each function on the same set of axes.

See Problem 4.

36. $y = \log_2 x$

37. $y = 2^x$

38. $y = \log_{\frac{1}{2}} x$

39. $y = \left(\frac{1}{2}\right)^x$

Describe how the graph of each function compares with the graph of the parent function, $y = \log_b x$.

See Problem 5.

40. $y = \log_5 x + 1$

41. $y = \log_7 (x - 2)$

42. $y = \log_3 (x - 5) + 3$

43. $y = \log_4 (x + 2) - 1$

B Apply

44. Think About a Plan The pH of a substance equals $-\log[H^+]$, where $[H^+]$ is the concentration of hydrogen ions, and it ranges from 0 to 14. A pH level of 7 is neutral. A level greater than 7 is basic, and a level less than 7 is acidic. The table shows the hydrogen ion concentration $[H^+]$ for selected foods. Is each food basic or acidic?

- How can you find the pH value of each food?
- What rule can you use to determine if the food is basic or acidic?

Approximate [H+] of Foods

Food	$[H^+]$
Apple juice	3.2×10^{-4}
Buttermilk	2.5×10^{-5}
Cream	2.5×10^{-7}
Ketchup	1.3×10^{-4}
Shrimp sauce	7.9×10^{-8}
Strained peas	1.0×10^{-6}

45. Chemistry Find the concentration of hydrogen ions in seawater, if the pH level of seawater is 8.5.

Write each equation in exponential form.

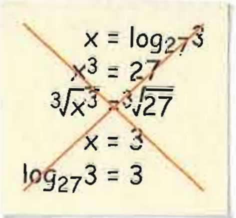
46. $\log_2 128 = 7$ **47.** $\log 0.0001 = -4$ **48.** $\log_6 6 = 1$ **49.** $\log_4 1 = 0$

50. $\log_7 16{,}807 = 5$ **51.** $\log_2 \frac{1}{2} = -1$ **52.** $\log_3 \frac{1}{9} = -2$ **53.** $\log 10 = 1$

Find the greatest integer that is less than the value of the logarithm. Use your calculator to check your answers.

54. $\log 5$ **55.** $\log 0.08$ **56.** $\log 17.52$ **57.** $\log(1.3 \times 10^7)$

58. Error Analysis Find the error in the following evaluation of $\log_{27} 3$. Then evaluate the logarithm correctly.



$x = \log_{27} 3$

$x^3 = 27$

$\sqrt[3]{x^3} = \sqrt[3]{27}$

$x = 3$

$\log_{27} 3 = 3$

59. Writing Explain why the base b in $y = \log_b x$ cannot equal 1.

60. Open-Ended Write a logarithmic function of the form $y = \log_b x$. Find its inverse function. Graph both functions on one set of axes.

Find the inverse of each function.

61. $y = \log_4 x$ **62.** $y = \log_{0.5} x$ **63.** $y = \log_{10} x$ **64.** $y = \log_2 2x$

65. $y = \log(x + 1)$ **66.** $y = \log 10x$ **67.** $y = \log_2 4x$ **68.** $y = \log(x - 6)$

Graph each logarithmic function.

69. $y = \log 2x$ **70.** $y = 2\log_2 x$ **71.** $y = \log_4(2x + 3)$ **72.** $y = \log_3(x + 5)$

Find the domain and the range of each function.

73. $y = \log_5 x$ **74.** $y = 3\log x$ **75.** $y = \log_2(x - 3)$ **76.** $y = 2\log(x - 2)$

You can write $5^3 = 125$ in logarithmic form using the fact that $\log_b b^x = x$.

$\log_5(5^3) = \log_5(125)$ Apply the log base 5 to each side.

$3 = \log_5 125$ Use $\log_b b^x = x$ to simplify.

Use this method to write each equation in logarithmic form. Show your work.

77. $3^4 = 81$ **78.** $x^4 = y$ **79.** $6^8 = a + 1$

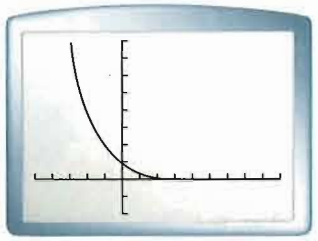
Challenge **Find the least integer greater than each number. Do not use a calculator.**

80. $\log_3 38$ **81.** $\log_{1.5} 2.5$ **82.** $\log_{\sqrt{7}} \sqrt{50}$ **83.** $\log_5 \frac{1}{47}$

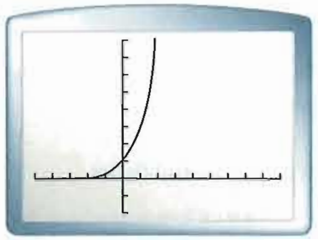
84. Match each function with the graph of its inverse.

a. $y = \log_3 x$ **b.** $y = \log_2 4x$ **c.** $y = \log_{\frac{1}{2}} x$

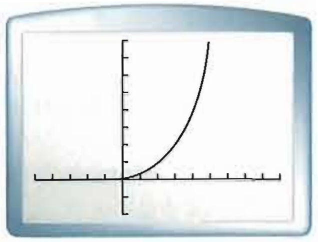
I.



II.



III.



Sunshine State Standards Practice

MA.912.A.8.1

85. Which is the logarithmic form of the exponential equation $2^3 = 8$?

Ⓐ $\log_8 2 = 3$ Ⓑ $\log_8 3 = 2$ Ⓒ $\log_3 8 = 2$ Ⓓ $\log_2 8 = 3$

MA.912.A.2.12

86. Dan will begin advertising his video production business online using a pay-per-click method, which charges \$30 as an initial fee, plus a fixed amount each time the ad is clicked. Dan estimates that with the cost of 8 cents per click, his ad will be clicked about 150 times per day. Which expression represents Dan's total estimated cost of advertising, in dollars, after x days?

Ⓕ $(30 + 0.08x)150$ Ⓖ $360x$ Ⓗ $30 + 1200x$ Ⓘ $30 + 12x$

MA.912.A.2.10

87. Which translation takes $y = |x|$ to $y = |x + 3| - 1$?

Ⓐ 3 units right, 1 unit down Ⓒ 3 units left, 1 unit down

Ⓑ 3 units right, 1 unit up Ⓓ 3 units left, 1 unit up

MA.912.A.6.3.

88. Short Response What is the expression $\sqrt[3]{(\sqrt{a})^7}$ written as a variable raised to a single rational exponent?

Mixed Review

Graph each function. See Lesson 7-2.

89. $y = 5^x - 100$ **90.** $y = -10(4)^{x+2}$ **91.** $y = -27(3)^{x-1} + 9$

Factor each expression. See Lesson 4-4.

92. $4x^2 - 8x + 3$ **93.** $4b^2 - 100$ **94.** $5x^2 + 13x - 6$

Get Ready! **To prepare for Lesson 7-4, do Exercises 95–98.** See Lesson 1-3.

Evaluate each expression for the given value of the variable.

95. $x^2 - x; x = 2$ **96.** $x^3 \cdot x^5; x = 2$ **97.** $\frac{x^8}{x^{10}}; x = 2$ **98.** $x^3 + x^2; x = 2$

Concept Byte

Use With Lesson 7-3

TECHNOLOGY

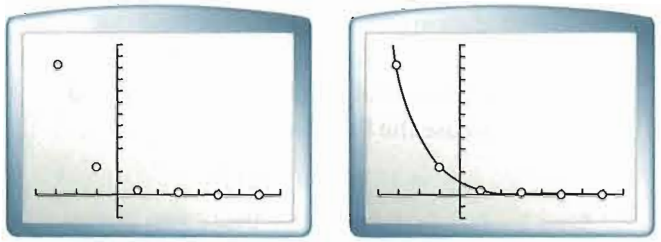
Fitting Curves to Data

Sunshine State Standard
MA.912.A.2.6 Identify linear, quadratic, cubic, logarithmic, and exponential functions.

Example 1

Which type of function models the data best—linear, logarithmic, or exponential?

Connect the points with a smooth curve. Since the points do not fall along a line, the function is not linear. The graph appears to approach a horizontal asymptote, so an exponential function models the data best.



Example 2

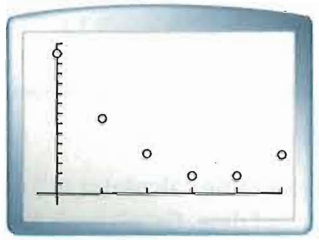
Which type of function models the data best—quadratic, logarithmic, or cubic?

Step 1 Press stat enter to enter the data in lists.

Step 2 Use the stat plot feature to draw a scatter plot.

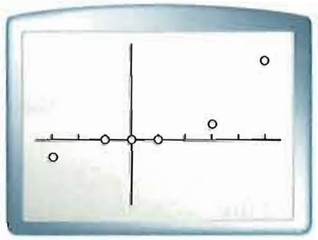
Step 3 If you connect the points with a smooth curve, the end behavior of the graph is up and up. The graph is not cubic or logarithmic, so the quadratic function best models the data.

x	y
0	14
1	7.5
2	4
3	1.8
4	1.8
5	3.9



Exercises

1. Which type of function models the data shown in the graphing calculator screen best—*linear, quadratic, logarithmic, cubic,* or *exponential*?



2. Which type of function models the data in the table best—*linear, quadratic, logarithmic, cubic,* or *exponential*?

x	y
–1	0
1	1.4
3	2.09
5	2.53
7	2.81
9	3.12

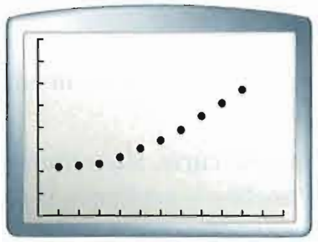
3. Reasoning Could you use a different model for the data in Exercises 1 and 2? Explain.

Example 3

The table shows the number of bacteria in a culture after the given number of hours. Find a good model for the data. Based on the model, how many bacteria will be in the culture after an additional ten hours?

Hour	Bacteria
1	2205
2	2270
3	2350
4	2653
5	3052
6	3417
7	3890
8	4522
9	5107
10	5724

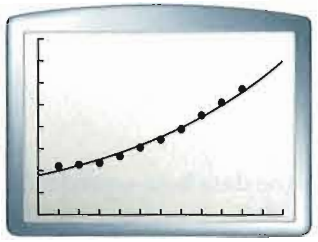
Step 1 Press stat enter to enter the data in lists. Use the stat plot feature to draw a scatter plot.



Step 2 Notice from the scatter plot that the data appears exponential. Find the equations for the best-fitting exponential function. Press stat ▷ **0** to use the **ExpReg** feature.

$$y = 1779.404(1.121)^x$$

Step 3 Graph the function. Press y= clear vars 5 ▷ ▷ enter to enter the **ExpReg** results. Press graph to display the function and the scatter plot together. Press zoom 9 to automatically adjust the window.



Step 4 In 10 more hours, there will be approximately $y = 1779.404(1.121)^{20} \approx 17{,}474$ bacteria in the culture.

Exercises

Use a graphing calculator to find the exponential or quadratic function that best fits each set of data. Graph each function.

4.

x	y
−1	4.9
0	3.8
1	5.0
2	8.1
3	13.3
4	70.2

5.

x	y
−3	0.1
−1	0.4
1	1.6
3	6.4
5	25.6
7	102.4

6.

x	y
1	3.5
2	2.11
3	1.30
4	0.73
5	0.28
6	0.08

7.

x	y
−1	0.04
0	0.1
1	0.5
2	2.5
3	12.5
4	62.5

8. **Writing** In Exercise 6 the function appears to level off. Explain why.

9. Find a quadratic model for the data in Exercise 6. Compare the graphs of the quadratic and exponential models for this data. Predict the y-values for both models when $x = -2$. Discuss the differences if any between the predictions.

7 Mid-Chapter Quiz

Do you know HOW?

Determine whether each function is an example of exponential growth or decay. Then find the *y*-intercept.

1. $y = 100(0.25)^x$ **2.** $y = 0.6\left(\frac{1}{10}\right)^x$ **3.** $y = \frac{7}{8}(18)^x$

Graph each function. Then find the domain, range, and *y*-intercept.

4. $y = -4(2)^x$ **5.** $y = \frac{1}{4}(10)^x$ **6.** $y = 8(0.25)^x$

7. Investment Suppose you deposit $600 into a savings account that pays 3.9% annual interest. How much will you have in the account after 3 years if no money is added or withdrawn?

8. Depreciation The initial value of a car is $25,000. After one year, the value of the car is $21,250. Write an exponential function to model the expected value of the car. Estimate the value of the car after 5 years.

Graph each function as a transformation of its parent function. Write the parent function.

9. $y = 3^x - 2$ **10.** $y = \frac{1}{2}(5)^{x-1} + 4$

11. $y = -(0.5)^{x+3}$ **12.** $y = -6\left(\frac{3}{4}\right)^x - 10$

Evaluate each expression to four decimal places.

13. e^5 **14.** $e^{\frac{3}{2}}$ **15.** e^{-4}

Find the amount in a continuously compounded account for the given conditions.

16. principal: $500; annual interest rate: 4.9%; time: 2.5 years

17. principal: $6000; annual interest rate: 6.8%; time: 10 years

Write each equation in logarithmic form.

18. $10^4 = 10{,}000$ **19.** $\frac{1}{4} = 4^{-1}$ **20.** $8 = \left(\frac{1}{2}\right)^{-3}$

Evaluate each logarithm.

21. $\log_8 64$ **22.** $\log_4(256)$ **23.** $\log_{\frac{1}{5}} 625$

Graph each logarithmic function. Find the domain and range.

24. $y = \log_5(x - 1)$ **25.** $y = 4\log x + 5$

26. Crafts For glass to be shaped, its temperature must stay above 1200°F. The temperature of a piece of glass is 2200°F when it comes out of the furnace. The table shows temperature readings for the glass. Write an exponential model for this data set and then find how long it takes for the piece of glass to cool to 1200°F.

Time (min)	Temp (°F)
0	2200
5	1700
10	1275
15	1000
20	850
25	650

Do you UNDERSTAND?

27. Error Analysis A student claims the *y*-intercept of the graph of the function $y = ab^x$ is the point $(0, b)$. What is the student's mistake? What is the actual *y*-intercept?

28. Writing Without graphing, how can you tell whether an exponential function represents exponential growth or exponential decay?

29. Compare and Contrast Compare the graph of $y = \log_3(x + 1)$ to the graph of its inverse $y = 3^x - 1$. How are the graphs alike? How are they different?

30. Vocabulary Explain how the continuously compounded interest formula differs from the annually compounded interest formula.

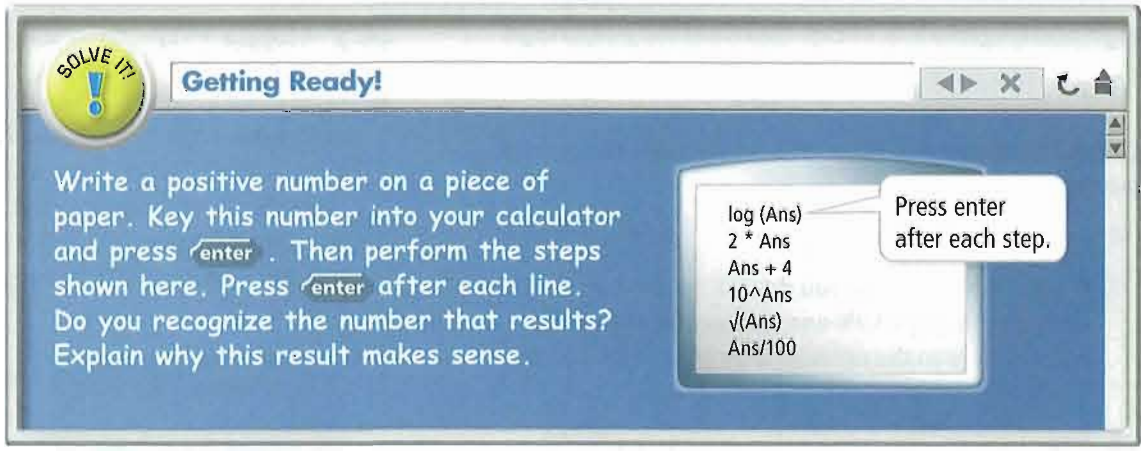
7-4 Properties of Logarithms

Sunshine State Standards

MA.912.A.8.2 Define and use the properties of logarithms to simplify logarithmic expressions and to find their approximate values.

MA.912.A.8.6 Use the change of base formula.

Objective To use the properties of logarithms



SOLVE IT! **Getting Ready!**

Write a positive number on a piece of paper. Key this number into your calculator and press enter. Then perform the steps shown here. Press enter after each line. Do you recognize the number that results? Explain why this result makes sense.

```
log (Ans)
2 * Ans
Ans + 4
10^Ans
√(Ans)
Ans/100
```

Press enter after each step.

Lesson Vocabulary
- Change of Base Formula

You can derive the properties of logarithms from the properties of exponents.

Essential Understanding Logarithms and exponents have corresponding properties.

Here's Why It Works You can use a product property of exponents to derive a product property of logarithms.

Let $x = \log_b m$ and $y = \log_b n$.

$m = b^x$ and $n = b^y$	Definition of logarithm
$mn = b^x \cdot b^y$	Write *mn* as a product of powers.
$mn = b^{x+y}$	Product Property of Exponents
$\log_b mn = x + y$	Definition of logarithm
$\log_b mn = \log_b m + \log_b n$	Substitute for *x* and *y*.

take note **Properties** Properties of Logarithms

For any positive numbers m, n, and b where $b \neq 1$, the following properties apply.

Product Property	$\log_b mn = \log_b m + \log_b n$
Quotient Property	$\log_b \frac{m}{n} = \log_b m - \log_b n$
Power Property	$\log_b m^n = n \log_b m$

Problem 1 Simplifying Logarithms

What is each expression written as a single logarithm?

Think

What must you do with the numbers that multiply the logarithms?
Apply the Power Property of Logarithms.

A $\log_4 32 - \log_4 2$

$\log_4 32 - \log_4 2 = \log_4 \frac{32}{2}$ Quotient Property of Logarithms

$= \log_4 16$ Divide.

$= \log_4 4^2$ Write 16 as a power of 4.

$= 2$ Simplify.

B $6\log_2 x + 5\log_2 y$

$6\log_2 x + 5\log_2 y = \log_2 x^6 + \log_2 y^5$ Power Property of Logarithms

$= \log_2 x^6y^5$ Product Property of Logarithms

Got It? **1.** What is each expression written as a single logarithm?

a. $\log_4 5x + \log_4 3x$

b. $2\log_4 6 - \log_4 9$

You can expand a single logarithm to involve the sum or difference of two or more logarithms.

Problem 2 Expanding Logarithms

What is each logarithm expanded?

Think

Can you apply the Power Property of Logarithms first?
No; the fourth power applies only to *x*.

A $\log \frac{4x}{y}$

$\log \frac{4x}{y} = \log 4x - \log y$ Quotient Property of Logarithms

$= \log 4 + \log x - \log y$ Product Property of Logarithms

B $\log_9 \frac{x^4}{729}$

$\log_9 \frac{x^4}{729} = \log_9 x^4 - \log_9 729$ Quotient Property of Logarithms

$= 4\log_9 x - \log_9 729$ Power Property of Logarithms

$= 4\log_9 x - \log_9 9^3$ Write 729 as a power of 9.

$= 4\log_9 x - 3$ Simplify.

Got It? **2.** What is each logarithm expanded?

a. $\log_3 \frac{250}{37}$

b. $\log_3 9x^5$

You have seen logarithms with many bases. The log key on a calculator finds $\log_{10}$ of a number. To evaluate a logarithm with any base, use the **Change of Base Formula**.

take note

Property Change of Base Formula

For any positive numbers m, b, and c, with $b \neq 1$ and $c \neq 1$,

$$\log_b m = \frac{\log_c m}{\log_c b}.$$

Here's Why It Works

$$\log_b m = \frac{(\log_b m)(\log_c b)}{\log_c b} \qquad \text{Multiply } \log_b m \text{ by } \frac{\log_c b}{\log_c b} = 1.$$

$$= \frac{\log_c b^{\log_b m}}{\log_c b} \qquad \text{Power Property of Logarithms}$$

$$= \frac{\log_c m}{\log_c b} \qquad b^{\log_b m} = m$$

Problem 3 Using the Change of Base Formula

What is the value of each expression?

Think

What common base has powers that equal 27 and 81?

3; $3^3 = 27$ and $3^4 = 81$.

A $\log_{81} 27$

Method 1 Use a common base.

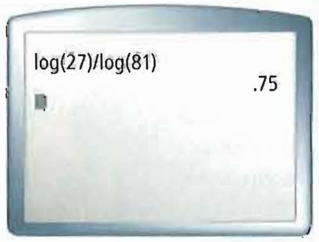
$$\log_{81} 27 = \frac{\log_3 27}{\log_3 81} \qquad \text{Change of Base Formula}$$

$$= \frac{3}{4} \qquad \text{Simplify.}$$

Method 2 Use a calculator.

$$\log_{81} 27 = \frac{\log 27}{\log 81} \qquad \text{Change of Base Formula}$$

$$= 0.75 \qquad \text{Use a calculator.}$$



log(27)/log(81)
.75

Think

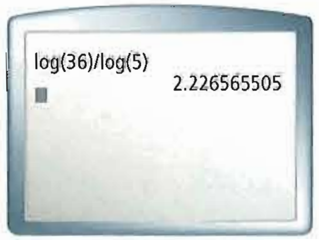
What would be a reasonable result?

$5^2 = 25$ and $5^3 = 125$, so $\log_5 36$ should be between 2 and 3.

B $\log_5 36$

$$\log_5 36 = \frac{\log 36}{\log 5} \qquad \text{Change of Base Formula}$$

$$\approx 2.23 \qquad \text{Use a calculator to evaluate.}$$



log(36)/log(5)
2.226565505

Got It? **3.** Use the Change of Base Formula. What is the value of each expression?

a. $\log_8 32$ **b.** $\log_4 18$

Problem 4 Using a Logarithmic Scale

Chemistry The pH of a substance equals $-\log[H^+]$, where $[H^+]$ is the concentration of hydrogen ions. $[H^+_a]$ for household ammonia is 10^{-11}. $[H^+_v]$ for vinegar is 6.3×10^{-3}. What is the difference of the pH levels of ammonia and vinegar?

Think	Write
Write the equation for pH.	$\text{pH} = -\log[H^+]$
Write the difference of the pH levels.	$-\log[H^+_a] - (-\log[H^+_v])$ $= -\log[H^+_a] + \log[H^+_v]$ $= \log[H^+_v] - \log[H^+_a]$
Substitute values for $[H^+_v]$ and $[H^+_a]$.	$= \log(6.3 \times 10^{-3}) - \log 10^{-11}$
Use the Product Property of Logarithms, and simplify.	$= \log 6.3 + \log 10^{-3} - \log 10^{-11}$ $= \log 6.3 - 3 + 11$
Use a calculator.	≈ 8.8
Write the answer.	The pH level of ammonia is about 8.8 greater than the pH level of vinegar.

Got It? **4. Reasoning** Suppose the hydrogen ion concentration for Substance A is twice that for Substance B. Which substance has the greater pH level? What is the greater pH level minus the lesser pH level? Explain.

Lesson Check

Do you know HOW?

Write each expression as a single logarithm.

1. $\log_4 2 + \log_4 8$

2. $\log_6 24 - \log_6 4$

Expand each logarithm.

3. $\log_3 \frac{x}{y}$ **4.** $\log m^2n^5$ **5.** $\log_2 \sqrt{x}$

Do you UNDERSTAND?

6. Vocabulary State which property or properties need to be used to write each expression as a single logarithm.

a. $\log_4 5 + \log_4 5$ **b.** $\log_5 4 - \log_5 6$

7. Reasoning If $\log x = 5$, what is the value of $\frac{1}{x}$?

8. Open-Ended Write log 150 as a sum or difference of two logarithms. Simplify if possible.

Practice and Problem-Solving Exercises

A Practice

Write each expression as a single logarithm.

See Problem 1.

9. $\log 7 + \log 2$

10. $\log_2 9 - \log_2 3$

11. $5 \log 3 + \log 4$

12. $\log 8 - 2 \log 6 + \log 3$

13. $4 \log m - \log n$

14. $\log 5 - k \log 2$

15. $\log_6 5 + \log_6 x$

16. $\log_7 x + \log_7 y - \log_7 z$

17. $\log_3 4 + \log_3 y + \log_3 8x$

Expand each logarithm.

See Problem 2.

18. $\log x^3y^5$

19. $\log_7 49xyz$

20. $\log_b \frac{b}{x}$

21. $\log a^2$

22. $\log_5 \frac{r}{s}$

23. $\log_3 (2x)^2$

24. $\log_3 7(2x - 3)^2$

25. $\log \frac{a^2b^3}{c^4}$

26. $\log_4 5\sqrt{x}$

27. $\log_8 8\sqrt{3a^5}$

28. $\log_5 \frac{25}{x}$

29. $\log 10m^4n^{-2}$

Use the Change of Base Formula to evaluate each expression.

See Problem 3.

30. $\log_2 9$

31. $\log_{12} 20$

32. $\log_7 30$

33. $\log_5 10$

34. $\log_4 7$

35. $\log_3 54$

36. $\log_5 62$

37. $\log_3 33$

38. Science The concentration of hydrogen ions in household dish detergent is 10^{-12}. What is the pH level of household dish detergent?

See Problem 4.

B Apply

Use the properties of logarithms to evaluate each expression.

39. $\log_2 4 - \log_2 16$

40. $\log_2 96 - \log_2 3$

41. $\log_3 27 - 2\log_3 3$

42. $\log_6 12 + \log_6 3$

43. $\log_4 48 - \frac{1}{2}\log_4 9$

44. $\frac{1}{2}\log_5 15 - \log_5 \sqrt{75}$

45. Think About a Plan The loudness in decibels (dB) of a sound is defined as $10 \log \frac{I}{I_0}$, where I is the intensity of the sound in watts per square meter (W/m^2). I_0, the intensity of a barely audible sound, is equal to $10^{-12}\ \text{W/m}^2$. Town regulations require the loudness of construction work not to exceed 100 dB. Suppose a construction team is blasting rock for a roadway. One explosion has an intensity of $1.65 \times 10^{-2}\ \text{W/m}^2$. Is this explosion in violation of town regulations?

- Which physical value do you need to calculate to answer the question?
- What values should you use for I and I_0?

46. Construction The foreman of a construction team puts up a sound barrier that reduces the intensity of the noise by 50%. By how many decibels is the noise reduced? Use the formula $L = 10 \log \frac{I}{I_0}$ to measure loudness. (*Hint:* Find the difference between the expression for loudness for intensity I and the expression for loudness for intensity $0.5I$.)

47. Error Analysis Explain why the expansion at the right of $\log_4\sqrt{\frac{t}{s}}$ is incorrect. Then do the expansion correctly.

$$\xcancel{\begin{aligned}\log_4\sqrt{\tfrac{t}{s}} &= \tfrac{1}{2}\log_4\tfrac{t}{s}\\ &= \tfrac{1}{2}\log_4 t - \log_4 s\end{aligned}}$$

48. Reasoning Can you expand $\log_3 (2x + 1)$? Explain.

49. Writing Explain why $\log (5 \cdot 2) \neq \log 5 \cdot \log 2$.

Determine if each statment is *true* or *false*. Justify your answer.

50. $\log_2 4 + \log_2 8 = 5$

51. $\log_3 \frac{3}{2} = \frac{1}{2} \log_3 3$

52. $\log(x - 2) = \frac{\log x}{\log 2}$

53. $\frac{\log_b x}{\log_b y} = \log_b \frac{x}{y}$

54. $(\log x)^2 = \log x^2$

55. $\log_4 7 - \log_4 3 = \log_4 4$

Write each logarithmic expression as a single logarithm.

56. $\frac{1}{4}\log_3 2 + \frac{1}{4}\log_3 x$

57. $\frac{1}{2}(\log_x 4 + \log_x y) - 3\log_x z$

58. $x \log_4 m + \frac{1}{y}\log_4 n - \log_4 p$

59. $\left(\frac{2\log_b x}{3} + \frac{3\log_b y}{4}\right) - 5\log_b z$

Expand each logarithm.

60. $\log \sqrt{\frac{2x}{y}}$

61. $\log \frac{s\sqrt{7}}{t^2}$

62. $\log \left(\frac{2\sqrt{x}}{5}\right)^3$

63. $\log \frac{m^3}{n^4 p^{-2}}$

64. $\log 4\sqrt{\frac{4r}{s^2}}$

65. $\log_b \frac{\sqrt{x}\sqrt[3]{y^2}}{\sqrt[5]{z^2}}$

66. $\log_4 \frac{\sqrt{x^5 y^7}}{zw^4}$

67. $\log \frac{\sqrt{x^2 - 4}}{(x + 3)^2}$

Write each logarithm as the quotient of two common logarithms. Do not simplify the quotient.

68. $\log_7 2$

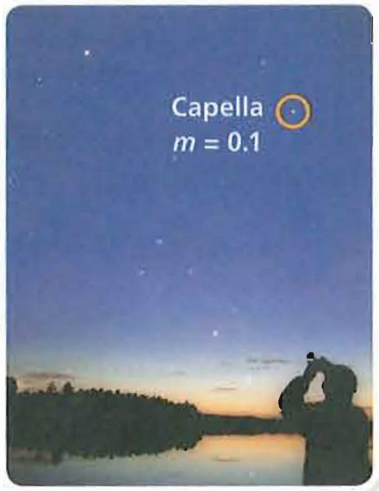
69. $\log_3 8$

70. $\log_5 140$

71. $\log_9 3.3$

72. $\log_4 3x$

Astronomy The apparent brightness of stars is measured on a logarithmic scale called magnitude, in which lower numbers mean brighter stars. The relationship between the ratio of apparent brightness of two objects and the difference in their magnitudes is given by the formula $m_2 - m_1 = -2.5 \log \frac{b_2}{b_1}$, where m is the magnitude and b is the apparent brightness.



73. How many times brighter is a magnitude 1.0 star than a magnitude 2.0 star?

74. The star Rigel has a magnitude of 0.12. How many times brighter is Capella than Rigel?

Challenge

Expand each logarithm.

75. $\log \sqrt{\frac{x\sqrt{2}}{y^2}}$

76. $\log_3 [(xy^{\frac{1}{3}}) + z^2]^3$

77. $\log_7 \frac{\sqrt{r + 9}}{s^2 t^{\frac{1}{3}}}$

Simplify each expression.

78. $\log_3(x + 1) - \log_3(3x^2 - 3x - 6) + \log_3(x - 2)$

79. $\log(a^2 - 10a + 25) + \frac{1}{2}\log\frac{1}{(a - 5)^3} - \log(\sqrt{a - 5})$

Sunshine State Standards Practice

MA.912.A.6.3

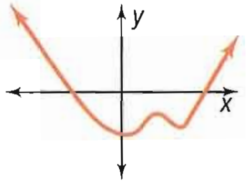
80. Which expression is NOT equivalent to $\sqrt[6]{16r^2}$?

Ⓐ $(16r^2)^{\frac{1}{6}}$ Ⓑ $4r^{\frac{1}{3}}$ Ⓒ $(4r)^{\frac{1}{3}}$ Ⓓ $\sqrt[3]{4r}$

MA.912.A.4.5

81. Assume that there are no more turning points beyond those shown. Which graph CANNOT be the graph of a fourth degree polynomial?

Ⓕ



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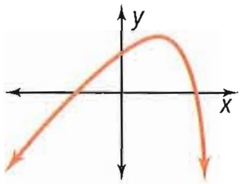

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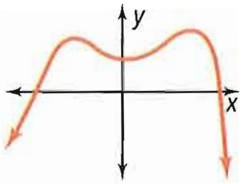
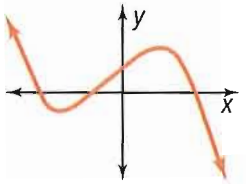


Ⓘ



MA.912.A.3.14

82. A florist is arranging a bouquet of daisies and tulips. He wants twice as many daisies as tulips in the bouquet. If the bouquet contains 24 flowers, how many daisies are in the bouquet?

Ⓐ 8 daisies Ⓑ 12 daisies Ⓒ 16 daisies Ⓓ 24 daisies

MA.912.A.8.2

83. Short Response Use the properties of logarithms to write log 18 in four different ways. Name each property you use.

Mixed Review

Write each equation in logarithmic form. See Lesson 7-3.

84. $49 = 7^2$ **85.** $\frac{1}{4} = 8^{-\frac{2}{3}}$ **86.** $5^{-3} = \frac{1}{125}$

Solve. Check for extraneous solutions. See Lesson 6-5.

87. $\sqrt[3]{y^4} = 16$ **88.** $\sqrt[3]{7x} - 4 = 0$ **89.** $2\sqrt{w-1} = \sqrt{w+2}$

Write a polynomial function with rational coefficients and the given roots. See Lesson 5-5.

90. $\sqrt{3}, -5$ **91.** $-i, 4i$ **92.** $-\sqrt{7}, 1 + 2i$

Get Ready! **To prepare for Lesson 7-5, do Exercises 93–95.**

Evaluate each logarithm. See Lesson 7-3.

93. $\log_{12} 144$ **94.** $\log_4 64$ **95.** $\log_{64} 4$

7-5 Exponential and Logarithmic Equations

Sunshine State Standard
MA.912.A.8.5 Solve logarithmic and exponential equations.

Objective To solve exponential and logarithmic equations

Getting Ready!

You are a winner on a TV game show. Which prize would you choose? Explain.

Prize A
$10,000 per week

Prize B
1¢ today,
2¢ tomorrow,
4¢ the next day,
and so on,
doubling each day

Lesson Vocabulary
- exponential equation
- logarithmic equation

Any equation that contains the form b^{cx}, such as $a = b^{cx}$ where the exponent includes a variable, is an **exponential equation**.

Essential Understanding You can use logarithms to solve exponential equations. You can use exponents to solve logarithmic equations.

Problem 1 Solving an Exponential Equation—Common Base

Plan
What common base is appropriate?
2 because 16 and 8 are both powers of 2.

Multiple Choice What is the solution of $16^{3x} = 8$?

Ⓐ $x = \frac{1}{4}$ Ⓑ $x = \frac{3}{7}$ Ⓒ $x = 1$ Ⓓ $x = 4$

$$16^{3x} = 8$$

$$(2^4)^{3x} = 2^3 \quad \text{Rewrite the terms with a common base.}$$

$$2^{12x} = 2^3 \quad \text{Power Property of Exponents}$$

$$12x = 3 \quad \text{If two numbers with the same base are equal, their exponents are equal.}$$

$$x = \frac{1}{4} \quad \text{Solve and simplify.}$$

The correct answer is A.

Got It? **1.** What is the solution of $27^{3x} = 81$?

When bases are not the same, you can solve an exponential equation by taking the logarithm of each side of the equation. If m and n are positive and $m = n$, then $\log m = \log n$.

Problem 2 Solving an Exponential Equation—Different Bases

What is the solution of $15^{3x} = 285$?

$15^{3x} = 285$

$\log 15^{3x} = \log 285$ Take the logarithm of each side.

$3x \log 15 = \log 285$ Power Property of Logarithms.

$x = \dfrac{\log 285}{3 \log 15}$ Divide each side by 3 log 15 to isolate x.

$x \approx 0.6958$ Use a calculator.

Check $15^{3x} = 285$

$15^{3(0.6958)} \approx 285.0840331 \approx 285$ ✓

Think

Which property of logarithms will help isolate x?

The rule $\log a^x = x \log a$ moves x out of the exponent position.

Got It? **2. a.** What is the solution of $5^{2x} = 130$?

b. Reasoning Why can't you use the same method you used in Problem 1 to solve Problem 2?

Problem 3 Solving an Exponential Equation With a Graph or Table

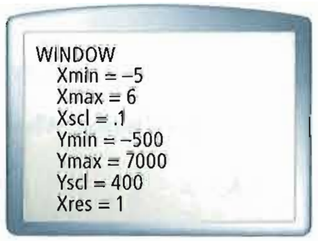
What is the solution of $4^{3x} = 6000$?

Method 1 Solve using a graph.

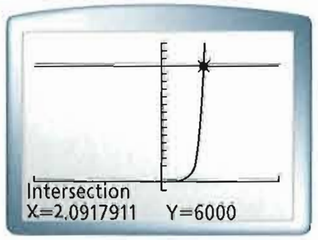
Use a graphing calculator. Graph the equations.

$Y_1 = 4^{3x}$

$Y_2 = 6000$



```
WINDOW
  Xmin = –5
  Xmax = 6
  Xscl = .1
  Ymin = –500
  Ymax = 7000
  Yscl = 400
  Xres = 1
```



```
Intersection
X=2.0917911    Y=6000
```

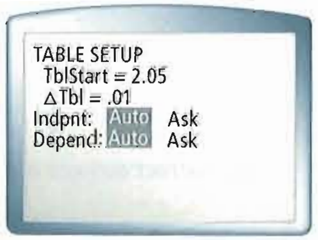
Adjust the window to find the point of intersection. The solution is $x \approx 2.09$.

Method 2 Solve using a table.

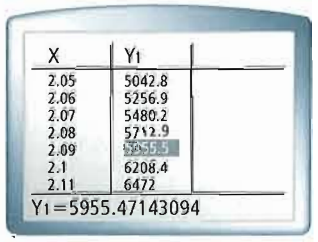
Use the table feature of a graphing calculator. Enter $Y_1 = 4^{3x}$.

Use the **TABLE SETUP** and **ΔTbl** features to locate the x-value that gives the y-value closest to 6000.

The solution is $x \approx 2.09$.



```
TABLE SETUP
 TblStart = 2.05
 ΔTbl = .01
Indpnt:  Auto   Ask
Depend:  Auto   Ask
```



```
  X    | Y1     |
 2.05  | 5042.8 |
 2.06  | 5256.9 |
 2.07  | 5480.2 |
 2.08  | 5712.9 |
 2.09  | 5955.5 |
 2.1   | 6208.4 |
 2.11  | 6472   |
Y1=5955.47143094
```

Think

How do you choose TblStart and ΔTbl values?

Start with 0 and 1, respectively. Adjust both values as you close in on the solution.

Got It? **3.** What is the solution of each exponential equation? Check your answer.

a. $7^{4x} = 800$

b. $5.2^{3x} = 400$

Problem 4 Modeling With an Exponential Equation

Resource Management **Wood is a sustainable, renewable, natural resource when you manage forests properly. Your lumber company has 1,200,000 trees. You plan to harvest 7% of the trees each year. How many years will it take to harvest half of the trees?**

Know	Need	Plan
• Number of trees • Rate of decay	Number of years it takes to harvest 600,000 trees	• Write an exponential equation. • Use logarithms to solve the equation.

Think

What equation should you use to model this situation? Since you are planning to harvest 7% of the trees each year, you should use $y = ab^x$, where b is the decay factor.

Step 1 Is an exponential model reasonable for this situation?

Yes, you are harvesting a fixed percentage each year.

Step 2 Define the variables and determine the model.

Let n = the number of years it takes to harvest half of the trees.

Let $T(n)$ = the number of trees remaining after n years.

A reasonable model is $T(n) = a(b)^n$.

Step 3 Use the model to write an exponential equation.

$$T(n) = 600{,}000$$

$$a = 1{,}200{,}000$$

$$r = -7\% = -0.07$$

$$b = 1 + r = 1 + (-0.07) = 0.93$$

So, $1{,}200{,}000(0.93)^n = 600{,}000$.

Step 4 Solve the equation. Use logarithms.

$$\begin{aligned}
1{,}200{,}000(0.93)^n &= 600{,}000 \\
0.93^n &= \frac{600{,}000}{1{,}200{,}000} && \text{Isolate the term with } n. \\
\log 0.93^n &= \log 0.5 && \text{Take the logarithm of each side.} \\
n \log 0.93 &= \log 0.5 && \text{Power Property of Logarithms.} \\
n &= \frac{\log 0.5}{\log 0.93} && \text{Solve for } n. \\
n &\approx 9.55 && \text{Use a calculator.}
\end{aligned}$$

It will take about 9.55 years to harvest half of the original trees.

Got It? **4.** After how many years will you have harvested half of the trees if you harvest 5% instead of 7% yearly?

A **logarithmic equation** is an equation that includes one or more logarithms involving a variable.

Problem 5 Solving a Logarithmic Equation

What is the solution of $\log(4x - 3) = 2$?

Plan

How do you convert between log form and exponential form?
Use the rule: $\log a = b$ if and only if $a = 10^b$.

Method 1 Solve using exponents.

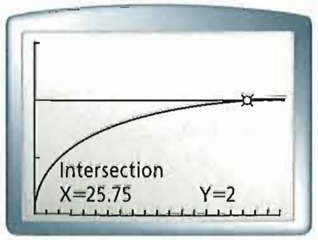
$\log(4x - 3) = 2$

$4x - 3 = 10^2$ Write in exponential form.

$4x = 103$ Simplify.

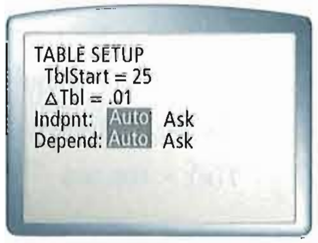
$x = \frac{103}{4} = 25.75$ Solve for x.

Method 2 Solve using a graph.
Graph the equations
Y_1 = LOG (4x − 3) and **Y_2 = 2.**
Find the point of intersection.
The solution is $x = 25.75$.

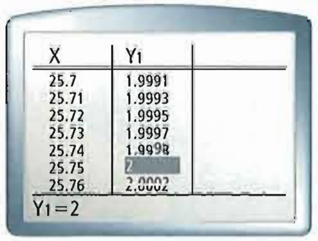


```
Intersection
X=25.75        Y=2
```

Method 3 Solve using a table.
Enter **Y_1 = LOG (4x − 3).**
Use the **TABLE SETUP** feature to find the x-value that corresponds to a y-value of 2 in the table.
The solution is $x = 25.75$.



```
TABLE SETUP
 TblStart = 25
 ΔTbl = .01
Indpnt:  Auto  Ask
Depend:  Auto  Ask
```



```
 X      | Y1
 25.7   | 1.9991
 25.71  | 1.9993
 25.72  | 1.9995
 25.73  | 1.9997
 25.74  | 1.9998
 25.75  | 2
 25.76  | 2.0002
Y1=2
```

Got It? **5.** What is the solution of $\log(3 - 2x) = -1$?

Problem 6 Using Logarithmic Properties to Solve an Equation

What is the solution of $\log(x - 3) + \log x = 1$?

$\log(x - 3) + \log x = 1$

$\log((x - 3)x) = 1$ Product Property of Logarithms

$(x - 3)x = 10^1$ Write in exponential form.

$x^2 - 3x - 10 = 0$ Simplify to a quadratic equation in standard form.

$(x - 5)(x + 2) = 0$ Factor the trinomial.

$x = 5$ or $x = -2$ Solve for x.

Think

What is the domain of the logarithm function?
Logs are defined only for positive numbers. The log of a negative number is undefined.

Check

$\log(x - 3) + \log(x) = 1$

$\log(-2 - 3) + \log(-2) \stackrel{?}{=} 1$ ✗

$\log(x - 3) + \log(x) = 1$

$\log(5 - 3) + \log(5) \stackrel{?}{=} 1$

$\log 2 + \log 5 \stackrel{?}{=} 1$

$0.3010 + 0.6990 = 1$ ✓

If $\log(x - 3) + \log(x) = 1$, $x = 5$.

Got It? **6.** What is the solution of $\log 6 - \log 3x = -2$?

Lesson Check

Do you know HOW?

Solve each equation.

1. $3^x = 9$

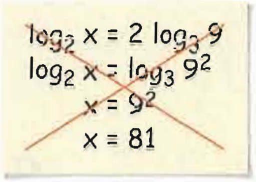
2. $2^{y+1} = 25$

3. $\log 4x = 2$

4. $\log x - \log 2 = 3$

Do you UNDERSTAND?

5. **Error Analysis** Describe and correct the error made in solving the equation.



$$\log_2 x = 2\log_3 9$$
$$\log_2 x = \log_3 9^2$$
$$x = 9^2$$
$$x = 81$$

6. **Reasoning** Is it possible for an exponential equation to have no solutions? If so, give an example. If not, explain why.

Practice and Problem-Solving Exercises

A Practice

Solve each equation. See Problem 1.

7. $2^x = 8$

8. $3^{2x} = 27$

9. $4^{3x} = 64$

10. $5^{3x} = \frac{1}{125}$

11. $2^{5x+1} = 32$

12. $3^{-2x+2} = 81$

13. $2^{3x} = 4^{x+1}$

14. $3^{x+2} = 27^{2x}$

Solve each equation. Round to the nearest ten-thousandth. Check your answers. See Problem 2.

15. $2^x = 3$

16. $4^x = 19$

17. $8 + 10^x = 1008$

18. $5 - 3^x = -40$

19. $9^{2y} = 66$

20. $12^{y-2} = 20$

21. $25^{2x+1} = 144$

22. $2^{3x-4} = 5$

Graphing Calculator **Solve by graphing. Round to the nearest ten-thousandth.** See Problem 3.

23. $4^{7x} = 250$

24. $5^{3x} = 500$

25. $6^x = 4565$

26. $1.5^x = 356$

Use a table to solve each equation. Round to the nearest hundredth.

27. $2^{x+3} = 512$

28. $3^{x-1} = 72$

29. $6^{2x} = 10$

30. $5^{2x} = 56$

31. The equation $y = 6.72(1.014)^x$ models the world population y, in billions of people, x years after the year 2000. Find the year in which the world population is about 8 billion. See Problem 4.

Solve each equation. Check your answers. See Problem 5.

32. $\log 2x = -1$

33. $2\log x = -1$

34. $\log(3x + 1) = 2$

35. $\log x + 4 = 8$

36. $\log 6x - 3 = -4$

37. $3\log x = 1.5$

38. $2\log(x + 1) = 5$

39. $\log(5 - 2x) = 0$

Solve each equation. See Problem 6.

40. $\log x - \log 3 = 8$

41. $\log 2x + \log x = 11$

42. $2\log x + \log 4 = 2$

43. $\log 5 - \log 2x = 1$

44. $3\log x - \log 6 + \log 2.4 = 9$

45. $\log(7x + 1) = \log(x - 2) + 1$

B Apply

46. **Think About a Plan** An earthquake of magnitude 9.1 occurred in 2004 in the Indian Ocean near Indonesia. It was about 74,900 times as strong as the greatest earthquake ever to hit Texas. Find the magnitude of the Texas earthquake. (Remember that an increase of 1.0 on the Richter scale means an earthquake is 30 times stronger.)
    - Can you write an exponential or logarithmic equation?
    - How does the solution of your equation help you find the magnitude?

47. Consider the equation $2^{\frac{x}{3}} = 80$.
    a. Solve the equation by taking the logarithm base 10 of each side.
    b. Solve the equation by taking the logarithm base 2 of each side.
    c. **Writing** Compare your result in parts (a) and (b). What are the advantages of each method? Explain.

48. **Seismology** An earthquake of magnitude 7.7 occurred in 2001 in Gujarat, India. It was about 4900 times as strong as the greatest earthquake ever to hit Pennsylvania. What is the magnitude of the Pennsylvania earthquake? (*Hint*: Refer to the Richter scale on page 453.)

49. As a town gets smaller, the population of its high school decreases by 6% each year. The senior class has 160 students now. In how many years will it have about 100 students? Write an equation. Then solve the equation without graphing.

Mental Math **Solve each equation.**

50. $2^x = \frac{1}{2}$

51. $3^x = 27$

52. $\log_9 3 = x$

53. $\log_4 64 = x$

54. $\log_8 2 = x$

55. $10^x = \frac{1}{100}$

56. $\log_7 343 = x$

57. $25^x = \frac{1}{5}$

58. **Demography** The table below lists the states with the highest and with the lowest population growth rates. Determine in how many years each event can occur. Use the model $P = P_0(1 + r)^x$, where P_0 is population from the table, as of July, 2007; x is the number of years after July, 2007, P is the projected population and r is the growth rate.
    a. Population of Idaho exceeds 2 million.
    b. Population of Michigan decreases by 1 million.
    c. Population of Nevada doubles.

State	Growth rate (%)	Population (in thousands)	State	Growth rate (%)	Population (in thousands)
1. Nevada	2.93	2,565	**46. New York**	0.08	19,298
2. Arizona	2.81	6,339	**47. Vermont**	0.08	621
3. Utah	2.55	2,645	**48. Ohio**	0.03	11,467
4. Idaho	2.43	1,499	**49. Michigan**	−0.30	10,072
5. Georgia	2.17	9,545	**50. Rhode Island**	−0.36	1,058

SOURCE: U.S. Census Bureau

59. Open-Ended Write and solve a logarithmic equation.

60. Reasoning The graphs of $y = 2^{3x}$ and $y = 3^{x+1}$ intersect at approximately (1.1201, 10.2692). What is the solution of $2^{3x} = 3^{x+1}$?

61. Reasoning If $\log 12^{0.5x} = \log 143.6$, then $12^{0.5x} =$ __?__.

Acoustics **In Exercises 62–63, the loudness measured in decibels (dB) is defined by loudness = $10 \log \frac{I}{I_0}$, where I is the intensity and $I_0 = 10^{-12}$ W/m^2.**

62. The human threshold for pain is 120 dB. Instant perforation of the eardrum occurs at 160 dB.

a. Find the intensity of each sound.

b. How many times as intense is the noise that will perforate an eardrum as the noise that causes pain?

63. The noise level inside a convertible driving along the freeway with its top up is 70 dB. With the top down, the noise level is 95 dB.

a. Find the intensity of the sound with the top up and with the top down.

b. By what percent does leaving the top up reduce the intensity of the sound?

Solve each equation. If necessary, round to the nearest ten-thousandth.

64. $8^x = 444$

65. $\frac{1}{2}\log x + \log 4 = 2$

66. $4\log_3 2 - 2\log_3 x = 1$

67. $\log x^2 = 2$

68. $9^{2x} = 42$

69. $\log_8(2x - 1) = \frac{1}{3}$

70. $\log(5x - 4) = 3$

71. $12^{4-x} = 20$

72. $5^{3x} = 125$

73. $\log 4 + 2\log x = 6$

74. $4^{3x} = 77.2$

75. $\log_7 3x = 3$

Use the properties of exponential and logarithmic functions to solve each system. Check your answers.

76. $\begin{cases} y = 2^{x+4} \\ y - 4^{x-1} = 0 \end{cases}$

77. $\begin{cases} 2^{x+y} = 16 \\ 4^{x-y} = 1 \end{cases}$

78. $\begin{cases} \log(2x - y) = 1 \\ \log(x + y) = 3\log 2 \end{cases}$

Challenge

Solve each equation.

79. $\log_7(2x - 3)^2 = 2$

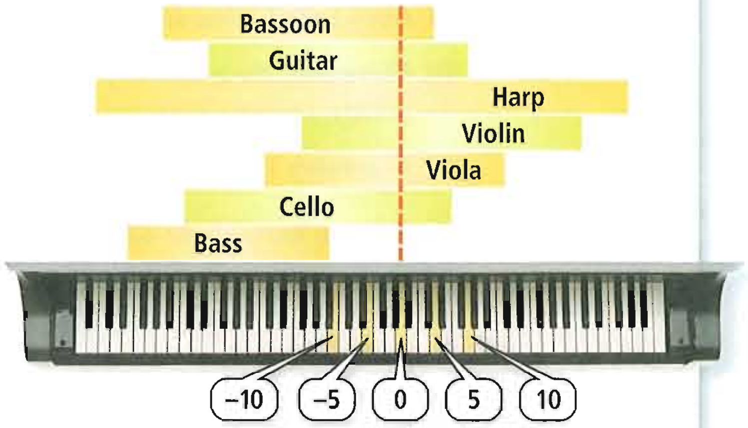
80. $\log_2(x^2 + 2x) = 3$

81. $\frac{3}{2}\log_2 4 - \frac{1}{2}\log_2 x = 3$

82. Meteorology In the formula $P = P_0\left(\frac{1}{2}\right)^{\frac{h}{4795}}$, P is the atmospheric pressure in millimeters of mercury at elevation h meters above sea level. P_0 is the atmospheric pressure at sea level. If P_0 equals 760 mm, at what elevation is the pressure 42 mm?

83. **Music** The pitch, or frequency, of a piano note is related to its position on the keyboard by the function $F(n) = 440 \cdot 2^{\frac{n}{12}}$, where F is the frequency of the sound waves in cycles per second and n is the number of piano keys above or below Concert A, as shown. If $n = 0$ at Concert A, which of the instruments shown in the diagram can sound notes at the given frequency?

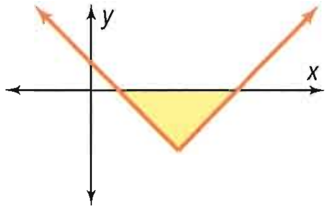
a. 590 **c.** 1440
b. 120 **d.** 2093



Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.3.6 **84.** The graph at the right shows the translation of the graph of the parent function $y = |x|$ down 2 units and 3 units to the right. What is the area of the shaded triangle in square units?



MA.912.A.8.5 **85.** What does x equal if $\log (1 + 3x) = 3$?

MA.912.A.8.6 **86.** Using the change of base formula, what is the value of x for which $\log_9 x = \log_3 5$?

MA.912.A.4.4 **87.** The polynomial $x^4 + 3x^3 + 16x^2 - 19x + 8$ is divided by the binomial $x - 1$. What is the coefficient of x^2 in the quotient?

MA.912.A.7.3 **88.** What positive value of b makes $x^2 + bx + 81$ a perfect square trinomial?

Mixed Review

Expand each logarithm. **See Lesson 7-4.**

89. $\log 2x^3y^{-2}$ **90.** $\log_3 \frac{x}{y}$ **91.** $\log_3 \sqrt{9x}$

Let $f(x) = 3x$ and $g(x) = x^2 - 1$. Perform each function operation. **See Lesson 6-6.**

92. $(g - f)(x)$ **93.** $(f \circ g)(x)$ **94.** $(g \circ f)(x)$

Find all the zeros of each function. **See Lesson 5-6.**

95. $y = x^3 - x^2 + x - 1$ **96.** $f(x) = x^4 - 16$ **97.** $f(x) = x^4 - 5x^2 + 6$

Get Ready! **To prepare for Lesson 7-6, do Exercises 98–100.**

Write each logarithmic expression as a single logarithm. **See Lesson 7-4.**

98. $\log_2 15 - \log_2 5$ **99.** $\log 3 + 4 \log x$ **100.** $5 \log_7 2 - 2 \log_7 y$

Concept Byte

For Use With Lesson 7-5

TECHNOLOGY

Using Logarithms for Exponential Models

Sunshine State Standards

MA.912.A.2.6 Identify and graph linear and exponential functions.

MA.912.A.8.3 Graph exponential functions.

You can transform an exponential function into a linear function by taking the logarithm of each side. Since linear models are easy to recognize, you can then determine whether an exponential function is a good model for a set of values.

$y = ab^x$	Write the general form of an exponential function.
$\log y = \log ab^x$	Take the logarithm of each side.
$\log y = \log a + x(\log b)$	Use the Product Property and the Power Property.

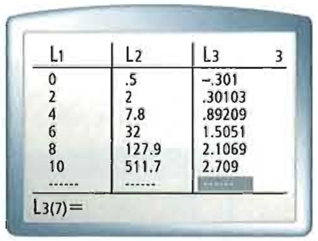
If $\log b$ and $\log a$ are constants, then $\log y = (\log b)x + \log a$ is a linear equation in slope-intercept form when you plot the points as $(x, \log y)$.

Activity

Determine whether an exponential function is a good model for the values in the table.

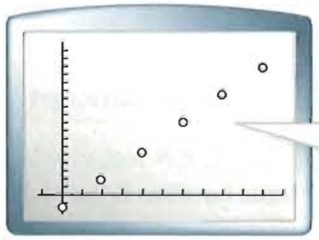
x	0	2	4	6	8	10
y	0.5	2	7.8	32	127.9	511.7

Step 1 Enter the values into **stat** lists $\mathbf{L_1}$ and $\mathbf{L_2}$. To enter the values of $\log y$, place the cursor in the heading of $\mathbf{L_3}$ and press **log** $\mathbf{L_2}$ **enter**.



```
L1      L2      L3      3
0       .5      -.301
2       2       .30103
4       7.8     .89209
6       32      1.5051
8       127.9   2.1069
10      511.7   2.709
------  ------  ------
L3(7)=
```

Step 2 To graph $\log y$, access the **stat plot** feature and press **1**. Then enter $\mathbf{L_3}$ next to **YLIST:**. Then press **zoom** **9**.



The points (*x*, log *y*) lie on a line, so an exponential model is appropriate.

Step 3 Press **stat** ▷ **0** **enter** to find the exponential function $y = 0.5(2)^x$.

Exercises

For each set of values, determine whether an exponential function is a good model. If so, find the exponential function.

1.

x	1	3	5	7	9
y	6	22	54	102	145

2.

x	−1	0	1	2	3
y	40.2	19.8	9.9	5.1	2.5

3. Writing Explain how you could determine whether a logarithmic function is a good model for a set of data.

7-6 Natural Logarithms

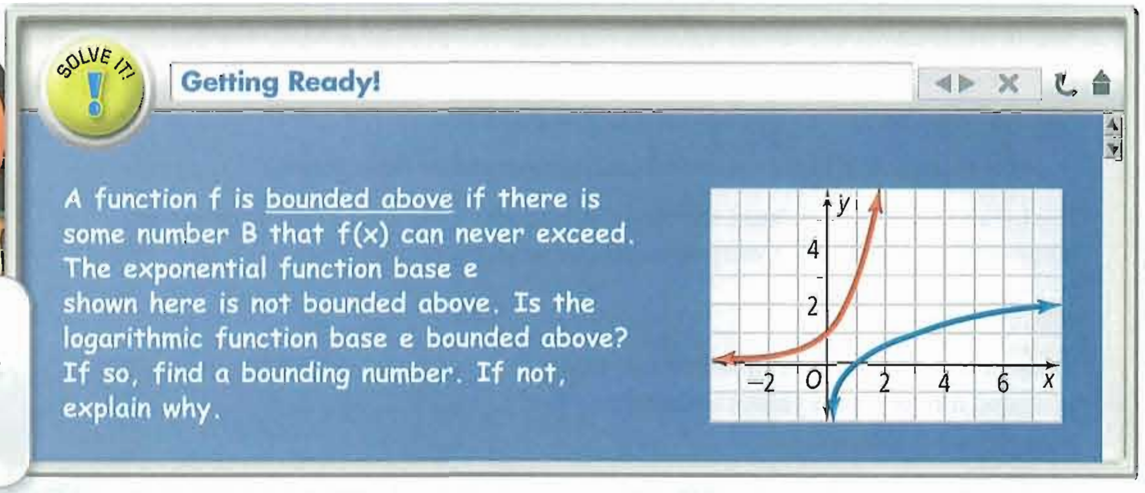
Sunshine State Standards

MA.912.A.8.1 Determine the relationship between exponential and logarithmic functions.

MA.912.A.8.5 Solve logarithmic and exponential equations.

Objectives To evaluate and simplify natural logarithmic expressions
To solve equations using natural logarithms

If you can find one number that bounds $y = \log_e x$ above, then you know there are many others.



Getting Ready!

A function f is <u>bounded above</u> if there is some number B that f(x) can never exceed. The exponential function base e shown here is not bounded above. Is the logarithmic function base e bounded above? If so, find a bounding number. If not, explain why.

Lesson Vocabulary
- natural logarithmic function

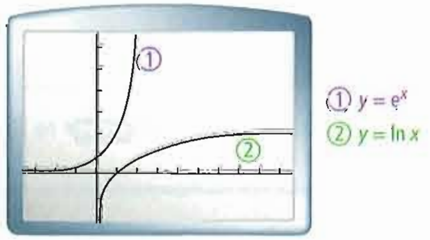
The function $y = e^x$ has an inverse, the **natural logarithmic function**, $y = \log_e x$, or $y = \ln x$.

Essential Understanding The functions $y = e^x$ and $y = \ln x$ are inverse functions. Just as before, this means that if $a = e^b$, then $b = \ln a$, and vice versa.

take note

Key Concept Natural Logarithmic Function

If $y = e^x$, then $x = \log_e y = \ln y$. The natural logarithmic function is the inverse of $x = \ln y$, so you can write it as $y = \ln x$.



Plan

Can you use properties of logarithms?
Yes; the properties you studied in Lesson 7-4 apply to logarithms with any base.

Problem 1 Simplifying a Natural Logarithmic Expression

What is $2 \ln 15 - \ln 75$ written as a single natural logarithm?

$2 \ln 15 - \ln 75 = \ln 15^2 - \ln 75$	Power Property of Logarithms
$= \ln \frac{15^2}{75}$	Quotient Property of Logarithms
$= \ln 3$	Simplify.

Got It? **1.** What is each expression written as a single natural logarithm?

a. $\ln 7 + 2\ln 5$ **b.** $3\ln x - 2\ln 2x$ **c.** $3\ln x + 2\ln y + \ln 5$

You can use the inverse relationship between the functions $y = \ln x$ and $y = e^x$ to solve certain logarithmic and exponential equations.

Problem 2 Solving a Natural Logarithmic Equation

Think

How do you go from logarithmic form to exponential form? Use the definition of logarithm: $\ln x = y$ if and only if $x = e^y$.

What are the solutions of $\ln (x - 3)^2 = 4$?

$$\begin{aligned} \ln (x-3)^2 &= 4 \\ (x-3)^2 &= e^4 && \text{Rewrite in exponential form.} \\ x - 3 &= \pm e^2 && \text{Find the square root of each side.} \\ x &= 3 \pm e^2 && \text{Solve for } x. \\ x &\approx 10.39 \text{ or } -4.39 && \text{Use a calculator.} \end{aligned}$$

Check

$$\ln (10.39 - 3)^2 \overset{?}{=} 4 \qquad \ln (-4.39 - 3)^2 \overset{?}{=} 4$$

$$4.0003 \approx 4 \ \checkmark \qquad 4.0003 \approx 4 \ \checkmark$$

Got It? **2.** What are the solutions of each equation? Check your answers.

a. $\ln x = 2$ **b.** $\ln (3x + 5)^2 = 4$ **c.** $\ln 2x + \ln 3 = 2$

Problem 3 Solving an Exponential Equation

Plan

How can you solve this equation? First get e^x by itself on one side of the equation. Then rewrite the equation in logarithmic form and solve for x.

What is the solution of $4e^{2x} + 2 = 16$?

$$\begin{aligned} 4e^{2x} + 2 &= 16 \\ 4e^{2x} &= 14 && \text{Subtract 2 from each side.} \\ e^{2x} &= 3.5 && \text{Divide each side by 4.} \\ 2x &= \ln 3.5 && \text{Rewrite in logarithmic form.} \\ x &= \frac{\ln 3.5}{2} && \text{Divide each side by 2.} \\ x &\approx 0.626 && \text{Use a calculator.} \end{aligned}$$

Check

$$\begin{aligned} 4e^{2x} + 2 &= 16 \\ 4e^{2(0.626)} + 2 &\overset{?}{=} 16 \\ 15.99 &\approx 16 \ \checkmark \end{aligned}$$

Got It? **3.** What is the solution of each equation? Check your answers.

a. $e^{x-2} = 12$ **b.** $2e^{-x} = 20$ **c.** $e^{3x} + 5 = 15$

Natural logarithms are useful because they help express many relationships in the physical world.

Problem 4 Using Natural Logarithms

Space **A spacecraft can attain a stable orbit 300 km above Earth if it reaches a velocity of 7.7 km/s. The formula for a rocket's maximum velocity v in kilometers per second is $v = -0.0098t + c \ln R$. The booster rocket fires for t seconds and the velocity of the exhaust is c km/s. The ratio of the mass of the rocket filled with fuel to its mass without fuel is R. Suppose the rocket shown in the photo has a mass ratio of 25, a firing time of 100 s and an exhaust velocity as shown. Can the spacecraft attain a stable orbit 300 km above Earth?**

Plan

How can you solve the problem?
Use the given formula to find the maximum velocity v of the spacecraft. If $v \geq 7.7$, the spacecraft can attain a stable orbit.

Let $R = 25$, $c = 2.8$, and $t = 100$. Find v.

$v = -0.0098t + c \ln R$	Use the formula.
$= -0.0098(100) + 2.8 \ln 25$	Substitute.
$\approx -0.98 + 2.8(3.219)$	Use a calculator.
≈ 8.0	Simplify.

The maximum velocity of 8.0 km/s is greater than the 7.7 km/s needed for a stable orbit. Therefore, the spacecraft can attain a stable orbit 300 km above Earth.

Got It? **4. a.** A booster rocket for a spacecraft has a mass ratio of about 15, an exhaust velocity of 2.1 km/s, and a firing time of 30 s. Can the spacecraft achieve a stable orbit 300 km above Earth?

b. **Reasoning** Suppose a rocket, as designed, cannot provide enough velocity to achieve a stable orbit. Could alterations to the rocket make a stable orbit achievable? Explain.

Lesson Check

Do you know HOW?

Write each expression as a single natural logarithm.

1. $4 \ln 3$

2. $\ln 18 - \ln 10$

3. $\ln 3 + \ln 4$

4. $-2 \ln 2$

Solve each equation.

5. $\ln 5x = 4$

6. $\ln (x - 7) = 2$

7. $2 \ln x = 4$

8. $\ln (2 - x) = 1$

Do you UNDERSTAND?

9. **Error Analysis** Describe the error made in solving the equation. Then find the correct solution.

$$\ln 4x = 5$$
$$e^{\ln 4x} = e^5$$
$$4x = 5$$
$$x = \frac{5}{4}$$
$$x = 1.25$$

10. **Reasoning** Can $\ln 5 + \log_2 10$ be written as a single logarithm? Explain your reasoning.

Practice and Problem-Solving Exercises

A Practice

Write each expression as a single natural logarithm.

See Problem 1.

11. $3 \ln 5$

12. $\ln 9 + \ln 2$

13. $\ln 24 - \ln 6$

14. $5 \ln m - 3 \ln n$

15. $\frac{1}{3}(\ln x + \ln y) - 4 \ln z$

16. $\ln a - 2 \ln b + \frac{1}{3}\ln c$

17. $4 \ln 8 + \ln 10$

18. $\ln 3 - 5 \ln 3$

19. $2 \ln 8 - 3 \ln 4$

Solve each equation. Check your answers.

See Problem 2.

20. $\ln 3x = 6$

21. $\ln x = -2$

22. $\ln(4x - 1) = 36$

23. $1.1 + \ln x^2 = 6$

24. $\ln \frac{x-1}{2} = 4$

25. $\ln 4r^2 = 3$

26. $2 \ln 2x^2 = 1$

27. $\ln(2m + 3) = 8$

28. $\ln(t - 1)^2 = 3$

Use natural logarithms to solve each equation.

See Problem 3.

29. $e^x = 18$

30. $e^{\frac{x}{5}} + 4 = 7$

31. $e^{2x} = 12$

32. $e^{\frac{x}{2}} = 5$

33. $e^{x+1} = 30$

34. $e^{2x} = 10$

35. $e^{3x} + 5 = 6$

36. $e^{\frac{x}{9}} - 8 = 6$

37. $7 - 2e^{\frac{x}{2}} = 1$

Space **For Exercises 38 and 39, use $v = -0.0098t + c \ln R$, where v is the velocity of the rocket, t is the firing time, c is the velocity of the exhaust, and R is the ratio of the mass of the rocket filled with fuel to the mass of the rocket without fuel.**

See Problem 4.

38. Find the velocity of a spacecraft whose booster rocket has a mass ratio of 20, an exhaust velocity of 2.7 km/s, and a firing time of 30 s. Can the spacecraft achieve a stable orbit 300 km above Earth?

39. A rocket has a mass ratio of 24 and an exhaust velocity of 2.5 km/s. Determine the minimum firing time for a stable orbit 300 km above Earth.

B Apply

40. **Think About a Plan** By measuring the amount of carbon-14 in an object, a paleontologist can determine its approximate age. The amount of carbon-14 in an object is given by $y = ae^{-0.00012t}$, where a is the amount of carbon-14 originally in the object, and t is the age of the object in years. In 2003, a bone believed to be from a dire wolf was found at the La Brea Tar Pits. The bone contains 14% of its original carbon-14. How old is the bone?

- What numbers should you substitute for y and t?
- What properties of logarithms and exponents can you use to solve the equation?

41. **Archaeology** A fossil bone contains 25% of its original carbon-14. What is the approximate age of the bone?

Simplify each expression.

42. $\ln 1$

43. $\frac{\ln e}{4}$

44. $\frac{\ln e^2}{2}$

45. $\ln e^{83}$

46. $\ln e$

47. $\ln e^2$

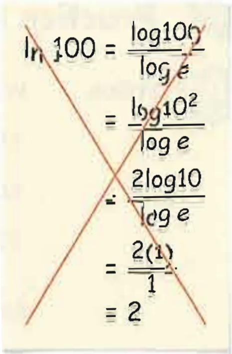
48. $\ln e^{10}$

49. $10 \ln e$

50. $\ln e^3$

51. $\frac{\ln e^4}{8}$

52. **Error Analysis** A student has broken the natural logarithm key on his calculator, so he decides to use the Change of Base Formula to find ln 100. Explain his error and find the correct answer.



$$\ln 100 = \frac{\log 100}{\log e} = \frac{\log 10^2}{\log e} = \frac{2\log 10}{\log e} = \frac{2(1)}{1} = 2$$

53. **Satellite** The battery power available to run a satellite is given by the formula $P = 50\,e^{-\frac{t}{250}}$, where P is power in watts and t is time in days. For how many days can the satellite run if it requires 15 watts of power?

Determine whether each statement is *always*, *sometimes*, or *never* true.

54. $\ln e^x \geq 1$

55. $\ln e^x = \ln e^x + 1$

56. $\ln t = \log_e t$

57. **Space** Use the formula for maximum velocity $v = -0.0098t + c \ln R$. Find the mass ratio of a rocket with an exhaust velocity of 3.1 km/s, a firing time of 50 s, and a maximum shuttle velocity of 6.9 km/s.

Biology **The formula $H = \frac{1}{r}(\ln P - \ln A)$ models the number of hours it takes a bacteria culture to decline, where H is the number of hours, r is the rate of decline, P is the initial bacteria population, and A is the reduced bacteria population.**

58. A scientist determines that an antibiotic reduces a population of 20,000 bacteria to 5000 in 24 hours. Find the rate of decline caused by the antibiotic.

59. A laboratory assistant tests an antibiotic that causes a rate of decline of 0.14. How long should it take for a population of 8000 bacteria to shrink to 500?

Challenge

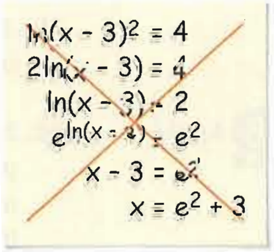
Solve each equation.

60. $\frac{1}{3}\ln x + \ln 2 - \ln 3 = 3$

61. $\ln(x + 2) - \ln 4 = 3$

62. $2e^{x-2} = e^x + 7$

63. **Error Analysis** Consider the solution to the equation $\ln(x - 3)^2 = 4$ at the right. In Problem 2 you saw that there are two solutions to this equation, $3 + e^2$ and $3 - e^2$. Why do you get only one solution using this method?



$$\ln(x - 3)^2 = 4$$
$$2\ln(x - 3) = 4$$
$$\ln(x - 3) = 2$$
$$e^{\ln(x-3)} = e^2$$
$$x - 3 = e^2$$
$$x = e^2 + 3$$

64. **Technology** In 2008, there were about 1.5 billion Internet users. That number is projected to grow to 3.5 billion in 2015.

- **a.** Let t represent the time, in years, since 2008. Write a function of the form $y = ae^{ct}$ that models the expected growth in the population of Internet users.
- **b.** In what year are there 2 billion Internet users?
- **c.** In what year are there 5 billion Internet users?
- **d.** Solve your equation for t.
- **e.** **Writing** Explain how you can use your equation from part (d) to verify your answers to parts (b) and (c).

65. Physics The function $T(t) = T_r + (T_i - T_r)e^{kt}$ models Newton's Law of Cooling. $T(t)$ is the temperature of a heated substance t minutes after it has been removed from a heat (or cooling) source. T_i is the substance's initial temperature, k is a constant for that substance, and T_r is room temperature.

a. The initial surface temperature of a beef roast is 236°F and room temperature is 72°F. If $k = -0.041$, how long will it take for this roast to cool to 100°F?

b. Graphing Calculator Write and graph an equation that you can use to check your answer to part (a). Use your graph to complete the table below.

Temperature (°F)	225	200	175	150	125	100	75
Minutes Later	■	■	■	■	■	■	■

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.8.7 **66.** An investment of \$750 will be worth \$1500 after 12 years of continuous compounding at a fixed interest rate. What percent is the interest rate?

MA.912.A.8.2 **67.** What is $\log 33{,}000 - \log 99 + \log 30$?

MA.912.A.2.8 **68.** If $f(x) = 5 - x^2$ and $g(x) = x^2 - 3$, what is $(g \circ f)(6)$?

MA.912.A.7.4 **69.** What is the positive root of $y = 2x^2 - 35x - 57$?

MA.912.A.1.6 **70.** What is the real part of $3 + 2i$?

MA.912.A.6.1 **71.** What is $\frac{\sqrt{36}}{\sqrt{4}}$?

Mixed Review

Solve each equation. See Lesson 7-5.

72. $3^{2x} = 6561$ **73.** $7^x - 2 = 252$ **74.** $25^{2x+1} = 144$

75. $\log 3x = 4$ **76.** $\log 5x + 3 = 3.7$ **77.** $\log 9 - \log x + 1 = 6$

Find the inverse of each function. Is the inverse a function? See Lesson 6-7.

78. $y = 5x + 7$ **79.** $y = 2x^3 + 10$ **80.** $y = -x^2 + 5$ **81.** $y = 3x + 2$

Get Ready! **To prepare for Lesson 8-1, do Exercises 82–84.**

For Exercises 82–84, y varies directly with x. See Lesson 2-2.

82. If $x = 2$ when $y = 4$, find y when $x = 5$.

83. If $x = 1$ when $y = 5$, find y when $x = 3$.

84. If $x = 10$ when $y = 3$, find y when $x = 4$.

Concept Byte

For Use With Lesson 7-6

EXTENSION

Exponential and Logarithmic Inequalities

Sunshine State Standards

MA.912.A.8.3 Graph exponential and logarithmic functions.

MA.912.A.8.7 Solve applications of exponential growth and decay.

You can use the graphing and table capabilities of your calculator to solve problems involving exponential and logarithmic inequalities.

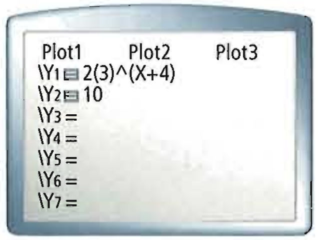
Example 1

Solve $2(3)^{x+4} > 10$ using a graph.

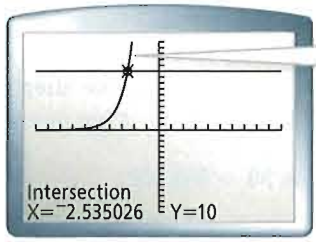
Step 1 Define **Y1** and **Y2**.

Step 2 Make a graph and find the point of intersection.

Step 3 Identify the x-values that make the inequality true.



```
Plot1     Plot2     Plot3
\Y1■2(3)^(X+4)
\Y2■10
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



```
Intersection
X=⁻2.535026   Y=10
```

$2(3)^{x+4} > 10$

The solution is $x > -2.535$.

Exercises

Solve each inequality using a graph.

1. $4(3)^{x+1} > 6$

2. $\log x + 3\log(x-1) < 4$

3. $3(2)^{x+2} \geq 5$

4. $x + 1 < 12\log x$

5. $2(3)^{x-4} > 7$

6. $\log x + 2\log(x-1) < 1$

7. $4(2)^{x-1} \leq 5$

8. $2\log x + 4\log(x+3) > 3$

9. $5(4)^{x-1} < 2$

10. **Bacteria Growth** Scientists are growing bacteria in a laboratory. They start with a known population of bacteria and measure how long it takes the population to double.

a. Write an exponential function that models the population in Sample A as a function of time in hours.

b. Write an exponential function that models the population in Sample B as a function of time in hours.

c. Write an inequality that models the population in Sample B overtaking the population in Sample A.

d. Use a graphing calculator to solve the inequality in part (c).

Bacteria Population

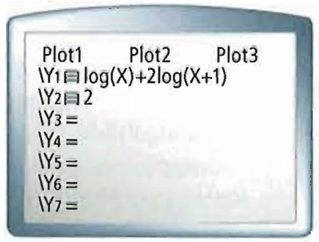
Sample	Initial Population	Doubling Time (in hours)
Sample A	200,000	1
Sample B	50,000	0.5

11. **Writing** Describe the solution sets to the inequality $x + c < \log x$ as c varies over the real numbers.

Example 2

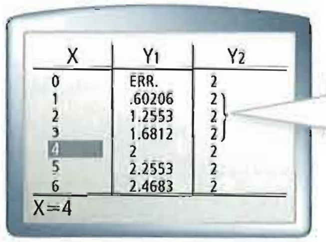
Solve $\log x + 2\log(x + 1) < 2$ using a table.

Step 1 Define **Y1** and **Y2**.



```
Plot1    Plot2    Plot3
\Y1 ▤ log(X)+2log(X+1)
\Y2 ▤ 2
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

Step 2 Make a table and examine the values.



```
 X  | Y1     | Y2
 0  | ERR.   | 2
 1  | .60206 | 2
 2  | 1.2553 | 2
 3  | 1.6812 | 2
 4  | 2      | 2
 5  | 2.2553 | 2
 6  | 2.4683 | 2
X=4
```

Y1 < Y2 for these *x*-values.

Step 3 Identify the *x*-values that make the inequality true.

The solution is $0 < x < 4$.

Exercises

Solve each inequality using a table.
(Hint: For more accurate results, set ΔTbl = 0.001.)

12. $\log x + \log(x + 1) < 3$

13. $3(2)^{x+1} > 5$

14. $\log x + 5\log(x - 1) \geq 3$

15. $5(3)^x \leq 2$

16. $3\log x + \log(x + 2) > 1$

17. $2(4)^{x+3} \leq 8$

Barometric Pressure **Average barometric pressure varies with the altitude of a location. The greater the altitude is, the lower the pressure. The altitude A is measured in feet above sea level. The barometric pressure P is measured in inches of mercury (in. Hg). The altitude can be modeled by the function $A(P) = 90{,}000 - 26{,}500 \ln P$.**

18. What is a reasonable domain of the function? What is the range of the function?

19. **Graphing Calculator** Use a graphing calculator to make a table of function values. Use **TblStart** = 30 and **ΔTbl** = −1.

20. Write an equation to find what average pressure the model predicts at sea level, or $A = 0$. Use your table to solve the equation.

21. Kilimanjaro is a mountain in Tanzania that formed from three extinct volcanoes. The base of the mountain is at 3000 ft above sea level. The peak is at 19,340 ft above sea level. On Kilimanjaro, $3000 \leq A(P) \leq 19{,}340$ is true for the altitude. Write an inequality from which you can find minimum and maximum values of normal barometric pressure on Kilimanjaro. Use a table and solve the inequality for P.

22. Denver, Colorado, is nicknamed the "Mile High City" because its elevation is about 1 mile, or 5280 ft, above sea level. The lowest point in Phoenix, Arizona, is 1117 ft above sea level. Write an inequality that describes the range of $A(P)$ as you drive from Phoenix to Denver. Then solve the inequality for P. (Assume that you never go lower than 1117 ft and you never go higher than 5280 ft.)

7-1 Exploring Exponential Models

Quick Review

The general form of an **exponential function** is $y = ab^x$, where x is a real number, $a \neq 0$, $b > 0$, and $b \neq 1$. When $b > 1$, the function models **exponential growth**, and b is the **growth factor**. When $0 < b < 1$, the function models **exponential decay**, and b is the **decay factor**. The y-intercept is $(0, a)$.

Example

Determine whether $y = 2(1.4)^x$ is an example of exponential growth or decay. Then, find the y-intercept.

Since $b = 1.4 > 1$, the function represents exponential growth.

Since $a = 2$, the y-intercept is $(0, 2)$.

Exercises

Determine whether each function is an example of exponential growth or decay. Then, find the y-intercept.

6. $y = 5^x$

7. $y = 2(4)^x$

8. $y = 0.2(3.8)^x$

9. $y = 3(0.25)^x$

10. $y = \frac{25}{7}\left(\frac{7}{5}\right)^x$

11. $y = 0.0015(10)^x$

12. $y = 2.25\left(\frac{1}{3}\right)^x$

13. $y = 0.5\left(\frac{1}{4}\right)^x$

Write a function for each situation. Then find the value of each function after five years. Round to the nearest dollar.

14. A $12,500 car depreciates 9% each year.

15. A baseball card bought for $50 increases 3% in value each year.

7-2 Properties of Exponential Functions

Quick Review

Exponential functions can be translated, stretched, compressed, and reflected.

The graph of $y = ab^{x-h} + k$ is the graph of the parent function $y = b^x$ stretched or compressed by a factor $|a|$, reflected across the x-axis if $a < 0$, and translated h units horizontally and k units vertically.

The **continuously compounded interest** formula is $A = Pe^{rt}$, where P is the principal, r is the annual interest rate, and t is time in years.

Example

How does the graph of $y = -3^x + 1$ compare to the graph of the parent function?

The parent function is $y = 3^x$.

Since $a = -1$, the graph is reflected across the x-axis.

Since $k = 1$, it is translated up 1 unit.

Exercises

How does the graph of each function compare to the graph of the parent function?

16. $y = 5(2)^{x+1} + 3$

17. $y = -2\left(\frac{1}{3}\right)^{x-2}$

Find the amount in a continuously compounded account for the given conditions.

18. principal: $1000
annual interest rate: 4.8%
time: 2 years

19. principal: $250
annual interest rate: 6.2%
time: 2.5 years

Evaluate each expression to four decimal places.

20. e^{-3}

21. e^{-1}

22. e^5

23. $e^{-\frac{1}{2}}$

7-3 Logarithmic Functions as Inverses

Quick Review

If $x = b^y$, then $\log_b x = y$. The **logarithmic function** is the inverse of the exponential function, so the graphs of the functions are reflections of one another across the line $y = x$. Logarithmic functions can be translated, stretched, compressed, and reflected, as represented by $y = a\log_b(x - h) + k$, similarly to exponential functions.

When $b = 10$, the logarithm is called a **common logarithm**, which you can write as $\log x$.

Example

Write $5^{-2} = 0.04$ in logarithmic form.

If $y = b^x$, then $\log_b y = x$.

$y = 0.04$, $b = 5$ and $x = -2$.

So, $\log_5 0.04 = -2$.

Exercises

Write each equation in logarithmic form.

24. $6^2 = 36$

25. $2^{-3} = 0.125$

26. $3^3 = 27$

27. $10^{-3} = 0.001$

Evaluate each logarithm.

28. $\log_2 64$

29. $\log_3 \frac{1}{9}$

30. $\log 0.00001$

31. $\log_2 1$

Graph each logarithmic function.

32. $y = \log_3 x$

33. $y = \log x + 2$

34. $y = 3\log_2(x)$

35. $y = \log_5(x + 1)$

How does the graph of each function compare to the graph of the parent function?

36. $y = 3\log_4(x + 1)$

37. $y = -\ln x + 2$

7-4 Properties of Logarithms

Quick Review

For any positive numbers, m, n, and b where $b \neq 1$, each of the following statements is true. Each can be used to rewrite a logarithmic expression.

- $\log_b mn = \log_b m + \log_b n$, by the Product Property
- $\log_b \frac{m}{n} = \log_b m - \log_b n$, by the Quotient Property
- $\log_b m^n = n\log_b m$, by the Power Property

Example

Write $2\log_2 y + \log_2 x$ as a single logarithm. Identify any properties used.

$$2\log_2 y + \log_2 x$$

$$= \log_2 y^2 + \log_2 x \quad \text{Power Property}$$

$$= \log_2 xy^2 \quad \text{Product Property}$$

Exercises

Write each expression as a single logarithm. Identify any properties used.

38. $\log 8 + \log 3$

39. $\log_2 5 - \log_2 3$

40. $4\log_3 x + \log_3 7$

41. $\log x - \log y$

42. $\log 5 - 2\log x$

43. $3\log_4 x + 2\log_4 x$

Expand each logarithm. State the properties of logarithms used.

44. $\log_4 x^2y^3$

45. $\log 4s^4t$

46. $\log_3 \frac{2}{x}$

47. $\log(x + 3)^2$

48. $\log_2(2y - 4)^3$

49. $\log \frac{z^2}{5}$

Use the Change of Base Formula to evaluate each expression.

50. $\log_2 7$

51. $\log_3 10$

7-5 Exponential and Logarithmic Equations

Quick Review

An equation in the form $b^{cx} = a$, where the exponent includes a variable, is called an **exponential equation**. You can solve exponential equations by taking the logarithm of each side of the equation. An equation that includes one or more logarithms involving a variable is called a **logarithmic equation**.

Example

Solve and round to the nearest ten-thousandth.

$$\begin{aligned} 6^{2x} &= 75 \\ \log 6^{2x} &= \log 75 && \text{Take the logarithm of both sides.} \\ 2x \log 6 &= \log 75 && \text{Power Property of Logarithms} \\ x &= \frac{\log 75}{2 \log 6} && \text{Divide both sides by 2 log 6.} \\ x &\approx 1.2048 && \text{Evaluate using a calculator.} \end{aligned}$$

Exercises

Solve each equation. Round to the nearest ten-thousandth.

52. $25^{2x} = 125$

53. $3^x = 36$

54. $7^{x-3} = 25$

55. $5^x + 3 = 12$

56. $\log 3x = 1$

57. $\log_2 4x = 5$

58. $\log x = \log 2x^2 - 2$

59. $2 \log_3 x = 54$

Solve by graphing. Round to the nearest ten-thousandth.

60. $5^{2x} = 20$

61. $3^{7x} = 160$

62. $6^{3x+1} = 215$

63. $0.5^x = 0.12$

64. A culture of 10 bacteria is started, and the number of bacteria will double every hour. In about how many hours will there be 3,000,000 bacteria?

7-6 Natural Logarithms

Quick Review

The inverse of $y = e^x$ is the **natural logarithmic function** $y = \log_e x = \ln x$. You solve natural logarithmic equations in the same way as common logarithmic equations.

Example

Use natural logarithms to solve $\ln x - \ln 2 = 3$.

$$\begin{aligned} \ln x - \ln 2 &= 3 \\ \ln \tfrac{x}{2} &= 3 && \text{Quotient Property} \\ \tfrac{x}{2} &= e^3 && \text{Rewrite in exponential form.} \\ \tfrac{x}{2} &\approx 20.0855 && \text{Use a calculator to find } e^3. \\ x &\approx 40.171 && \text{Simplify.} \end{aligned}$$

Exercises

Solve each equation. Check your answers.

65. $e^{3x} = 12$

66. $\ln x + \ln(x + 1) = 2$

67. $2 \ln x + 3 \ln 2 = 5$

68. $\ln 4 - \ln x = 2$

69. $4e^{(x-1)} = 64$

70. $3 \ln x + \ln 5 = 7$

71. An initial investment of $350 is worth $429.20 after six years of continuous compounding. Find the annual interest rate.

7 Chapter Test

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

Determine whether each function is an example of exponential growth or decay. Then find the y-intercept.

1. $y = 3(0.25)^x$

2. $y = 2(6)^{-x}$

3. $y = 0.1(10)^x$

4. $y = 3e^x$

Describe how the graph of each function is related to the graph of its parent function. Then find the domain, range, and asymptotes.

5. $y = 3^x + 2$

6. $y = \left(\frac{1}{2}\right)^{x+1}$

7. $y = -(2)^{x+2}$

Write each equation in logarithmic form.

8. $5^4 = 625$

9. $e^0 = 1$

Evaluate each logarithm.

10. $\log_2 8$

11. $\log_7 7$

12. $\log_5 \frac{1}{125}$

13. $\log_{11} 1$

Graph each logarithmic function. Compare each graph to the graph of its parent function. List each function's domain, range, y-intercept, and asymptotes.

14. $y = \log_3 (x - 1)$

15. $y = \frac{1}{2}\log_3 (x + 2)$

16. $y = 1 - \log_2 x$

Write each logarithmic expression as a single logarithm.

17. $\log_2 4 + 3 \log_2 9$

18. $3 \log a - 2 \log b$

Expand each logarithm.

19. $\log_7 \frac{a}{b}$

20. $\log 3x^3y^2$

Use the properties of logarithms to evaluate each expression.

21. $\log_9 27 - \log_9 9$

22. $2 \log 5 + \log 40$

Solve each equation.

23. $(27)^{3x} = 81$

24. $3^{x-1} = 24$

25. $2e^{3x} = 16$

26. $2 \log x = -4$

Use the Change of Base Formula to rewrite each expression using common logarithms.

27. $\log_3 16$

28. $\log_2 10$

29. $\log_7 8$

30. $\log_4 9$

Use the properties of logarithms to simplify and solve each equation. Round to the nearest thousandth.

31. $\ln 2 + \ln x = 1$

32. $\ln (x + 1) + \ln (x - 1) = 4$

33. $\ln (2x - 1)^2 = 7$

34. $3 \ln x - \ln 2 = 4$

Do you UNDERSTAND?

35. **Writing** Show that solving the equation $3^{2x} = 4$ by taking the common logarithm of each side is equivalent to solving it by taking the logarithm with base 3 of each side.

36. **Open-Ended** Give an example of an exponential function that models exponential growth and an example of an exponential function that models exponential decay.

37. **Investment** You put $1500 into an account that pays 7% annual interest compounded continuously. How long will it be before you have $2000 in your account?

7

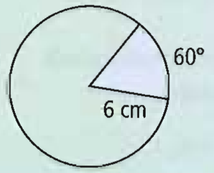
Sunshine State Standards Practice

TIPS FOR SUCCESS

Some problems ask you to find lengths of arcs or areas of sectors. Read the question at the right. Then follow the tips to answer the sample question.

TIP 1

Draw a diagram



Let $f(x) = \pi x^2$ represent the area of a circle with radius x. Let $g(x) = \frac{60x}{360}$ represent the area of a 60° sector of a circle with area x. A circle with radius 6 centimeters has a sector measuring 60°. What is the area of this sector $(g \circ f)(x)$?

Ⓐ $6\pi \text{ cm}^2$

Ⓑ $12\pi \text{ cm}^2$

Ⓒ $24\pi \text{ cm}^2$

Ⓓ $36\pi \text{ cm}^2$

TIP 2

Find $f(x)$ first.

$$\begin{aligned} f(x) &= \pi x^2 \\ &= \pi \cdot 6^2 \\ &= 36\pi \end{aligned}$$

Think It Through

$$g(x) = \frac{60x}{360}$$

$$\begin{aligned} (g \circ f)(x) &= \frac{60(36\pi)}{360} \\ &= \frac{36\pi}{6} \\ &= 6\pi \end{aligned}$$

The correct answer is A.

Vocabulary Builder

As you solve problems, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. growth factor

B. asymptote

C. logarithmic function

D. exponential equation

I. the inverse of an exponential function

II. a line that a graph approaches as x or y increases in absolute value

III. the value of b in $y = ab^x$, when $b > 1$

IV. an equation of the form $b^{cx} = a$, where the exponent includes a variable

Multiple Choice

Read each question. Then write the letter of the correct answer on your page.

1. The population of a town is modeled by the equation $P = 16{,}581e^{0.02t}$ where P represents the population t years after 2000. According to the model, what will the population of the town be in 2020?

Ⓐ 16,916

Ⓑ 17,258

Ⓒ 20,252

Ⓓ 24,736

2. If $i = \sqrt{-1}$, then which expression is equal to $9i(13i)$?

Ⓕ -117

Ⓖ $117i$

Ⓗ 117

Ⓘ $-117i$

3. Which expression is equivalent to $\log_5 32$?

Ⓐ $\log 5 + \log 32$

Ⓑ $\log 5 - \log 32$

Ⓒ $(\log 5)(\log 32)$

Ⓓ $\frac{\log 32}{\log 5}$

4. The table shows the height of a ball that was tossed into the air. Which equation best models the relationship between time t and the height of the ball h?

Time (seconds)	0	0.25	0.5	0.75
Height (feet)	4	10.5	15	17.5

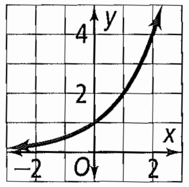
Ⓕ $h = 26t + 4$

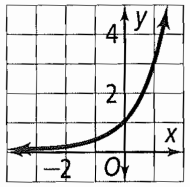
Ⓖ $h = -16t^2 + 30t + 4$

Ⓗ $h = 4t^2$

Ⓘ $h = -16t^2 + 4$

5. Which is the graph of $y = 3^x$?

Ⓐ


Ⓒ


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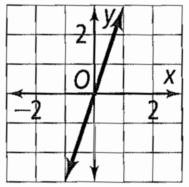
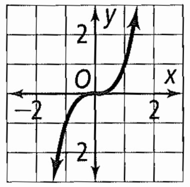


Ⓓ


6. Which exponential function is equivalent to $y = \log_3 x$?

Ⓕ $y = 3^x$

Ⓗ $y = x^3$

Ⓖ $y = \frac{x}{3}$

Ⓘ $x = 3^y$

7. Simplify $\left(\frac{3x^{\frac{1}{2}}y^4}{x^2y^3}\right)^2$.

Ⓐ $\frac{3y^2}{x^3}$

Ⓒ $\frac{9y^2}{x^3}$

Ⓑ $3x^3y^2$

Ⓓ $9x^3y^2$

8. Solve the mass energy equivalence formula $e = mc^2$ for c.

Ⓕ $c = e^2m$

Ⓗ $c = \sqrt{\frac{e}{m}}$

Ⓖ $c = \sqrt{\frac{m}{e}}$

Ⓘ $c = \sqrt{(e - m)}$

9. What is the quotient of $(x^3 + 2x^2 - x + 6) \div (x + 3)$?

Ⓐ $x^2 + 5x + 14$, R 42

Ⓒ $x^3 + 5x^2 + 14x + 42$

Ⓑ $x^2 - x + 2$

Ⓓ $x^2 + x - 2$

10. On a certain night, a restaurant employs x servers at \$25 per hour and y bus persons at \$8 per hour. The total hourly cost for the restaurant's 12 employees that night is \$249. The following system of equations can be used to find the number of servers and the number of bus persons at work.

$$\begin{cases} 25x + 8y = 249 \\ x + y = 12 \end{cases}$$

Based on the solution of the system of equations, which of the following can you conclude?

Ⓕ Fewer than 2 bus persons were working.

Ⓖ More than ten servers were working.

Ⓗ 50% of the people working were bus persons.

Ⓘ 75% of the people working were servers.

11. Which polynomial equation has the real roots of $-3, 1, 1$, and $\frac{3}{2}$?

Ⓐ $x^4 - \frac{1}{2}x^3 - \frac{13}{2}x^2 + \frac{21}{2}x - \frac{9}{2} = 0$

Ⓑ $x^4 - \frac{1}{2}x^3 - \frac{17}{2}x^2 - 10x - \frac{9}{2} = 0$

Ⓒ $x^4 + x^3 - 5x^2 + 3x - \frac{3}{2} = 0$

Ⓓ $(x - 3)(x + 1)(x + 1)\left(x + \frac{3}{2}\right) = 0$

12. What is the factored form of $2x^3 + 5x^2 - 12x$?

Ⓕ $x(2x - 3)(x + 4)$

Ⓖ $(2x^2 - 3)(x + 4)$

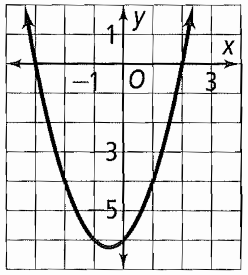
Ⓗ $x(2x + 4)(x - 3)$

Ⓘ $(2x - 4)(x + 3)$

13. Simplify $5\sqrt[3]{x^2} + 3\sqrt[3]{x^2}$.

Ⓐ $8\sqrt[3]{x^2}$ Ⓑ $8\sqrt[6]{x^2}$ Ⓒ $8\sqrt[3]{x^4}$ Ⓓ $8\sqrt[6]{x^4}$

14. What is the equation of the function graphed below?



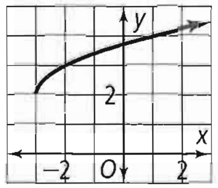
Ⓕ $y = (x + 2)(x - 3)$

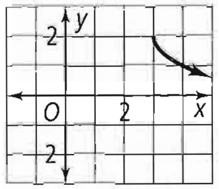
Ⓗ $y = (x + 3)(x - 2)$

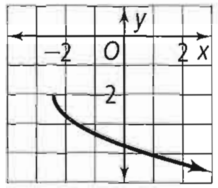
Ⓖ $y = (x - 6)^2$

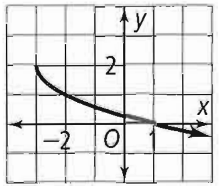
Ⓘ $y = (x - 1)(x + 5)$

15. Which graph shows $y = -\sqrt{x + 3} + 2$, a transformation of the radical parent function $y = \sqrt{x}$?

Ⓐ


Ⓒ


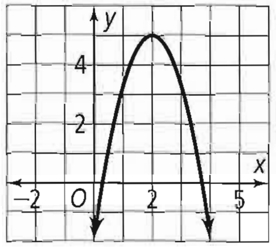
Ⓑ


Ⓓ


16. Which function would have a graph whose end behavior is up and up?

Ⓕ $f(x) = x^3 + 1$
Ⓗ $f(x) = 5x^2 - 7$
Ⓖ $f(x) = -2x^4$
Ⓘ $f(x) = -3x$

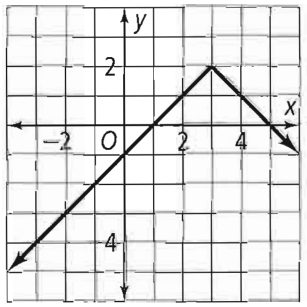
17. What is the domain of the parabola?



Ⓐ $0 < x < 4$
Ⓒ $x \leq 5$
Ⓑ $0 \leq x \leq 4$
Ⓓ all real numbers

18. A transformation of the parent absolute value function is shown in the graph. Which equation represents the graph?

Ⓕ $y = -|x + 3| + 2$
Ⓖ $y = |-x - 3| + 2$
Ⓗ $y = -|x + 2| + 3$
Ⓘ $y = -|x - 3| + 2$



19. Which set of numbers could not be the solution to a quadratic equation with real coefficients?

Ⓐ 4, −4
Ⓒ 0
Ⓑ $-4i, 4$
Ⓓ $2i, -2i$

20. To use a toddler swing, a child must weigh at least 15 lb and no more than 35 lb. Which absolute value inequality describes acceptable weights of a child who uses the swing?

Ⓕ $|x - 10| \leq 25$
Ⓗ $|25 - 10| \leq x$
Ⓖ $|x - 25| \leq 10$
Ⓘ $|x - 25| \geq 10$

GRIDDED RESPONSE

21. What number do you add to each side of the equation when you solve $1 = x^2 + 3x$ by completing the square?

22. What is the solution of the equation $\log_9 x = \log_6 x$?

23. A savings account pays 4.62% annual interest, compounded continuously. After approximately how many years will a principal of $500 double?

24. The graph of a polynomial has x-intercepts at $(-3, 0)$, $(-1, 0)$, and $(1, 0)$. What is the least possible degree of the polynomial?

25. Evaluate $\log_4 8$.

26. Solve $4^{2x} = 32$.

27. How many different real solutions are there for the equation $4x^2 = -4x - 4$?

28. Use the Fundamental Theorem of Algebra to determine the total number of complex zeros of $f(x) = x^2 - 3x^5 + 4x - x^7 - 44$.

29. y varies directly with x and $y = 30$ when $x = 4$. What is y when $x = 7$?

30. What is the y-intercept of the line that passes through the point $(-2, 7)$ and is parallel to $y = 3x + 5$?

31. What is the solution of $\sqrt{x + 2} = x$?

32. Simplify $9^{\frac{5}{2}}$.

33. What is the x-intercept of $f(x) = x^3 - x^2 + 1$? Round the answer to the nearest hundredth.

CHAPTER 8

Get Ready!

Lesson 2-2

Using Direct Variation

For each direct variation, find the constant of variation. Then find the value of y when $x = -3$.

1. $y = 4$ when $x = 3$

2. $y = 1$ when $x = -1.5$

3. $y = -5$ when $x = \frac{3}{2}$

4. $y = -16$ when $x = 7$

Lesson 4-4

Factoring Quadratic Expressions

Factor each expression.

5. $x^2 + x - 6$

6. $4x^2 + 17x + 15$

7. $9x^2 - 25$

8. $x^2 - 12x + 36$

9. $3x^2 + 10x + 8$

10. $x^2 - 5x + 6$

Lesson 4-5

Solving Quadratic Equations

Solve each equation.

11. $x^2 + 7x - 8 = 0$

12. $\frac{1}{4}x^2 + \frac{7}{2}x = -12$

13. $3x^2 = 18x - 24$

14. $9x^2 + 6x = 0$

15. $4x^2 + 16 = 34x$

16. $x^2 - 13x - 30 = 0$

Looking Ahead Vocabulary

17. If you need to drive 30 miles, you have many options. For instance, you can drive 15 miles per hour for 2 hours, 30 miles per hour for 1 hour, or 60 miles per hour for half an hour. Notice that when you double your speed, it takes half as much time to get to your destination. Mathematicians describe this kind of relationship as an *inverse variation*. Why do you suppose they use the word *inverse* to describe it?

18. Suppose you are hiking on a trail and find that the bridge over the river has been washed out, making a gap or *discontinuity* in the trail. Graphs can have gaps too. Sketch what you think a graph with a discontinuity might look like.

CHAPTER 8

Rational Functions

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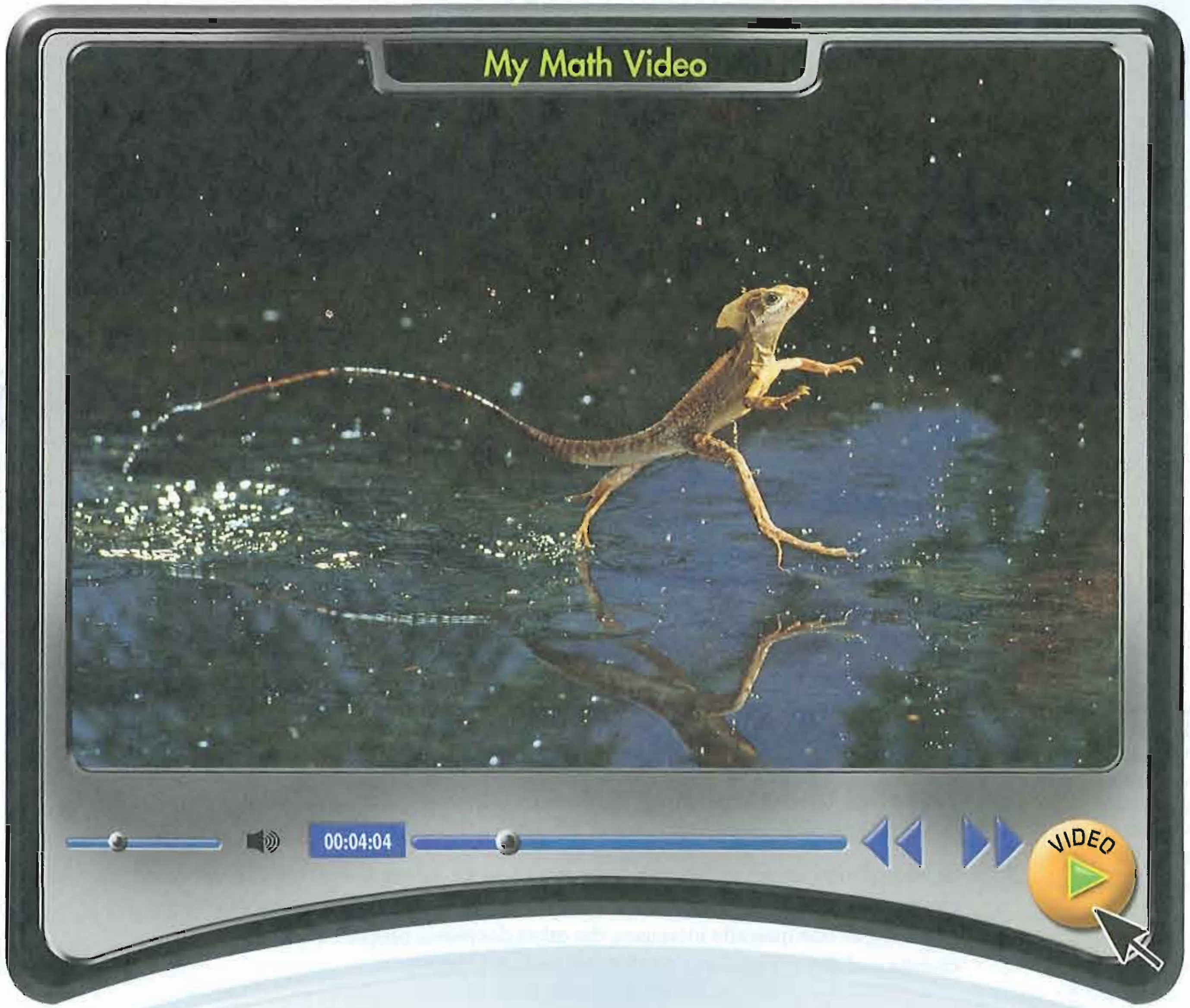
Rational functions help explain how surface tension allows some animals to tread across a pond's surface.

How can you graph rational functions and solve rational equations? You will learn how in this chapter.

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
combined variation, *p. 501*	variación combinada
complex fraction, *p. 536*	fracción compleja
continuous graph, *p. 516*	gráfica continua
discontinuous graph, *p. 516*	gráfica discontinua
inverse variation, *p. 498*	variación inversa
joint variation, *p. 501*	variación conjunta
point of discontinuity, *p. 516*	punto de discontinuidad
rational equation, *p. 542*	ecuación racional
rational expression, *p. 527*	expresión racional
rational function, *p. 515*	función racional
reciprocal function, *p. 507*	función recíproca



BIG ideas

1 **Proportionality**

Essential Question Are two quantities inversely proportional if an increase in one corresponds to a decrease in the other?

2 **Function**

Essential Question What kinds of asymptotes are possible for a rational function?

3 **Equivalence**

Essential Question Are a rational expression and its simplified form equivalent?

Chapter Preview

8-1 Inverse Variation

Sunshine State Standard
MA.912.A.2.12 Solve problems using direct, inverse, and joint variations.

Objectives To recognize and use inverse variation
To use joint and other variations

Mulch is a ground cover that helps keep moisture in the soil.

SOLVE IT! **Getting Ready!**

You have 20 bags of mulch. You plan to spread the mulch from all the bags to make a rectangular layer that is 3-in. thick. How many square feet can you cover? If ℓ and w represent the length and width of the rectangle in feet, what equation relates ℓ and w? Justify your reasoning.

Lesson Vocabulary
- inverse variation
- combined variation
- joint variation

Among all rectangles with a given area, the longer the length of one side, the shorter the length of an adjacent side.

Essential Understanding If a product is constant, a decrease in the value of one factor must accompany an increase in the value of the other factor.

As an equation, direct variation has the form $y = kx$, where $k \neq 0$. **Inverse variation** can have the form $xy = k$, $y = \frac{k}{x}$, or $x = \frac{k}{y}$, where $k \neq 0$. When two quantities vary inversely, as one quantity increases, the other decreases proportionally. For both inverse and direct variation, k is the constant of variation.

Problem 1 Identifying Direct and Inverse Variations

Is the relationship between the variables a *direct variation*, an *inverse variation*, or *neither*? Write function models for the direct and inverse variations.

Think
How can you tell if the quantities vary directly or inversely?
If the product of corresponding x- and y-values is constant, they vary inversely. If the ratio of corresponding x- and y-values is constant, they vary directly.

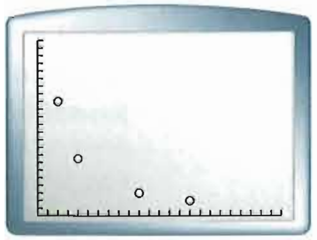
A

x	y
2	15
4	7.5
10	3
15	2

As x increases, y decreases. This might be an inverse relationship. A plot confirms that an inverse relationship is possible. Test to see whether xy is constant.

$2 \cdot 15 = 30 \qquad 4 \cdot 7.5 = 30$

$10 \cdot 3 = 30 \qquad 15 \cdot 2 = 30$



The product of each pair is 30, so $xy = 30$ and y varies inversely with x. The constant of variation is 30 and the function model is $y = \frac{30}{x}$.

Think

Could you model this data with a direct variation?
No; the values of y are decreasing as x increases.

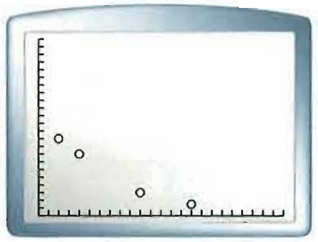
B

x	y
2	10
4	8
10	3
15	1.5

A plot of the points suggests that an inverse relationship is possible. Test to see whether the products of x and y are constant.

$2 \cdot 10 = 20, 4 \cdot 8 = 32, 10 \cdot 3 = 30$, and
$15 \cdot 1.5 = 22.5$

Since the products are not constant, the relationship is not an inverse variation.



Got It? **1.** Is the relationship between the variables a *direct variation*, an *inverse variation*, or *neither*? Write function models for the direct and inverse variations.

a.

x	y
0.2	8
0.5	20
1.0	40
1.5	60

b.

x	y
0.2	40
0.5	16
1.0	8.0
2.0	4.0

c.

x	y
0.5	40
1.2	12
2	10
2.5	6

Dynamic Activity
Direct and Inverse Variation

Problem 2 Determining an Inverse Variation

Suppose x and y vary inversely, and $x = 4$ when $y = 12$.

A What function models the inverse variation?

$y = \frac{k}{x}$ Write the general function form for inverse variation.

$12 = \frac{k}{4}$ Substitute for x and y.

$k = 48$ Solve for k.

The function is $y = \frac{48}{x}$.

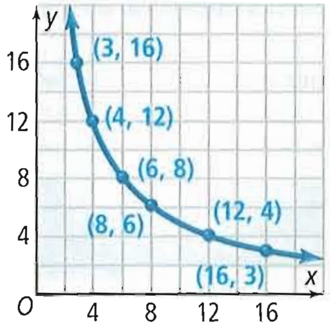
Plan

Is it reasonable to connect the points of this function with a smooth curve?
Yes, $\frac{48}{x}$ is defined for every real number except $x = 0$.

B What does the graph of this function look like?

Make a table of values. Sketch a graph.

x	y
3	16
4	12
6	8
8	6
12	4
16	3



C What is y when $x = 10$?

$y = \frac{48}{x}$ Write the function.

$y = \frac{48}{10}$ Evaluate y for $x = 10$.

$y = 4.8$, when $x = 10$.

Got It? **2.** Suppose x and y vary inversely, and $x = 8$ when $y = -7$.

a. What is the function that models the inverse variation?
b. What does the graph of this function look like?
c. What is y when $x = 2$?

Problem 3 Modeling an Inverse Variation

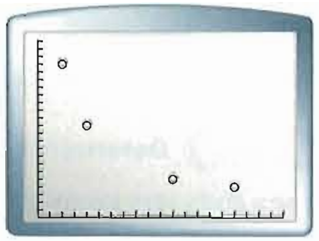
Your math class has decided to pick up litter each weekend in a local park. Each week there is approximately the same amount of litter. The table shows the number of students who worked each of the first four weeks of the project and the time needed for the pickup.

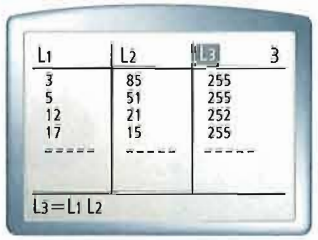
Park Cleanup Project

Number of students (n)	3	5	12	17
Time in minutes (t)	85	51	21	15

A What function models the data?

Step 1 Investigate the data. The more students who help, the less time the cleanup takes. An inverse variation seems appropriate. If this is an inverse variation, then $nt = k$. From the table, nt (or **L1 · L2**) is almost always 255.





```
L1      L2      L3      3
 3       85      255
 5       51      255
 12      21      252
 17      15      255
 -----   -----   -----
L3=L1 L2
```

Think

Can you still use inverse variation to model the data if $12 \times 21 = 252$?
Often, you cannot describe real life data exactly with a function rule. But 252 is close enough to 255 for inverse variation to still be a good model.

Step 2 Determine the model. $nt = 255$

B How many students should there be to complete the project in at most 30 minutes each week?

$nt = 255$	Use the model from part A.
$n(30) = 255$	Substitute for t.
$n = \frac{255}{30} = 8.5$	Solve for n.

There should be at least 9 students to do the job in at most 30 minutes.

Got It? **3.** After a major storm, your math class volunteers to remove debris from yards. The table shows the time t in minutes that it takes a group of n students to remove the debris from an average-sized yard.

Number of students (n)	1	3	5	14
Time in minutes (t)	225	75	45	16

a. What function models the time needed to clear the debris from an average-sized yard relative to the number of students who do the work?
b. How many students should there be to clear debris from an average-sized yard in at most 25 minutes?

You have seen many variation formulas in geometry. Some, like the formula for the perimeter of a square, are simple direct variations. Others, like the volume of a cone, relate three or more variables.

When one quantity varies with respect to two or more quantities, you have a **combined variation**. When one quantity varies directly with two or more quantities, you have **joint variation**. The volume of a cone varies jointly with the area of the base and the height of the cone, $V = kBh$.

take note

Key Concept Combined Variations

Combined Variation	Equation Form
z varies jointly with x and y.	$z = kxy$
z varies jointly with x and y and inversely with w.	$z = \frac{kxy}{w}$
z varies directly with x and inversely with the product wy.	$z = \frac{kx}{wy}$

Problem 4 Using Combined Variation

Multiple Choice **The number of bags of grass seed n needed to reseed a yard varies directly with the area a to be seeded and inversely with the weight w of a bag of seed. If it takes two 3-lb bags to seed an area of 3600 ft^2, how many 3-lb bags will seed 9000 ft^2?**

Ⓐ 3 bags Ⓑ 4 bags Ⓒ 5 bags Ⓓ 6 bags

Plan

How can you write the model?
Write the constant of variation and direct variation variable in the numerator. Write the inverse variation variable in the denominator.

$n = \frac{ka}{w}$	n varies directly with a and inversely with w.
$2 = \frac{3600k}{3}$	Substitute for n, a, and w.
$\frac{(2)(3)}{3600} = k$	Solve for k.
$k = \frac{6}{3600} = \frac{1}{600}$	Simplify.

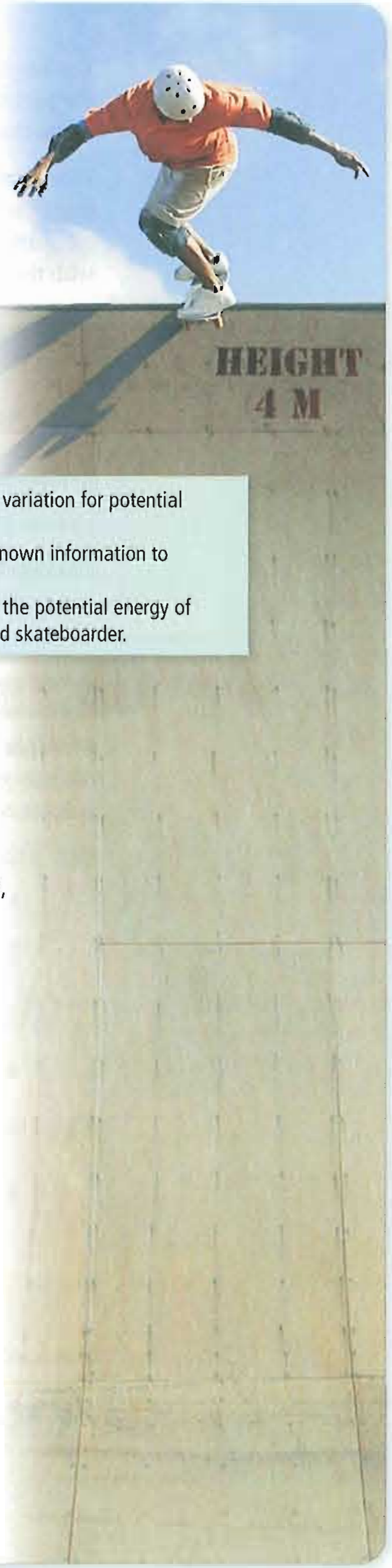
The combined variation equation is $n = \frac{a}{600w}$.

$n = \frac{a}{600w}$	Use the combined variation equation.
$= \frac{9000}{600 \cdot 3}$	Substitute for a and w.
$= 5$	

You need five 3-lb bags to seed 9000 ft^2. The correct choice is C.

Got It? **4.** The number of bags of mulch you need to cover a planting area varies jointly with the area to be mulched a in square feet and the depth of the mulch d in feet. If you need 10 bags to mulch 120 ft^2 to a depth of 3 in., how many bags do you need to mulch 200 ft^2 to a depth of 4 in.?

Problem 5 Applying Combined Variation



Physics **Gravitational potential energy *PE* is a measure of energy. *PE* varies directly with an object's mass *m* and its height *h* in meters above the ground. Physicists use *g* to represent the constant of variation, which is gravity.**

The skateboarder in the photo has a mass of 58 kg and a potential energy of 2273.6 joules. What is the gravitational potential energy of a 65-kg skateboarder on the halfpipe shown?

Know	Need	Plan
• The mass of each skateboarder • The height of each skateboarder • The potential energy of the first skateboarder	The potential energy of the second skateboarder	• Write the variation for potential energy. • Use the known information to find *g*. • Then find the potential energy of the second skateboarder.

Step 1 Write the formula for potential energy. Potential energy varies directly with mass and height. $PE = gmh$

Step 2 Use the given data to find g.

$PE = gmh$	Potential energy formula
$2273.6 = g(58)(4)$	Substitute 2273.6 for PE, 58 for m, and 4 for h.
$9.8 = g$	Solve for g.

Step 3 Use the formula to find the potential energy of the second skateboarder.

$PE = 9.8mh$	Potential energy formula
$= 9.8(65)(4)$	Evaluate for $m = 65$ and $h = 4$.
$= 2548$	Simplify.

The second skateboarder has 2548 joules of potential energy.

Got It? **5. a.** How much potential energy would a 41-kg diver have standing on a 10-m diving platform?

b. **Reasoning** An 80-kg diver stands on a 6-m diving platform. At what height should a 40-kg diver stand to have equal potential energy? Do you need to find the potential energy of either diver to solve this? Explain your reasoning.

Lesson Check

Do you know HOW?

Is the relationship between the variables in each table a *direct variation*, an *inverse variation*, or *neither*? Write equations to model the direct and inverse variations.

1.

x	y
1	6
3	2
12	0.5
15	0.4

2.

u	v
−3	−15
5	25
6	30
16	80

Do you UNDERSTAND?

3. Compare and Contrast Describe the difference between direct variation and inverse variation.

4. Writing Describe how the variables in the given equation are related.

$$p = \frac{kqrt}{s}$$

5. Error Analysis A student described the relationship between the variables in the equation below as d varies directly with r and inversely with t. Correct the error in relating the variables.

$$d = \frac{k\sqrt[3]{r}}{t^2}$$

Practice and Problem-Solving Exercises

A Practice

Is the relationship between the values in each table a *direct variation*, an *inverse variation*, or *neither*? Write equations to model the direct and inverse variations.

See Problem 1.

6.

x	y
3	15
8	40
10	50
22	110

7.

x	y
3	15
5	8.4
7	6
10.5	4

8.

x	y
0.5	1
2.1	8.4
3.5	7
11	22

9.

x	y
0.1	3
3	0.1
6	0.05
24	0.0125

Suppose that x and y vary inversely. Write a function that models each inverse variation. Graph the function and find y when $x = 10$.

See Problem 2.

10. $x = 1$ when $y = 11$

11. $x = -13$ when $y = 100$

12. $x = 1$ when $y = 1$

13. $x = 1$ when $y = 5$

14. $x = 1.2$ when $y = 3$

15. $x = 2.5$ when $y = 100$

16. $x = 20$ when $y = -4$

17. $x = 5$ when $y = -\frac{1}{3}$

18. $x = -\frac{4}{15}$ when $y = -105$

19. Fundraising In a bake sale, you recorded the number of muffins sold and the amount of sales in a table as shown.

a. What is a function that relates the sales and the number of muffins?

b. How many muffins would you have to sell to make at least $250.00 in sales?

See Problem 3.

Number of muffins (m)	Sales (s)
5	$12.50
8	$20.00
13	$32.50
20	$50.00

20. Painting The number of buckets of paint n needed to paint a fence varies directly with the total area a of the fence and inversely with the amount of paint p in a bucket. It takes three 1-gallon buckets of paint to paint 72 square feet of fence. How many 1-gallon buckets will be needed to paint 90 square feet of fence?

See Problem 4.

21. Potential Energy On Earth with a gravitational acceleration g, the potential energy stored in an object varies directly with its mass m and its vertical height h. What is the equation of the potential energy of a 2 kg skateboard that is sliding down a ramp?

See Problem 5.

22. Think About a Plan The table shows data about how the life span s of a mammal relates to its heart rate r. The data could be modeled by an equation of the form $rs = k$. Estimate the life span of a cat with a heart rate of 126 beats/min.

- How can you estimate a constant of the inverse variation?
- What expression would you use to find the life span?

Heart Rate and Life Span

Mammal	Heart rate (beats/min)	Life span (min)
Mouse	634	1,576,800
Rabbit	158	6,307,200
Lion	76	13,140,000

Source: *The Handy Science Answer Book*

23. Physics The force F of gravity on a rocket varies directly with its mass m and inversely with the square of its distance d from Earth. Write a model for this combined variation.

B Apply

24. The spreadsheet shows data that could be modeled by an equation of the form $PV = k$. Estimate P when $V = 62$.

	A	B
1	P	V
2	140.00	100
3	147.30	95
4	155.60	90
5	164.70	85
6	175.00	80
7	186.70	75

25. Chemistry The formula for the Ideal Gas Law is $PV = nRT$, where P is the pressure in kilopascals (kPA), V is the volume in liters (L), T is the temperature in Kelvin (K), n is the number of moles of gas, and $R = 8.314$ is the universal gas constant.

a. What volume is needed to store 5 moles of helium gas at 350 K under the pressure 190 kPA?

b. A 10 L cylinder is filled with hydrogen gas to a pressure of 5,000 kPA. The temperature of gas is 300 K. How many moles of hydrogen gas are in the cylinder?

Write the function that models each variation. Find z when $x = 4$ and $y = 9$.

26. z varies directly with x and inversely with y. When $x = 6$ and $y = 2$, $z = 15$.

27. z varies jointly with x and y. When $x = 2$ and $y = 3$, $z = 60$.

28. z varies inversely with the product of x and y. When $x = 2$ and $y = 4$, $z = 0.5$.

Each pair of values is from a direct variation. Find the missing value.

29. $(2, 5), (4, y)$

30. $(4, 6), (x, 3)$

31. $(3, 7), (8, y)$

32. $(x, 12), (4, 1.5)$

Each ordered pair is from an inverse variation. Find the constant of variation.

33. $(6, 3)$

34. $(0.9, 4)$

35. $\left(\frac{3}{8}, \frac{2}{3}\right)$

36. $(\sqrt{2}, \sqrt{18})$

Each pair of values is from an inverse variation. Find the missing value.

37. $(2, 5), (4, y)$

38. $(4, 6), (x, 3)$

39. $(3, 7), (8, y)$

40. $(x, 12), (4, 1.5)$

Challenge

41. Writing Explain why 0 cannot be in the domain of an inverse variation.

42. Reasoning Suppose that (x_1, y_1) and (x_2, y_2) are values from an inverse variation. Show that $\frac{x_1}{x_2} = \frac{y_2}{y_1}$.

43. Open-Ended The height h of a cylinder varies directly with its volume V and inversely with the square of its radius r. Find at least four ways to change the volume and radius of a cylinder so that its height is quadrupled.

Sunshine State Standards Practice

MA.912.A.2.12

44. Which equation represents inverse variation between x and y?

Ⓐ $x = \frac{y}{z}$ Ⓑ $x = -\frac{15z}{y}$ Ⓒ $z = -\frac{15y}{x}$ Ⓓ $xz = 5y$

MA.912.A.1.6

45. How can you rewrite the expression $(8 - 5i)^2$ in the form $a + bi$?

Ⓕ $39 + 80i$ Ⓖ $39 - 80i$ Ⓗ $89 + 80i$ Ⓘ $89 - 80i$

MA.912.A.4.10

46. The height of a ball thrown straight up from the ground with a velocity of 96 ft/s is given by the quadratic function $h(t) = -16t^2 + 96t$. What is the maximum height the ball reaches?

Ⓐ 6 ft Ⓑ 128 ft Ⓒ 144 ft Ⓓ 160 ft

MA.912.A.6.4

47. Which expression is NOT equivalent to $\sqrt[6]{81x^4y^8}$?

Ⓕ $(3xy^2)^{\frac{2}{3}}$ Ⓖ $(3x)^{\frac{2}{3}}y^{\frac{4}{3}}$ Ⓗ $(3x^2y^2)^{\frac{1}{3}}$ Ⓘ $\sqrt[3]{9x^2y^4}$

MA.912.A.2.11

48. Short Response What is the inverse of $y = 4x^2 + 5$? Is the inverse a function?

Mixed Review

Solve each equation. Check your answers.

See Lesson 7-6.

49. $\ln 4 + \ln x = 5$

50. $\ln x - \ln 3 = 4$

51. $2\ln x + 3\ln 4 = 4$

Multiply and simplify.

See Lesson 6-2.

52. $-5\sqrt{6x} \cdot 3\sqrt{6x^3}$

53. $3\sqrt[3]{2x^2} \cdot 7\sqrt[3]{32x^4}$

54. $\sqrt{5x^3} \cdot \sqrt{40xy^7}$

Simplify each radical expression. Use absolute value bars where they are needed.

See Lesson 6-1.

55. $\sqrt{x^{10}y^{100}}$

56. $\sqrt[3]{-64a^3b^6}$

57. $\sqrt[4]{64m^8n^4}$

58. $\sqrt[n]{x^n}$

Get Ready! **To prepare for Lesson 8-2, do Exercises 59–64.**

Graph each equation. Then describe the transformation of the parent function $f(x) = |x|$.

See Lesson 2-7.

59. $y = |x| + 2$

60. $y = |x + 2|$

61. $y = |x| - 3$

62. $y = |x - 3|$

63. $y = |x + 4| - 5$

64. $y = |x - 10| + 7$

Concept Byte

For Use With Lesson 8-2

Graphing Rational Functions

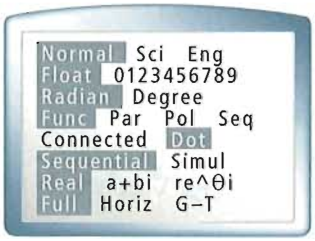
Sunshine State Standard
MA.912.A.2.6 Identify and graph common functions (including but not limited to linear, rational, quadratic, cubic, radical, absolute value).

You can use your graphing calculator to graph *rational functions* and other members of the reciprocal function family. It is sometimes preferable to use the **DOT** plotting mode rather than **CONNECTED** plotting mode. The **CONNECTED** mode can join branches of a graph that should be separated. Try both modes to get the best graph.

Example

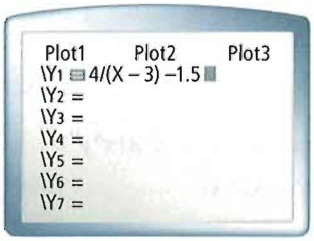
Graph $y = \frac{4}{x-3} - 1.5$.

Step 1 Press the mode key. Scroll down to highlight the word **DOT**. Then press enter.



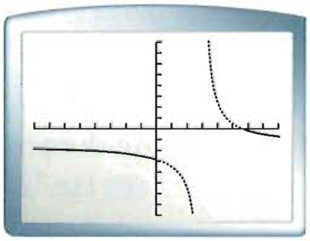
```
Normal  Sci  Eng
Float  0123456789
Radian  Degree
Func  Par  Pol  Seq
Connected  Dot
Sequential  Simul
Real  a+bi  re^θi
Full  Horiz  G–T
```

Step 2 Enter the function. Use parentheses to enter the denominator accurately.



```
Plot1    Plot2    Plot3
\Y1 ≡4/(X–3)–1.5
\Y2 =
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

Step 3 Graph the function.



Exercises

1. a. Graph the parent reciprocal function $y = \frac{1}{x}$.
   b. Examine both negative and positive values of x. Describe what happens to the y-values as x approaches zero.
   c. What happens to the y-values as x increases? As x decreases?

2. a. Change the mode on your calculator to **CONNECTED**. Graph the function from the example.
   b. Press trace and trace the function. What happens between $x \approx 2.8$ and $x \approx 3.2$?
   c. **Reasoning** How does your graph differ from the graph in the example? Explain the differences.

Use a graphing calculator to graph each function. Then sketch the graph.

3. $y = \frac{7}{x}$
4. $y = \frac{3}{x+4} - 2$
5. $y = \frac{x+2}{(x+1)(x+3)}$
6. $y = \frac{4x+1}{x-3}$
7. $y = \frac{2}{x-2}$
8. $y = \frac{1}{x+2} + 3$
9. $y = \frac{2x}{x+3}$
10. $y = \frac{x^2}{x^2-5}$
11. $y = \frac{20}{x^2+5}$

8-2 The Reciprocal Function Family

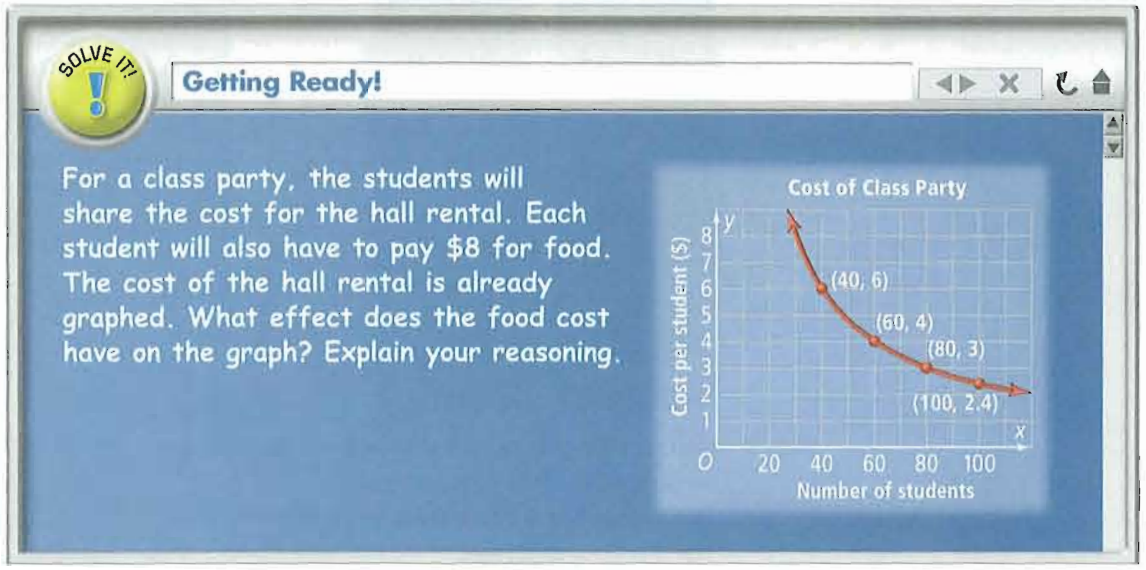
Sunshine State Standards

MA.912.A.2.10 Describe and graph transformations of functions.

MA.912.A.2.12 Solve problems using direct, inverse, and joint variations.

Objectives To graph reciprocal functions

To graph translations of reciprocal functions



For a class party, the students will share the cost for the hall rental. Each student will also have to pay $8 for food. The cost of the hall rental is already graphed. What effect does the food cost have on the graph? Explain your reasoning.

Dynamic Activity
Graphing Translations of Inverse Variations

Lesson Vocabulary
- reciprocal function
- branch

Functions that model inverse variation have the form $f(x) = \frac{a}{x}$, where $x \neq 0$. They belong to a family whose parent is the **reciprocal function** $f(x) = \frac{1}{x}$, where $x \neq 0$.

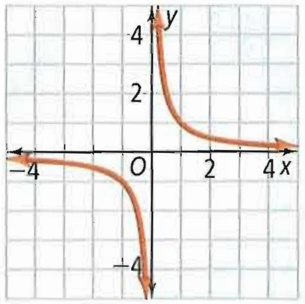
Essential Understanding Transformations of the parent reciprocal function include stretches, compressions (or shrinks), reflections, and horizontal and vertical translations.

take note Key Concept General Form of the Reciprocal Function Family

The general form of a member of the reciprocal function family is $y = \frac{a}{x-h} + k$, where $x \neq h$.

The inverse variation functions, $y = \frac{a}{x}$, are stretches, shrinks, and reflections of the parent reciprocal function, depending on the value of a.

The graph of the parent reciprocal function $y = \frac{1}{x}$ is shown at the right.



Problem 1 Graphing an Inverse Variation Function

Think

What values should you choose for x?
Choose values of x that divide nicely into 8. Make a table of points that are easy to graph.

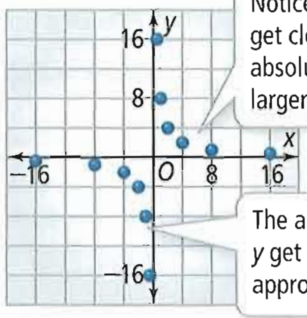
What is the graph of $y = \frac{8}{x}, x \neq 0$? Identify the x- and y-intercepts and the asymptotes of the graph. Also, state the domain and range of the function.

Step 1 Make a table of values that includes positive and negative values of x.

x	y
-16	$-\frac{1}{2}$
-8	-1
-4	-2
-2	-4
-1	-8
$-\frac{1}{2}$	-16

x	y
$\frac{1}{2}$	16
1	8
2	4
4	2
8	1
16	$\frac{1}{2}$

Step 2 Graph the points.

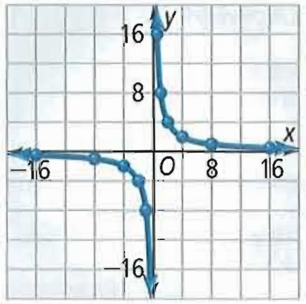


Notice how the y-values get closer to zero as the absolute values of x get larger.

The absolute values of y get very large as x approaches zero.

Step 3 Connect the points with a smooth curve. x cannot be zero, so there is no y-intercept.
The numerator is never zero, so y is never 0.
There is no x-intercept.

The x-axis is a horizontal asymptote.
The y-axis is a vertical asymptote.
Knowing the asymptotes provides you with the basic shape of the graph.
The domain is the set of all real numbers except $x = 0$.
The range is the set of all real numbers except $y = 0$.



Got It? **1. a.** What is the graph of $y = \frac{12}{x}$? Identify the x- and y-intercepts and the asymptotes of the graph. Also, state the domain and range of the function.

b. Reasoning Would the function $y = \frac{6}{x}$ have the same domain and range as $y = \frac{8}{x}$ or $y = \frac{12}{x}$? Explain.

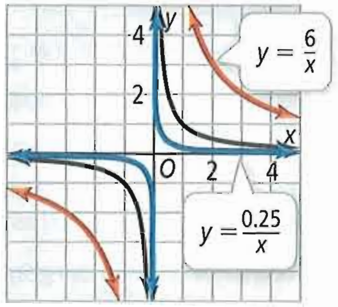
Each part of the graph of a reciprocal function is a **branch**. The branches of the parent function $y = \frac{1}{x}$ are in Quadrants I and III. Stretches and compressions of the parent function remain in the same quadrants. Reflections are in Quadrants II and IV.

Problem 2 Identifying Reciprocal Function Transformations

For each given value of a, how do the graphs of $y = \frac{1}{x}$ and $y = \frac{a}{x}$ compare? What is the effect of a on the graph?

A $a = 6$

The graph (in red) of $y = \frac{6}{x}$ is a stretch of the graph of $y = \frac{1}{x}$ (in black) by the factor 6.

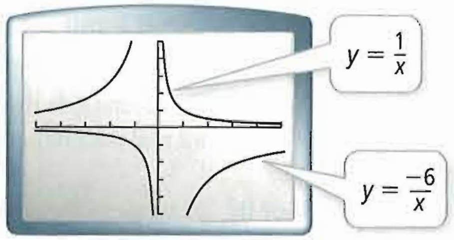


B $a = 0.25$

The graph (in blue) of $y = \frac{0.25}{x}$ is a shrink of the graph of $y = \frac{1}{x}$ (in black) by the factor $\frac{1}{4}$.

C $a = -6$

The graph of $y = \frac{-6}{x}$ is the stretch by the factor 6 in part A followed by a reflection across the x-axis.



Think

How does the negative sign affect the graph?
The y-values have signs that are opposite those in part A. The graph in A reflects across the x-axis.

Got It? **2.** For each given value of a, how do the graphs of $y = \frac{1}{x}$ and $y = \frac{a}{x}$ compare? What is the effect of a on the graph?

a. $a = \frac{1}{2}$ **b.** $a = 2$ **c.** $a = -\frac{1}{2}$

You can translate any reciprocal function horizontally or vertically just as you can other functions.

take note Key Concept The Reciprocal Function Family

Parent function	$y = \frac{1}{x}, x \neq 0$
Stretch ($\lvert a \rvert > 1$) Shrink ($0 < \lvert a \rvert < 1$) Reflection ($a < 0$) across x-axis	$y = \frac{a}{x}, x \neq 0$
Translation (horizontal by h; vertical by k) with vertical asymptote $x = h$ horizontal asymptote $y = k$	$y = \frac{1}{x - h} + k; x \neq h$
Combined	$y = \frac{a}{x - h} + k; x \neq h$

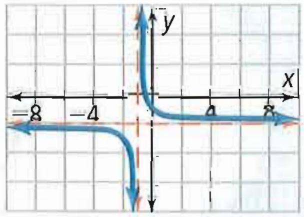
When you graph a translated reciprocal function, a good first step is to draw the asymptotes.

Problem 3 Graphing a Translation

Think

How do you find the asymptotes?
The asymptotes of $y = \frac{1}{x}$ (the axes) translate 1 unit to the left and 2 units down.

What is the graph of $y = \frac{1}{x+1} - 2$? Identify the domain and range.



Step 1 Draw the asymptotes (red).

For, $y = \frac{1}{x+1} - 2$, $h = -1$ and $k = -2$.

The vertical asymptote is $x = -1$.
The horizontal asymptote is $y = -2$.

Step 2 Translate the graph of $y = \frac{1}{x}$.

The graph of $y = \frac{1}{x}$ contains the points (1, 1) and (−1, −1). Translate these points 1 unit to the left and 2 units down to (0, −1) and (−2, −3), respectively. Draw the branches through these points (blue).

The domain is the set of all real numbers except $x = -1$. The range is the set of all real numbers except $y = -2$.

Got It? **3.** What is the graph of $y = \frac{1}{x-4} + 6$? Identify the domain and range.

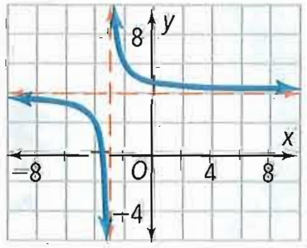
If you know the asymptotes of the graph of a reciprocal function and the value of a, you can write the equation of the function.

Problem 4 Writing the Equation of a Transformation

Plan

How can you get started?
Identify the asymptotes of the graph.

Multiple Choice **This graph of a function is a translation of the graph of $y = \frac{2}{x}$. What is an equation for the function?**



Ⓐ $y = \frac{2}{x+3} + 4$

Ⓑ $y = \frac{2}{x+3} - 4$

Ⓒ $y = \frac{2}{x-3} + 4$

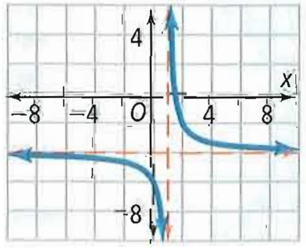
Ⓓ $y = \frac{2}{x-3} - 4$

The asymptotes are $x = -3$ and $y = 4$. Thus $h = -3$ and $k = 4$.

$y = \frac{a}{x-h} + k$	Use the general form.
$y = \frac{2}{x-(-3)} + 4$	Substitute for a, h, and k.
$y = \frac{2}{x+3} + 4$	Simplify.

The correct choice is A.

Got It? **4.** This graph of a function is a translation of the graph of $y = \frac{2}{x}$. What is an equation for the function?



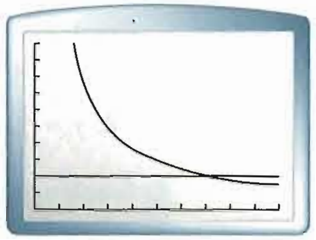
Problem 5 Using a Reciprocal Function

Clubs **The rowing club is renting a 57-passenger bus for a day trip. The cost of the bus is \$750. Five passengers will be chaperones. If the students who attend share the bus cost equally, what function models the cost per student C with respect to the number of students n who attend? What is the domain of the function? How many students must ride the bus to make the cost per student no more than \$20?**

Know	Need	Plan
• The bus holds 57 passengers. • The bus costs \$750. • Five riders are chaperones who pay nothing for the bus.	• A function for the cost per student • The number of students needed so that the cost does not exceed \$20 per student	• Write a reciprocal function for the situation. • Graph the function and solve an inequality using the \$20 limit.

To share the cost equally, divide 750 by the number of students, n, who attend.

The function that models the cost per student is $C = \frac{750}{n}$.



The bus has a capacity of 57 passengers and there will be 5 chaperones. The maximum number of students is $57 - 5 = 52$.

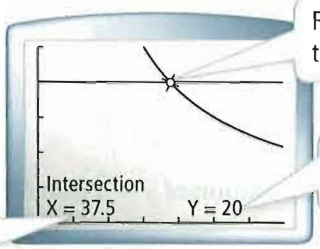
The domain is the integers from 1 to 52.

Think

Is the domain $x \leq 52$?
No; the domain is the possible numbers of students, so only positive integers make sense.

Use a graphing calculator to solve the inequality $\frac{750}{n} \leq 20$. Let **Y1** $= \frac{750}{x}$ and **Y2** $= 20$.

Change the window dimensions to get a closer look at the graph. Use the **intersect** feature.



For all values greater than or equal to 38, the cost is less than \$20.

If $x = 37$, the cost will be more than \$20.

The number of people must be a whole number.

At least 38 students must ride the bus.

Got It? **5.** The junior class is renting a laser tag facility with a capacity of 325 people. The cost for the facility is \$1200. The party must have 13 adult chaperones.

a. If every student who attends shares the facility cost equally, what function models the cost per student C with respect to the number of students n who attend? What is the domain of the function? How many students must attend to make the cost per student no more than \$7.50?

b. The class wants to promote the event by giving away 30 spots to students in a drawing. How does the model change? Now how many paying students must attend so the cost for each is no more than \$7.50?

Lesson Check

Do you know HOW?

1. Graph the equation $y = \frac{3}{x}$.

Describe the transformation from the graph of $y = \frac{1}{x}$ to the graph of the given function.

2. $y = \frac{1}{x} + 5$

3. $y = \frac{-4}{x}$

4. What are the asymptotes of the graph of $y = \frac{5}{x+2} - 7$?

Do you UNDERSTAND?

5. Vocabulary What transformation changes the graph of $y = \frac{1}{x}$ into the graph of $y = \frac{1}{2x}$?

6. Open Ended Write an equation of a stretch and a reflection of the graph $y = \frac{1}{x}$ across the x-axis.

7. Writing Explain how you can tell if a function $y = \frac{a}{x}$ is a stretch or compression of the parent function $y = \frac{1}{x}$.

Practice and Problem-Solving Exercises

A Practice

Graph each function. Identify the x- and y-intercepts and the asymptotes of the graph. Also, state the domain and the range of the function. **See Problem 1.**

8. $y = \frac{2}{x}$ **9.** $y = \frac{15}{x}$ **10.** $y = \frac{-3}{x}$ **11.** $y = -\frac{10}{x}$ **12.** $y = \frac{10}{x}$

Graphing Calculator **Graph the equations $y = \frac{1}{x}$ and $y = \frac{a}{x}$ using the given value of a. Then identify the effect of a on the graph.** **See Problem 2.**

13. $a = 2$ **14.** $a = -4$ **15.** $a = 0.5$ **16.** $a = 12$ **17.** $a = 0.75$

Sketch the asymptotes and the graph of each function. Identify the domain and range. **See Problem 3.**

18. $y = \frac{1}{x} - 3$ **19.** $y = \frac{-2}{x} - 3$ **20.** $y = \frac{1}{x-2} + 5$ **21.** $y = \frac{1}{x-3} + 4$

22. $y = \frac{2}{x+6} - 1$ **23.** $y = \frac{10}{x+1} - 8$ **24.** $y = \frac{1}{x} - 2$ **25.** $y = \frac{-8}{x+5} - 6$

Write an equation for the translation of $y = \frac{2}{x}$ that has the given asymptotes. **See Problem 4.**

26. $x = 0$ and $y = 4$ **27.** $x = -2$ and $y = 3$ **28.** $x = 4$ and $y = -8$

29. Construction The weight P in pounds that a beam can safely carry is inversely proportional to the distance D in feet between the supports of the beam. For a certain type of wooden beam, $P = \frac{9200}{D}$. What distance between supports is needed to carry 1200 lb? **See Problem 5.**

B Apply

30. Think About a Plan A high school decided to spend $750 on student academic achievement awards. At least 5 awards will be given, they should be equal in value, and each award should not be less than $50. Write and sketch a function that models the relationship between the number a of awards and the cost c of each award. What are the domain and range of the function?
- Which equation describes the relationship between a and c?
- What information can you use to determine the domain and range?

31. **Open-Ended** Write an equation for a horizontal translation of $y = \frac{2}{x}$. Then write an equation for a vertical translation of $y = \frac{2}{x}$. Identify the horizontal and vertical asymptotes of the graph of each function.

Sketch the graph of each function.

32. $xy = 3$ **33.** $xy + 5 = 0$ **34.** $3xy = 1$ **35.** $5xy = 2$ **36.** $10xy = -4$

37. **Writing** Explain how knowing the asymptotes of a translation of $y = \frac{1}{x}$ can help you graph the function. Include an example.

38. **Multiple Choice** The formula $p = \frac{69.1}{a + 2.3}$ models the relationship between atmospheric pressure p in inches of mercury and altitude a in miles.

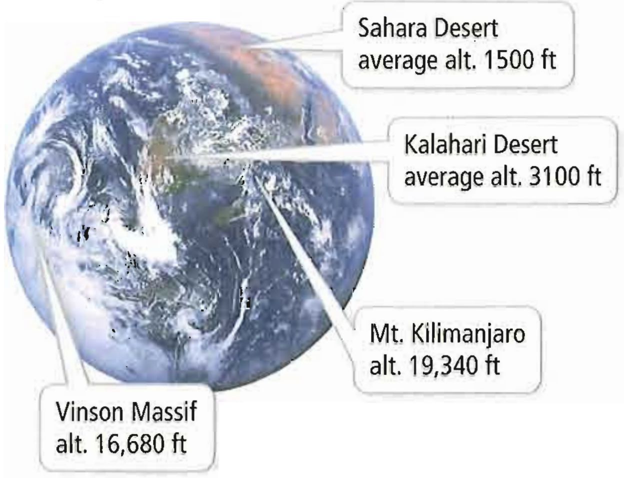
Use the data shown with the photo. At which location does the model predict the pressure to be about 23.93 in. of mercury? (*Hint:* 1 mi = 5280 ft.)

Ⓐ Sahara Desert

Ⓑ Kalahari Desert

Ⓒ Mt. Kilimanjaro

Ⓓ Vinson Massif



Sahara Desert average alt. 1500 ft

Kalahari Desert average alt. 3100 ft

Mt. Kilimanjaro alt. 19,340 ft

Vinson Massif alt. 16,680 ft

Graphing Calculator **Graph each pair of functions. Find the approximate point(s) of intersection.**

39. $y = \frac{6}{x - 2}, y = 6$ **40.** $y = -\frac{1}{x - 3} - 6, y = 6.2$ **41.** $y = \frac{3}{x + 1}, y = -4$

42. **Reasoning** How will the domain and the range of the parent function $y = \frac{1}{x}$ change after the translation of its graph by 3 units up and by 5 units to the left?

43. **a.** **Gasoline Mileage** Suppose you drive an average of 10,000 miles each year. Your gasoline mileage (mi/gal) varies inversely with the number of gallons of gasoline you use each year. Write and graph a model for your average mileage m in terms of the gallons g of gasoline used.

b. After you begin driving on the highway more often, you use 50 gal less per year. Write and graph a new model to include this information.

c. Calculate your old and new mileage assuming that you originally used 400 gal of gasoline per year.

Challenge

Reasoning **Compare each pair of graphs and find any points of intersection.**

44. $y = \frac{1}{x}$ and $y = \left|\frac{1}{x}\right|$ **45.** $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ **46.** $y = \left|\frac{1}{x}\right|$ and $y = \frac{1}{x^2}$

47. Find two reciprocal functions such that the minimum distance from the origin to the graph of each function is $4\sqrt{2}$.

48. Write each equation in the form $y = \frac{k}{x - b} + c$, and sketch the graph.

a. $y = \frac{2}{3x - 6}$ **b.** $y = \frac{1}{2 - 4x}$ **c.** $y = \frac{3 - x}{x + 2}$ **d.** $xy - y = 1$

Sunshine State Standards Practice

MA.912.A.2.10 **49.** What is an equation for the translation of $y = \frac{2}{x}$ that has asymptotes at $x = 3$ and $y = -5$?

Ⓐ $y = \frac{2}{x - 3} - 5$ Ⓑ $y = \frac{2}{x + 3} + 5$ Ⓒ $y = \frac{2}{x + 5} - 3$ Ⓓ $y = \frac{2}{x - 5} + 3$

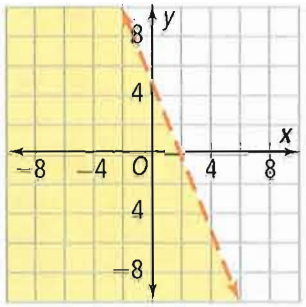
MA.912.A.2.5 **50.** The graph at the right shows which inequality?

Ⓕ $y < -2.5x + 5$

Ⓖ $2.5x + y \geq 5$

Ⓗ $-2.5x + y < 5$

Ⓘ $5x + y \leq 5$



MA.912.A.2.12 **51.** If p and q vary inversely, and $p = 10$ when $q = -4$, what is q when $p = -2$?

Ⓐ 20 Ⓑ $\frac{4}{5}$ Ⓒ $-\frac{4}{5}$ Ⓓ −20

MA.912.A.8.3 **52.** Which equation represents the inverse of the graph at the right?

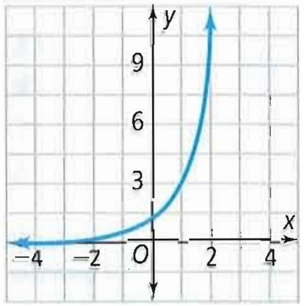
Ⓕ $y = \log_3 x$

Ⓖ $x = \log_3 y$

Ⓗ $y = \log_x 3$

Ⓘ $x = \log_y 3$

MA.912.A.8.1 **53. Short Response** What is b if the graph of $y = 27b^x$ includes the point $(-1, 81)$?



Mixed Review

Suppose that x and y vary inversely. Write a function that models each inverse variation and find y when $x = -5$. **See Lesson 8-1.**

54. $x = 2$ when $y = 12$ **55.** $x = 25$ when $y = 2$ **56.** $x = 12$ when $y = 4$

Without graphing, determine whether the function represents exponential growth or exponential decay. Then find the y-intercept. **See Lesson 7-1.**

57. $y = 3(4)^x$ **58.** $y = 0.1(2)^x$ **59.** $y = 5(0.8)^x$ **60.** $y = 3\left(\frac{1}{2}\right)^x$

Multiply. **See Lesson 6-3.**

61. $(5\sqrt{3} - 2)^2$ **62.** $(4 + 2\sqrt{3})(6 - 3\sqrt{3})$ **63.** $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$

Get Ready! **To prepare for Lesson 8-3, do Exercises 64–67.**

Factor each expression. **See Lesson 4-4.**

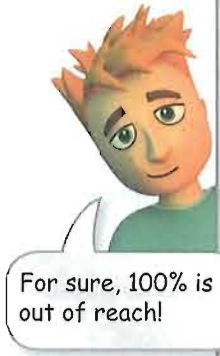
64. $x^2 - 6x + 8$ **65.** $x^2 + 6x - 27$ **66.** $2x^2 + x - 28$ **67.** $2x^2 - 19x + 24$

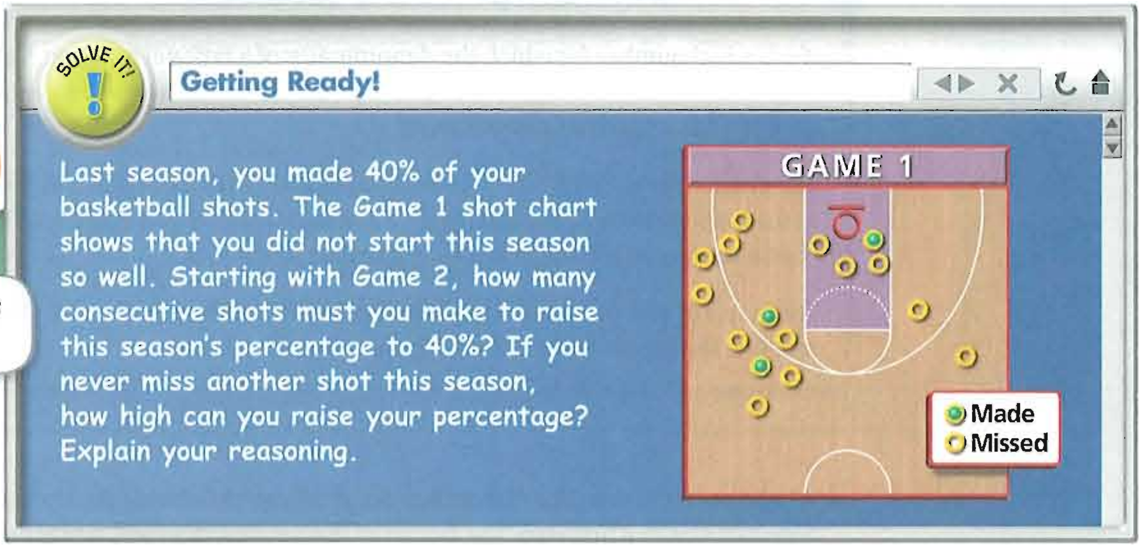
8-3 Rational Functions and Their Graphs

Sunshine State Standard

MA.912.A.5.6 Identify removable and non-removable discontinuities and vertical, horizontal, and oblique asymptotes of a graph of a rational function, find the zeros, and graph the function.

Objectives To identify properties of rational functions
To graph rational functions





Getting Ready!

Last season, you made 40% of your basketball shots. The Game 1 shot chart shows that you did not start this season so well. Starting with Game 2, how many consecutive shots must you make to raise this season's percentage to 40%? If you never miss another shot this season, how high can you raise your percentage? Explain your reasoning.

Dynamic Activity
Rational Functions

Lesson Vocabulary
- rational function
- continuous graph
- discontinuous graph
- point of discontinuity
- removable discontinuity
- non-removable discontinuity

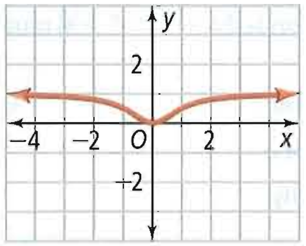
You use a ratio of polynomial functions to form a *rational function*, like $y = \frac{x + 3}{x + 16}$.

Essential Understanding If a function has a polynomial in its denominator, its graph has a gap at each zero of the polynomial. The gap could be a one-point hole in the graph, or it could be the location of a vertical asymptote for the graph.

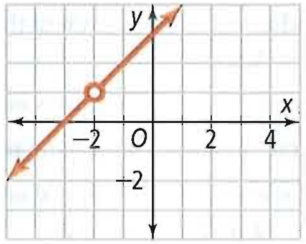
A **rational function** is a function that you can write in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions. The domain of $f(x)$ is all real numbers except those values for which $Q(x) = 0$.

Here are graphs of three rational functions:

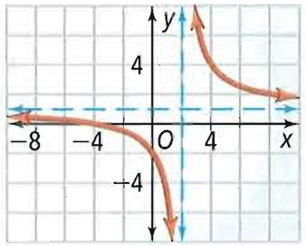
$y = \frac{x^2}{x^2 + 1}$



$y = \frac{(x + 3)(x + 2)}{(x + 2)}$



$y = \frac{x + 4}{x - 2}$



For the first rational function, $y = \frac{x^2}{x^2 + 1}$, there is no value of x that makes the denominator 0. The graph is a **continuous graph** because it has no jumps, breaks, or holes. You can draw the graph and your pencil never leaves the paper.

For the second rational function, $y = \frac{(x + 3)(x + 2)}{x + 2}$, x cannot be -2. For $y = \frac{x + 4}{x - 2}$, x cannot be 2. The second and third graphs are **discontinuous graphs**.

Key Concept Point of Discontinuity

If a is a real number for which the denominator of a rational function $f(x)$ is zero, then a is not in the domain of $f(x)$. The graph of $f(x)$ is not continuous at $x = a$ and the function has a **point of discontinuity** at $x = a$.

The graph of $y = \frac{(x + 3)(x + 2)}{x + 2}$ has a **removable discontinuity** at $x = -2$. The hole in the graph is called a removable discontinuity because you could make the function continuous by redefining it at $x = -2$ so that $f(-2) = 1$.

The graph of $y = \frac{x + 4}{x - 2}$ has a **non-removable discontinuity** at $x = 2$. There is no way to redefine the function at 2 to make the function continuous.

When you are looking for discontinuities, it is helpful to factor the numerator and denominator as a first step. The factors of the denominator will reveal the points of discontinuity. The discontinuity caused by $(x - a)^n$ in the denominator is removable if the numerator also has $(x - a)^n$ as a factor.

Problem 1 Finding Points of Discontinuity

What are the domain and points of discontinuity of each rational function? Are the points of discontinuity removable or non-removable? What are the *x*- and *y*-intercepts?

A $y = \frac{x + 3}{x^2 - 4x + 3}$

Factor the numerator and denominator to check for common factors.

$$y = \frac{x + 3}{x^2 - 4x + 3} = \frac{x + 3}{(x - 3)(x - 1)}$$

The function is undefined where $x - 3 = 0$ and where $x - 1 = 0$, at $x = 3$ and $x = 1$. The domain of the function is the set of all real numbers except $x = 1$ and $x = 3$.

There are non-removable points of discontinuity at $x = 1$ and $x = 3$.

The x-intercept occurs where the numerator equals 0, at $x = -3$.

To find the y-intercept, let $x = 0$ and simplify.

$$y = \frac{0 + 3}{(0 - 3)(0 - 1)} = \frac{3}{(-3)(-1)} = \frac{3}{3} = 1$$

Think

Are the discontinuities removable?

There are no common factors in the numerator and denominator. Any discontinuity is non-removable.

Think

When is the denominator zero?
x^2 is at least 0, so $x^2 + 1$ is always greater than 0.

B $y = \frac{x - 5}{x^2 + 1}$

You cannot factor the numerator or the denominator. Also, there are no values of x that make the denominator 0. The domain of the function is all real numbers, and there are no discontinuities.

The x-intercept occurs where the numerator equals 0, at $x = 5$.

To find the y-intercept, let $x = 0$ and simplify: $y = \frac{0 - 5}{0^2 + 1} = \frac{-5}{1} = -5$

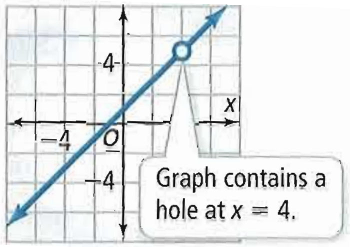
C $y = \frac{x^2 - 3x - 4}{x - 4}$

Factor the numerator and denominator: $y = \frac{x^2 - 3x - 4}{x - 4} = \frac{(x - 4)(x + 1)}{(x - 4)}$

The function is undefined where $x - 4 = 0$, at $x = 4$. The domain of the function is the set of all real numbers except $x = 4$.

Because $y = x + 1$, except at $x = 4$, there is a removable discontinuity at $x = 4$.

At $x = 4$, $y = x + 1 = 4 + 1 = 5$, so you can redefine the function to remove the discontinuity.



Graph contains a hole at $x = 4$.

$$y = \begin{cases} \frac{x^2 - 3x - 4}{x - 4}, & \text{if } x \neq 4 \\ 5, & \text{if } x = 4 \end{cases}$$

The x-intercept occurs where the numerator equals 0, at $x = -1$.

To find the y-intercept, let $x = 0$ and simplify.

$$y = \frac{0^2 - 3 \cdot 0 - 4}{0 - 4} = \frac{0 - 0 - 4}{-4} = \frac{-4}{-4} = 1$$

Got It? **1.** What are the domain and points of discontinuity of the rational function? Are the points of discontinuity *removable* or *non-removable*? What are the x- and y-intercepts of the rational function?

a. $y = \frac{1}{x^2 - 16}$ **b.** $y = \frac{x^2 - 1}{x^2 + 3}$ **c.** $y = \frac{x + 1}{x^2 + 3x + 2}$

In Chapter 7, you learned that an asymptote is a line that a graph approaches as x or y increases in absolute value. If a rational function has a non-removable discontinuity at $x = a$, the graph of the rational function will have a vertical asymptote at $x = a$.

take note

Key Concept Vertical Asymptotes of Rational Functions

The graph of the rational function $f(x) = \frac{P(x)}{Q(x)}$ has a vertical asymptote at each real zero of $Q(x)$ if $P(x)$ and $Q(x)$ have no common zeros. If $P(x)$ and $Q(x)$ have $(x - a)^m$ and $(x - a)^n$ as factors, respectively and $m < n$, then $f(x)$ also has a vertical asymptote at $x = a$.

Problem 2 Finding Vertical Asymptotes

Plan

How can you locate a vertical asymptote? Find factors $x - a$ of the denominator that have no matching factor in the numerator. $x = a$ is a vertical asymptote.

What are the vertical asymptotes for the graph of $y = \frac{(x + 1)}{(x - 2)(x - 3)}$?

Since 2 and 3 are zeros of the denominator and neither is a zero of the numerator, the lines $x = 2$ and $x = 3$ are vertical asymptotes.

Got It? **2.** What are the vertical asymptotes for the graph of the rational function?

a. $y = \frac{x - 2}{(x - 1)(x + 3)}$ **b.** $y = \frac{x - 2}{(x - 2)(x + 3)}$ **c.** $y = \frac{x^2 - 1}{x + 1}$

While the graph of a rational function can have any number of vertical asymptotes, it can have no more than one horizontal asymptote.

take note

Key Concept Horizontal Asymptote of a Rational Function

To find the horizontal asymptote of the graph of a rational function, compare the degree of the numerator m to the degree of the denominator n.

If $m < n$, the graph has horizontal asymptote $y = 0$ (the x-axis).

If $m > n$, the graph has no horizontal asymptote.

If $m = n$, the graph has horizontal asymptote $y = \frac{a}{b}$ where a is the coefficient of the term of greatest degree in the numerator and b is the coefficient of the term of greatest degree in the denominator.

Problem 3 Finding Horizontal Asymptotes

Plan

How can you find the horizontal asymptote when the numerator and denominator have equal degree? Find the quotient, q, of the leading coefficients of the numerator and denominator. $y = q$ is the horizontal asymptote.

What is the horizontal asymptote for the rational function?

A $y = \frac{2x}{x - 3}$

The degree of the numerator and denominator are the same.

The horizontal asymptote is $y = \frac{2}{1}$ or $y = 2$.

B $y = \frac{x - 2}{x^2 - 2x - 3}$

The degree of the numerator is less than the degree of the denominator. The horizontal asymptote is $y = 0$.

C $y = \frac{x^2}{2x - 5}$

The degree of the numerator is greater than the degree of the denominator. There is no horizontal asymptote.

Got It? **3.** What is the horizontal asymptote for the rational function?

a. $y = \frac{-2x + 6}{x - 5}$ **b.** $y = \frac{x - 1}{x^2 + 4x + 4}$ **c.** $y = \frac{x^2 + 2x - 3}{x - 2}$

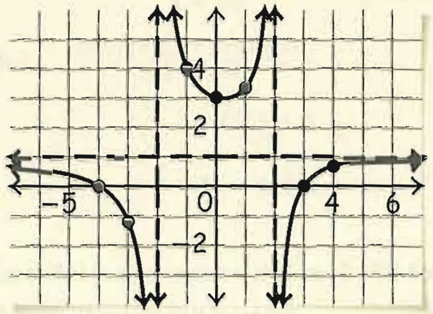
Essential Understanding You can get a reasonable graph for a rational function by finding all intercepts and asymptotes. Sometimes you will also have to plot a few extra points to get a good sense of the shape of the graph.

Problem 4 Graphing a Rational Function

What is the graph of the rational function $y = \frac{x^2 + x - 12}{x^2 - 4}$?

Plan

How can you graph this function?
Find the horizontal and vertical asymptotes and the x- and y-intercepts. Look for holes and find additional points to help get a better sense of the graph.

Think	Write
The degrees of the numerator and denominator are equal.	$y = \frac{x^2 + x - 12}{x^2 - 4}$ horizontal asymptote: $y = \frac{1}{1} = 1$
Factor the numerator and the denominator. They have no common factor. The graph has no holes. It has two vertical asymptotes at the zeros of the denominator.	$y = \frac{(x + 4)(x - 3)}{(x + 2)(x - 2)}$ vertical asymptotes: $x = -2$, $x = 2$
Find the x- and y-intercepts. The x-intercepts occur where $y = 0$. The y-intercepts occur where $x = 0$.	When the numerator equals zero, $y = 0$. x-intercepts: $(-4, 0)$ and $(3, 0)$ $y = \frac{(0 + 4)(0 - 3)}{(0 + 2)(0 - 2)} = 3$ y-intercept: $(0, 3)$
Find a few more points on the graph.	More points on the graph: $(-3, -\frac{6}{5})$, $(-1, 4)$, $(1, \frac{10}{3})$ and $(4, \frac{2}{3})$
Graph the asymptotes. Then plot the intercepts and additional points. Use the points to sketch the graph.	

Got It? **4.** What is the graph of the rational function $y = \frac{x + 3}{x^2 - 6x + 5}$?

Problem 5 Using a Rational Function

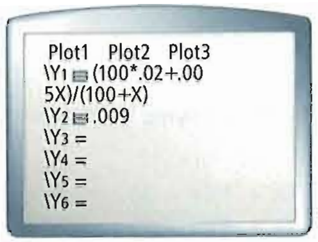
GRIDDED RESPONSE

Chemistry You work in a pharmacy that mixes different concentrations of saline solutions for its customers. The pharmacy has a supply of two concentrations, 0.5% and 2%. The function $y = \frac{(100)(0.02) + x(0.005)}{100 + x}$ gives the concentration of the saline solution after adding x milliliters of the 0.5% solution to 100 milliliters of the 2% solution. How many milliliters of the 0.5% solution must you add for the combined solution to have a concentration of 0.9%?

Plan

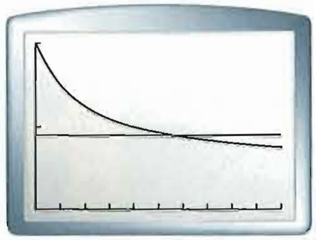
How can you use a calculator to solve the problem?
Graph $y = \frac{(100)(0.02) + x(0.005)}{100 + x}$ and $y = 0.009$ in the calculator and find the point of intersection.

Step 1 Use a graphing calculator to graph **Y1** $= \frac{(100)(0.02) + x(0.005)}{100 + x}$ and **Y2** $= 0.009$.



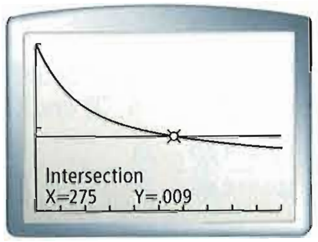
```
Plot1  Plot2  Plot3
\Y1 ▤ (100*.02+.00
5X)/(100+X)
\Y2 ▤ .009
\Y3 =
\Y4 =
\Y5 =
\Y6 =
```

```
WINDOW
  Xmin=0
  Xmax=500
  Xscl=50
  Ymin=0
  Ymax=.02
  Yscl=.01
  Xres=1
```



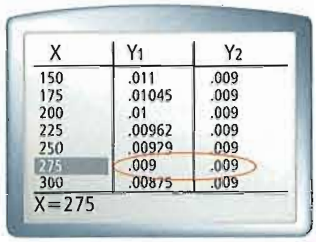
Step 2 Find the point of intersection of the two functions.

Graphic Solution



```
Intersection
X=275     Y=.009
```

Table Solution



```
X    | Y1     | Y2
150  | .011   | .009
175  | .01045 | .009
200  | .01    | .009
225  | .00962 | .009
250  | .00929 | .009
275  | .009   | .009
300  | .00875 | .009
X=275
```

You should add 275 mL of the 0.5% solution to get a 0.9% solution. Write 275 in the grid.

Check $y = \frac{(100)(0.02) + x(0.005)}{100 + x}$

$y \overset{?}{=} \frac{(100)(0.02) + (275)(0.005)}{100 + 275}$ Substitute 275 for x.

$y \overset{?}{=} \frac{2 + 1.375}{375}$

$y = 0.009$ ✔

Got It? 5. **a.** You want to mix a 10% orange juice drink with 100% pure orange juice to make a 40% orange juice drink. The function $y = \frac{(2)(1.0) + x(0.1)}{2 + x}$ gives the concentration y of orange juice in the drink after you add x gallons of the 10% drink to 2 gallons of pure juice. How much of the 10% drink must you add to get a drink that is 40% juice?

b. Reasoning If you wanted a drink that is 80% orange juice, would you need to add half as much as your answer in part (a)? Explain.

Lesson Check

Do you know HOW?

Find any points of discontinuity for each rational function.

1. $y = \dfrac{x + 5}{x^2 + 9x + 20}$

2. $y = \dfrac{x^2 + 2x}{x^2 - 7x - 18}$

3. $y = \dfrac{x - 1}{(x + 1)^2}$

4. $y = \dfrac{x^2 - x - 2}{3x^2 - 7x + 2}$

Find the vertical asymptotes of the graph of each rational function.

5. $y = \dfrac{x - 3}{x + 5}$

6. $y = \dfrac{x - 3}{x^2 + 5x + 6}$

7. $y = \dfrac{2x + 2}{x^2 - 1}$

8. $y = \dfrac{x^2 + 2x + 3}{x^2 + 2x - 3}$

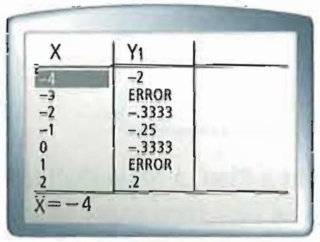
Sketch the graph of each rational function.

9. $y = \dfrac{3x}{x - 4}$

10. $y = \dfrac{x + 3}{(x - 1)(x - 6)}$

Do you UNDERSTAND?

For Exercises 11 and 12, use the following table. The table shows data for a rational function.



X	Y1
-4	-2
-3	ERROR
-2	-.3333
-1	-.25
0	-.3333
1	ERROR
2	.2

X=−4

11. What do the **Y1** values for **X = −3** and **X = 1** tell you about the rational function?

12. **Reasoning** Assume that there are no more **ERROR** values in the **Y1** column. What is the lowest possible degree of the denominator? Explain how you know.

Practice and Problem-Solving Exercises

A Practice

Find the domain, points of discontinuity, and x- and y- intercepts of each rational function. Determine whether the discontinuities are removable or non-removable. **See Problem 1.**

13. $y = \dfrac{2x^2 + 5}{x^2 - 2x}$

14. $y = \dfrac{x^2 + 2x}{x^2 + 2}$

15. $y = \dfrac{3x - 3}{x^2 - 1}$

16. $y = \dfrac{6 - 3x}{x^2 - 5x + 6}$

Find the vertical asymptotes and holes for the graph of each rational function. **See Problem 2.**

17. $y = \dfrac{3}{x + 2}$

18. $y = \dfrac{x + 5}{x + 5}$

19. $y = \dfrac{x + 3}{(2x + 3)(x - 1)}$

20. $y = \dfrac{(x + 3)(x - 2)}{(x - 2)(x + 1)}$

21. $y = \dfrac{x^2 - 4}{x + 2}$

22. $y = \dfrac{x + 5}{x^2 + 9}$

Find the horizontal asymptote of the graph of each rational function. **See Problem 3.**

23. $y = \dfrac{5}{x + 6}$

24. $y = \dfrac{x + 2}{2x^2 - 4}$

25. $y = \dfrac{x + 1}{x + 5}$

26. $y = \dfrac{x^2 + 2}{2x^2 - 1}$

27. $y = \dfrac{5x^3 + 2x}{2x^5 - 4x^3}$

28. $y = \dfrac{3x - 4}{4x + 1}$

Sketch the graph of each rational function. **See Problem 4.**

29. $y = \dfrac{x^2 - 4}{3x - 6}$

30. $y = \dfrac{4x}{x^3 - 4x}$

31. $y = \dfrac{x + 4}{x - 4}$

32. $y = \dfrac{x(x + 1)}{x + 1}$

33. $y = \dfrac{x + 6}{(x - 2)(x + 3)}$

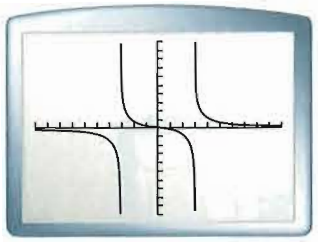
34. $y = \dfrac{3x}{(x + 2)^2}$

35. **Pharmacology** How many milliliters of the 0.5% solution must be added to the 2% solution to get a 0.65% solution? Use the rational function given in Problem 5. **See Problem 5.**

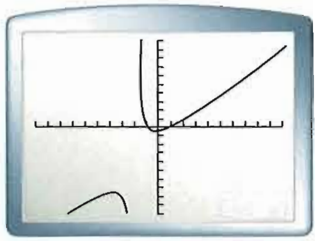
B Apply

Find the vertical and horizontal asymptotes, if any, of the graph of each rational function.

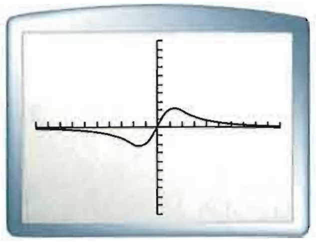
36.



37.



38.



39. Think About a Plan A basketball player has made 21 of her last 30 free throws—a percentage of 70%. How many more consecutive free throws does she need to raise her free throw percentage to 75%?

- How can you model the player's free throw percentage as a rational function? (*Hint:* Let $x =$ the number of additional free throws needed.)
- How can a graph help you answer this question?

40. Grades A student earns an 82% on her first test. How many consecutive 100% test scores does she need to bring her average up to 95%? Assume that each test has equal impact on the average grade.

41. Error Analysis A student listed the asymptotes of the function $y = \frac{x^2 - 3x + 2}{x^2 + 6x + 5}$ as shown at the right. Explain the student's error. What are the correct asymptotes?

~~vertical asymptotes:~~
~~x = 1, x = 2~~
~~horizontal asymptotes:~~
~~y = −1, y = −5~~

Sketch the graph of each rational function.

42. $y = \frac{2x + 3}{x - 5}$

43. $y = \frac{x^2 + 6x + 9}{x + 3}$

44. $y = \frac{4x^2 - 100}{2x^2 + x - 15}$

45. $y = -\frac{x}{(x - 1)^2}$

46. Business CDs can be manufactured for $.19 each. The development cost is $210,000. The first 500 discs are samples and will not be sold.

a. Write a function for the average cost of a disc that is not a sample. Graph the function.

b. What is the average cost if 5000 discs are produced? If 15,000 discs are produced?

c. How many discs must be produced to bring the average cost under $10?

d. What are the vertical and horizontal asymptotes of the graph of the function?

47. Writing Describe the conditions that will produce a rational function with a graph that has no vertical asymptotes.

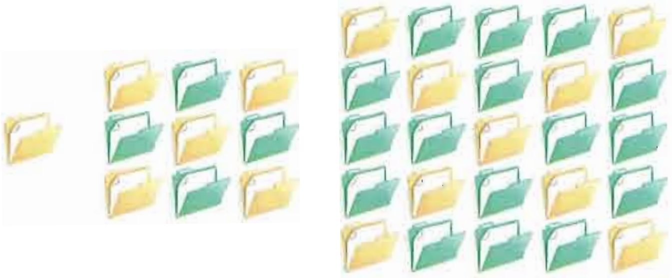
C Challenge

48. Reasoning Look for a pattern in the sequence of file folders below.

a. Write a model for the number of yellow folders $Y(n)$ at each step n.

b. Write a model for the number of green folders $G(n)$ at each step n.

c. Write a model for the ratio of $Y(n)$ to $G(n)$. Use it to predict the ratio of yellow folders to green folders in the next figure. Verify your answer.



49. Write a rational function with the following characteristics.

a. Vertical asymptotes at $x = 1$ and $x = -3$, horizontal asymptote at $y = 1$, zeros at 3 and 4

b. Vertical asymptotes at $x = 0$ and $x = 3$, horizontal asymptote at $y = 0$, a zero at -4

c. Vertical asymptotes at $x = -2$ and $x = 2$, horizontal asymptote at $y = 3$, only one zero at -1.

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.5.6 **50.** What is the x-coordinate of the hole in the graph of $y = \frac{x^2 - 9}{2x^2 - x - 15}$?

MA.912.A.2.12 **51.** Suppose z varies directly with x and inversely with y. If z is 1.5 when x is 9 and y is 4, what is z when x is 6 and y is 0.5?

MA.912.A.7.6 **52.** What is the y-coordinate of the vertex of the parabola $y = -3(x - 4)^2 + 5$?

MA.912.A.4.8 **53.** What is the real solution of $54x^3 - 16 = 0$ written as a fraction?

MA.912.A.8.6 **54.** Using the Change of Base Formula, what is the value of $\log_7 15$ rounded to the nearest hundredth?

Mixed Review

Sketch the asymptotes and the graph of each equation. Identify the domain and range. **See Lesson 8-2.**

55. $y = \frac{3}{x} + 4$

56. $y = \frac{2}{x + 3}$

57. $y = \frac{-1}{x + 1} + 1$

58. $y = \frac{5}{x - 7} - 3$

59. $y = \frac{4}{x}$

60. $y = \frac{-2}{x - 1} + 2$

Find the inverse of each function. Determine if the inverse is a function. **See Lesson 6-7.**

61. $y = 2x - 3$

62. $y = 6 - x$

63. $y = 2x^2$

64. $y = \frac{x^2}{5}$

65. $y = \frac{1}{x + 2}$

66. $y = \sqrt{x - 2} + 1$

Solve each inequality. Graph the solution. **See Lesson 1-5.**

67. $6a - 17 < 47$

68. $2(x + 9) \geq 90$

69. $5(x - 11) + 13 \geq 47$

70. $6 + y < 3y - 2$

71. $49 > 7x + 28$

72. $12 - 2b > 3(b - 3) - 4$

Get Ready! **To prepare for Lesson 8-4, do Exercises 73–76.**

Factor each expression. **See Lesson 4-4.**

73. $2x^2 - 3x + 1$

74. $4x^2 - 9$

75. $5x^2 + 6x + 1$

76. $10x^2 - 10$

Concept Byte

For Use With Lesson 8-3

TECHNOLOGY

Oblique Asymptotes

Sunshine State Standards
MA.912.A.5.6 Identify the oblique asymptote of the graph of a rational function.
LA.910.4.2.1 Write in a variety of technical/informational forms.

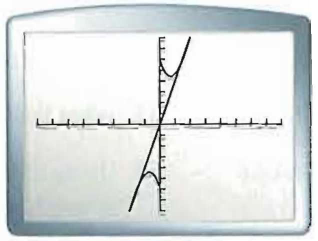
In Lesson 8-2, you saw that the graphs of some rational functions have horizontal and vertical asymptotes. The graphs of some rational functions can have *oblique asymptotes*. **Oblique asymptotes** are asymptotes that are neither horizontal nor vertical. These asymptotes only occur in rational functions in which the degree of the numerator is one greater than the degree of the denominator.

Example 1

Compare the graphs of $y = \frac{6x^2 + 1}{3x}$ and $y = 2x$ using a graphing calculator.

The graph of $y = \frac{6x^2 + 1}{3x}$ gets closer to $y = 2x$ as $|x|$ gets increasingly large.

The graph of $y = 2x$ is an oblique asymptote of $y = \frac{6x^2 + 1}{3x}$.



Example 2

Use a spreadsheet to find the differences between $f(x) = \frac{6x^2 + 1}{3x}$ and $g(x) = 2x$ for the values from 1 to 10.

Step 1 Label each column in Row 1.

Step 2 Enter the x-values in column A.

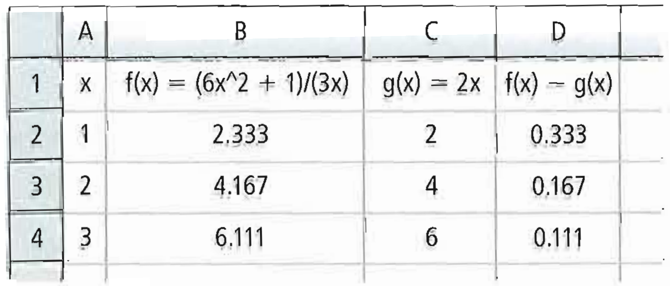
Step 3 Enter the formulas for $f(x)$, $g(x)$, and $f(x) - g(x)$ into cells B2, C2, and D2.

$B2 = (6 \times (A2)^2 + 1))/(3 \times A2)$

$C2 = 2 \times A2$

$D2 = B2 - C2$

Step 4 In columns B, C, and D fill the formulas down to find the values of $f(x)$, $g(x)$, and $f(x) - g(x)$ for each corresponding x-value.



	A	B	C	D
1	x	f(x) = (6x^2 + 1)/(3x)	g(x) = 2x	f(x) − g(x)
2	1	2.333	2	0.333
3	2	4.167	4	0.167
4	3	6.111	6	0.111

As the values of x get larger, the value $6x^2$ becomes much larger than the constant term in the numerator. As a result, the constant term has a smaller effect on the value of the function. So, as x increases, the value of $\frac{6x^2 + 1}{3x}$ gets closer to the value of $\frac{6x^2}{3x} = 2x$, for $x \neq 0$. The spreadsheet confirms this conclusion by showing that the difference between these two values gets closer to zero as x gets larger.

Sometimes it is not as easy to find the oblique asymptote. For example, the asymptote of $y = \frac{2x^2 - 3x + 3}{x - 2}$ is not $\frac{2x^2}{x}$ or $2x$. You can use polynomial division to find the oblique asymptote of any rational function.

Example 3

Determine the oblique asymptote of $y = \frac{2x^2 - 3x + 3}{x - 2}$.

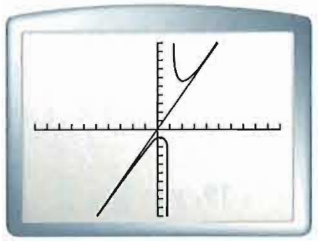
Divide the numerator by the denominator.

$$\begin{array}{r} 2x + 1 \\ x - 2\overline{)2x^2 - 3x + 3} \\ \underline{2x^2 - 4x} \\ x + 3 \\ \underline{x - 2} \\ 5 \end{array}$$

Ignore the remainder. The asymptote is the quotient, $y = 2x + 1$.

Check

Use a graphing calculator to check your answer.



Exercises

Graphing Calculator For each function determine the oblique asymptote. Check with a graphing calculator.

1. $y = \frac{x^2 - 1}{x}$
2. $y = -\frac{2x^2}{3x + 2}$
3. $y = \frac{4 - x^3}{4x^2 - 1}$
4. $y = \frac{2x^4 + 99,999}{x^3}$
5. **Technical Writing** Write a step-by-step manual for classmates to use so they can use a spreadsheet to explore the differences between $f(x)$ and $g(x)$ as the value of x increases.

$$f(x) = \frac{12x^2 - 7}{3x + 4} \qquad g(x) = 4x$$

6. **Open-Ended** Write three rational functions that have an oblique asymptote of $y = 2x$. Graph to check your work.

Describe the asymptotes of the graph of each function.

7. $f(x) = \frac{2x + 1}{x^2 - 1}$
8. $f(x) = \frac{x^2 - 9}{x + 3}$
9. $f(x) = \frac{5x + 11}{4x + 6}$
10. $f(x) = \frac{4x^2 + x - 3}{7x - 1}$
11. $y = \frac{2x^2 - 7x - 5}{2x + 3}$
12. $y = \frac{6x^2 + 14x + 7}{2x + 3}$

8 Mid-Chapter Quiz

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

If $z = 30$ when $x = 3$ and $y = 2$, write the function that models the relationship.

1. z varies jointly with x and y.

2. z varies directly with x and inversely with y.

3. z varies inversely with the product of x and y.

Is the relationship between the values in the table a *direct variation*, an *inverse variation*, or *neither*?

4.

x	y
22	104
35	174
48	239
54	269

5.

x	y
2	−4.8
6	−14.4
12	−28.8
19	−45.6

6.

x	y
15	2.4
18	2
20	1.8
45	0.8

Suppose that x and y vary inversely. Write a function that models the inverse variation.

7. $x = 13$ when $y = 17$

8. $x = -12$ when $y = 4$

9. $x = -52$ when $y = \frac{1}{4}$

Explain how the graph of y_2 is related to the graph of y_1.

10. $y_1 = \frac{4}{x}$ and $y_2 = \frac{9}{x}$

11. $y_1 = \frac{1}{x}$ and $y_2 = \frac{1}{x} + 5$

12. $y_1 = \frac{1}{x-1} + 2$ and $y_2 = \frac{1}{x+1} - 2$

Find any holes and vertical or horizontal asymptotes for the graph of each rational function.

13. $y = \frac{1}{x^2 + 3x - 10}$

14. $y = \frac{x+2}{(x+2)(x-3)}$

15. $y = \frac{x-1}{x^2 - 2x + 1}$

16. $y = \frac{5x-2}{x+2}$

Sketch the graph of each rational function. Then identify the domain and range.

17. $y = \frac{-2}{x}$

18. $y = \frac{5}{x+3} - 4$

19. $y = \frac{x^2 - 9}{2x + 6}$

20. $y = \frac{3x}{x^3 - x}$

21. $y = \frac{x+3}{x-3}$

22. $y = \frac{x^2 - 2x}{x - 2}$

Do you UNDERSTAND?

Open-Ended **Write a rational function with the given characteristics.**

23. a vertical asymptote at $x = 8$ and a horizontal asymptote at $y = 0$

24. a vertical asymptote at $x = -4$ and a horizontal asymptote at $y = 3$

25. a hole at $x = -5$ and a vertical asymptote at $x = 2$

26. **Reasoning** How many inverse variation functions have (2, 3) as a solution?

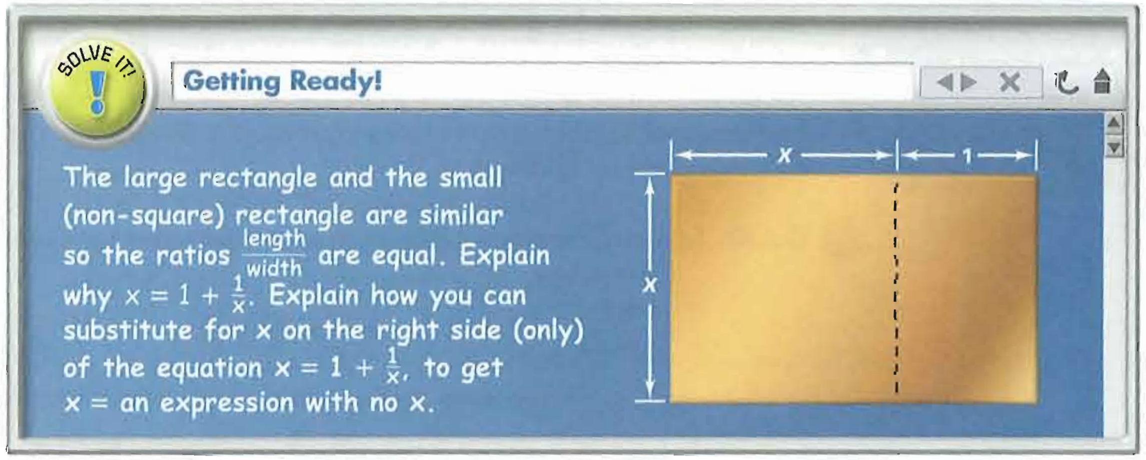
27. **Reasoning** The graph of an inverse variation function contains the point (a, b). Using a and b, identify 3 other points on the graph.

28. **Reasoning** Graph the equations $y = \frac{x^2 + x - 6}{x^2 - 5x + 6}$ and $y = \frac{x+3}{x-3}$. Are they equivalent? Explain.

8-4 Rational Expressions

Sunshine State Standard
MA.912.A.5.2 Add, subtract, multiply, and divide rational expressions.

Objectives To simplify rational expressions
To multiply and divide rational expressions



Getting Ready!

The large rectangle and the small (non-square) rectangle are similar so the ratios $\frac{\text{length}}{\text{width}}$ are equal. Explain why $x = 1 + \frac{1}{x}$. Explain how you can substitute for x on the right side (only) of the equation $x = 1 + \frac{1}{x}$, to get x = an expression with no x.

Lesson Vocabulary
- rational expression
- simplest form

The expression $1 + \frac{1}{x}$ in the Solve It is equivalent to the *rational expression* $\frac{x+1}{x}$. A **rational expression** is the quotient of two polynomials. You will find that, at different times, it is helpful to think of rational expressions as ratios, as fractions, or as quotients.

Essential Understanding You can use much of what you know about multiplying and dividing fractions to multiply and divide rational expressions.

A rational expression is in **simplest form** when its numerator and denominator are polynomials that have no common divisors.

In simplest form	**Not in simplest form**
$\frac{x+1}{x-1}, \frac{x^2+3x+2}{x+3}$	$\frac{x}{x^2}, \frac{3(x-3)}{x-3}, \frac{x^2-x-6}{x^2+x-2}$

You simplify a rational expression by dividing out the common factors in the numerator and the denominator. Factoring the numerator and denominator will help you find the common divisors.

A rational expression and any simplified form must have the same domain in order to be equivalent.

$$\frac{x^2-x-6}{x^2+x-2} = \frac{(x-3)(x+2)}{(x-1)(x+2)} \text{ and } \frac{x-3}{x-1},\ x \neq -2, \text{ are equivalent.}$$

In the example above, you must exclude -2 from the domain of $\frac{x-3}{x-1}$ because -2 is not in the domain of $\frac{x^2-x-6}{x^2+x-2}$. Note that this restriction is not evident from the simplified expression $\frac{x-3}{x-1}$.

Problem 1 Simplifying a Rational Expression

What is $\frac{x^2 + 7x + 10}{x^2 - 3x - 10}$ in simplest form? State any restrictions on the variable.

$$\frac{x^2 + 7x + 10}{x^2 - 3x - 10} = \frac{(x + 2)(x + 5)}{(x + 2)(x - 5)}$$ Factor the numerator and denominator.

$$= \frac{\cancel{(x + 2)}(x + 5)}{\cancel{(x + 2)}(x - 5)}$$ Divide out common factors.

$$= \frac{x + 5}{x - 5}$$ Simplify.

The simplified form is $\frac{x + 5}{x - 5}$ for $x \neq 5$ and $x \neq -2$. The restriction $x \neq -2$ is not evident from the simplified form, but is needed to prevent the denominator of the original expression from being zero.

Think

Is there more than one restriction?
Yes, before you divided the common factors out, $(x + 2)$ was one of the factors of the denominator so $x \neq -2$.

Got It? **1.** What is the rational expression in simplest form? State any restrictions on the variables.

a. $\frac{24x^3y^2}{-6x^2y^3}$ **b.** $\frac{x^2 + 2x - 8}{x^2 - 5x + 6}$ **c.** $\frac{12 - 4x}{x^2 - 9}$

You can use what you know about simplifying rational expressions when you multiply and divide them.

Problem 2 Multiplying Rational Expressions

What is the product $\frac{x^2 + x - 6}{x - 5} \cdot \frac{x^2 - 25}{x^2 + 4x + 3}$ in simplest form? State any restrictions on the variable.

Plan

How is multiplying rational expressions like multiplying fractions?
To multiply rational expressions, you multiply the numerators and multiply the denominators.

$$\frac{x^2 + x - 6}{x - 5} \cdot \frac{x^2 - 25}{x^2 + 4x + 3}$$

$$= \frac{(x + 3)(x - 2)}{x - 5} \cdot \frac{(x + 5)(x - 5)}{(x + 3)(x + 1)}$$ Factor all polynomials.

$$= \frac{\cancel{(x + 3)}(x - 2)}{\cancel{x - 5}} \cdot \frac{(x + 5)\cancel{(x - 5)}}{\cancel{(x + 3)}(x + 1)}$$ Divide out common factors.

$$= \frac{(x - 2)(x + 5)}{x + 1}$$ Simplify.

The product is $\frac{(x - 2)(x + 5)}{x + 1}$ for $x \neq -3$, $x \neq 5$, and $x \neq -1$. The restrictions $x \neq -3$ and $x \neq 5$ are not evident from the simplified form, but are needed to prevent the denominators in the original product from being zero.

Got It? **2.** What is the product $\frac{2x - 8}{x^2 - 16} \cdot \frac{x^2 + 5x + 4}{x^2 + 8x + 16}$ in simplest form? State any restrictions on the variable.

To divide rational expressions, you multiply by the reciprocal of the divisor, just as you do when you divide rational numbers.

Problem 3 Dividing Rational Expressions

What is the quotient $\frac{2 - x}{x^2 + 2x + 1} \div \frac{x^2 + 3x - 10}{x^2 - 1}$ in simplest form? State any restrictions on the variable.

Plan

How do you start?
Think of division as multiplying by the reciprocal.

Think	Write
To divide, you multiply by the reciprocal.	$\frac{2 - x}{x^2 + 2x + 1} \div \frac{x^2 + 3x - 10}{x^2 - 1}$ $= \frac{2 - x}{x^2 + 2x + 1} \cdot \frac{x^2 - 1}{x^2 + 3x - 10}$
The expressions may have common factors. So, factor the numerators and denominators.	$= \frac{2 - x}{(x + 1)(x + 1)} \cdot \frac{(x + 1)(x - 1)}{(x + 5)(x - 2)}$
Factor −1 from (2 − *x*) to get a second (*x* − 2).	$= \frac{-1(x - 2)}{(x + 1)(x + 1)} \cdot \frac{(x + 1)(x - 1)}{(x + 5)(x - 2)}$
Divide out common factors.	$= \frac{-1\cancel{(x - 2)}}{\cancel{(x + 1)}(x + 1)} \cdot \frac{\cancel{(x + 1)}(x - 1)}{(x + 5)\cancel{(x - 2)}}$
Rewrite the remaining factors.	$= \frac{-1(x - 1)}{(x + 1)(x + 5)}$
Identify the restrictions from the denominator of the simplified expression and from any other denominator used.	$x \neq -1$, $x \neq -5$, $x \neq 1$, and $x \neq 2$

Got It? **3. a.** What is the quotient $\frac{x^2 + 5x + 4}{x^2 + x - 12} \div \frac{x^2 - 1}{2x^2 - 6x}$ in simplest form? State any restrictions on the variable.

b. Reasoning Without doing the calculation, what is greatest number of restrictions the quotient $\frac{x^2 + 8x + 7}{x^2 - x - 12} \div \frac{x^2 + 2x - 8}{x^2 + 13x + 24}$ could have? Explain.

Problem 4 Using Rational Expressions to Solve a Problem

Construction **Your community is building a park. It wants to fence in a play space for toddlers. It wants the maximum area for a given amount of fencing. Which shape, a square or a circle, provides a more efficient use of fencing?**

One measure of efficiency is the ratio of *area fenced* to *fencing used,* or area to perimeter. Which of the two shapes has the greater ratio?

Square		**Circle**
$\text{Area} = s^2$	Define area and perimeter.	$\text{Area} = \pi r^2$
$\text{Perimeter} = 4s$		$\text{Perimeter} = 2\pi r$
$s = \frac{P}{4}$	Express *s* and *r* in terms of a common variable, *P*.	$r = \frac{P}{2\pi}$
$\frac{\text{Area}}{\text{Perimeter}} = \frac{s^2}{P}$	Write the ratios.	$\frac{\text{Area}}{\text{Perimeter}} = \frac{\pi r^2}{P}$
$= \frac{\left(\frac{P}{4}\right)^2}{P}$	Substitute for *s* and *r*.	$= \frac{\pi\left(\frac{P}{2\pi}\right)^2}{P}$
$= \frac{P}{16}$	Simplify.	$= \frac{P}{4\pi}$

Think

How can you compare $\frac{P}{16}$ and $\frac{P}{4\pi}$ without evaluating *P*?

Since the numerators are the same, the fraction with the smaller denominator is the larger fraction.

Since $\frac{P}{4\pi} > \frac{P}{16}$, a circle provides a more efficient use of fencing.

Check Assume $P = 40$ ft. The area of the circle is $\pi\left(\frac{40}{2\pi}\right)^2 \approx 127 \text{ ft}^2$. The area of the square is $\left(\frac{40}{4}\right)^2 = 100 \text{ ft}^2$. The area of the circle is greater.

Got It? **4.** Which shape of play space provides for a more efficient use of fencing, a square or an equilateral triangle? (*Hint:* The area of an equilateral triangle in terms of one side is $\frac{1}{2}(s)\left(\frac{\sqrt{3}}{2}s\right)$. The perimeter of an equilateral triangle is 3*s*.)

Lesson Check

Do you know HOW?

Simplify each rational expression. State any restrictions on the variables.

1. $\frac{4z - 12}{8z + 24}$

2. $\frac{3x - 3}{x^2 - x}$

Multiply or divide. State any restrictions on the variables.

3. $\frac{x^2 + 3x - 10}{x^2 + 4x - 12} \cdot \frac{3x + 18}{x + 3}$

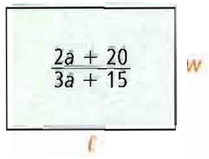
4. $\frac{x^2 - 7x + 10}{x^2 - 8x + 15} \div \frac{4 - x^2}{x^2 + 3x - 18}$

Do you UNDERSTAND?

5. **Vocabulary** Is the equation $y = \frac{x + 1}{x^2 + 1}$ in simplest form? Explain how you can tell.

6. **Error Analysis** A student claims that $x = 2$ is the only solution of the equation $\frac{x}{x - 2} = \frac{2}{x - 2}$. Is the student correct? Explain.

7. **Reasoning** The width of the rectangle is $\frac{a + 10}{3a + 24}$. Write an expression for the length of the rectangle in simplest form.



Practice and Problem-Solving Exercises

A Practice

Simplify each rational expression. State any restrictions on the variables.

See Problem 1.

8. $-\frac{5x^3y}{15xy^3}$

9. $\frac{2x}{4x^2 - 2x}$

10. $\frac{6c^2 + 9c}{3c}$

11. $\frac{49 - z^2}{z + 7}$

12. $\frac{x^2 + 8x + 16}{x^2 - 2x - 24}$

13. $\frac{12 - x - x^2}{x^2 - 8x + 15}$

Multiply. State any restrictions on the variables.

See Problem 2.

14. $\frac{4x^2}{5y} \cdot \frac{7y}{12x^4}$

15. $\frac{2x^4}{10y^{-2}} \cdot \frac{5y^3}{4x^3}$

16. $\frac{8y - 4}{10y - 5} \cdot \frac{5y - 15}{3y - 9}$

17. $\frac{2x + 12}{3x - 9} \cdot \frac{6 - 2x}{3x + 8}$

18. $\frac{x^2 - 4}{x^2 - 1} \cdot \frac{x + 1}{x^2 + 2x}$

19. $\frac{x^2 - 5x + 6}{x^2 - 4} \cdot \frac{x^2 + 3x + 2}{x^2 - 2x - 3}$

Divide. State any restrictions on the variables.

See Problem 3.

20. $\frac{7x}{4y^3} \div \frac{21x^3}{8y}$

21. $\frac{3x^3}{5y^2} \div \frac{6y^{-3}}{5x^{-5}}$

22. $\frac{6x + 6y}{y - x} \div \frac{18}{5x - 5y}$

23. $\frac{3y - 12}{2y + 4} \div \frac{6y - 24}{8 + 4y}$

24. $\frac{x^2}{x^2 + 2x + 1} \div \frac{3x}{x^2 - 1}$

25. $\frac{y^2 - 5y + 6}{y^3} \div \frac{y^2 + 3y - 10}{4y^2}$

See Problem 4.

26. **Industrial Design** A storage tank will have a circular base of radius r and a height of r. The tank can be either cylindrical or hemispherical (half a sphere).
    - **a.** Write and simplify an expression for the ratio of the volume of the hemispherical tank to its surface area (including the base). For a sphere, $V = \frac{4}{3}\pi r^3$ and $SA = 4\pi r^2$.
    - **b.** Write and simplify an expression for the ratio of the volume of the cylindrical tank to its surface area (including the bases).
    - **c.** Compare the ratios of volume to surface area for the two tanks.
    - **d.** Compare the volumes of the two tanks.

B Apply

Simplify each rational expression. State any restrictions on the variables.

27. $\frac{x^2 - 5x - 24}{x^2 - 7x - 30}$

28. $\frac{2y^2 + 8y - 24}{2y^2 - 8y + 8}$

29. $\frac{xy^3 - 9xy}{12xy^2 + 12xy - 144x}$

30. **Open-Ended** Write three rational expressions that simplify to $\frac{x}{x + 1}$.

31. **Think About a Plan** A cereal company wants to use the most efficient packaging for their new product. They are considering a cylindrical-shaped box and a cube-shaped box. Compare the ratios of the volume to the surface area of the containers to determine which packaging will be more efficient.
- How can you measure the cereal box's efficiency?
- What formulas will you need to use to solve this problem?

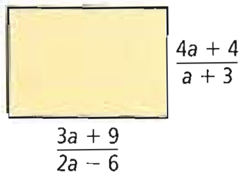
Multiply or divide. State any restrictions on the variables.

32. $\frac{6x^3 - 6x^2}{x^4 + 5x^3} \div \frac{3x^2 - 15x + 12}{2x^2 + 2x - 40}$

33. $\frac{2x^2 - 6x}{x^2 + 18x + 81} \cdot \frac{9x + 81}{x^2 - 9}$

34. $\frac{x^2 - x - 2}{2x^2 - 5x + 2} \div \frac{x^2 - x - 12}{2x^2 + 5x - 3}$

35. $\frac{2x^2 + 5x + 2}{4x^2 - 1} \cdot \frac{2x^2 + x - 1}{x^2 + x - 2}$



36. **Reasoning** Write a simplified expression for the area of the rectangle at the right. State all restrictions on a.

37. **Manufacturing** A toy company is considering a cube or sphere-shaped container for packaging a new product. The height of the cube would equal the diameter of the sphere. Compare the volume-to-surface area ratios of the containers. Which packaging will be more efficient? For a sphere, $SA = 4\pi r^2$.

Decide whether the given statement is *always*, *sometimes*, or *never* true.

38. Rational expressions contain exponents.

39. Rational expressions contain logarithms.

40. Rational expressions are undefined for values of the variables that make the denominator 0.

41. Restrictions on variables change when a rational expression is simplified.

Simplify. State any restrictions on the variables.

42. $\frac{(x^2 - x)^2}{x(x - 1)^{-2}(x^2 + 3x - 4)}$

43. $\frac{2x + 6}{(x - 1)^{-1}(x^2 + 2x - 3)}$

44. $\frac{54x^3y^{-1}}{3x^{-2}y}$

Challenge

45. a. **Reasoning** Simplify $\frac{(2x^n)^2 - 1}{2x^n - 1}$, where x is an integer and n is a positive integer. (*Hint:* Factor the numerator.)
   b. Use the result from part (a) to show that the value of the given expression is always an odd integer.

Use the fact that $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d}$ to simplify each rational expression. State any restrictions on the variables.

46. $\frac{\frac{8x^2y}{x + 1}}{\frac{6xy^2}{x + 1}}$

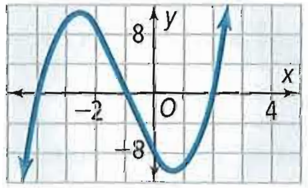
47. $\frac{\frac{3a^3b^3}{a - b}}{\frac{4ab}{b - a}}$

48. $\frac{\frac{9m + 6n}{m^2n^2}}{\frac{12m + 8n}{5m^2}}$

49. $\frac{\frac{x^2 - 1}{x^2 - 9}}{\frac{x^2 + 3x - 4}{x^2 + 8x + 15}}$

Sunshine State Standards Practice

MA.912.A.4.5 **50.** Which function is graphed at the right?



Ⓐ $y = (x + 4)(x - 1)(x + 2)$

Ⓑ $y = (x - 4)(x - 1)(x + 2)$

Ⓒ $y = (x - 4)(x + 1)(x - 2)$

Ⓓ $y = (x + 4)(x + 1)(x - 2)$

MA.912.A.8.1 **51.** Which function generates the table of values at the right?

Ⓕ $y = \log_{\frac{1}{2}} x$

Ⓖ $y = -\log_2 x$

Ⓗ $y = \log_2 x$

Ⓘ $y = \left(\frac{1}{2}\right)^x$

x	y
$\frac{1}{2}$	−1
1	0
2	1
4	2

MA.912.A.5.2 **52.** Which expression equals $\frac{x}{x^2 - 2x - 3} \cdot \frac{2x - 6}{x^2 - 4x + 3}$?

Ⓐ $\frac{2x - 1}{(x - 1)(x + 3)(x + 1)}$

Ⓑ $\frac{2x + 1}{(x - 1)(x + 1)(x - 3)}$

Ⓒ $\frac{2x}{(x - 1)(x + 1)(x - 3)}$

Ⓓ $\frac{2x}{(x + 3)(x - 1)(x + 1)}$

MA.912.A.8.5 **53. Short Response** What is the solution of the equation $3^{-x} = \frac{1}{243}$?

Mixed Review

Find the vertical asymptotes and holes for the graph of each rational function. **See Lesson 8-3.**

54. $y = \frac{x - 3}{x - 3}$ **55.** $y = \frac{x - 1}{(3x + 2)(x + 1)}$ **56.** $y = \frac{(x - 4)(x + 5)}{(x + 3)(x - 4)}$

Evaluate each logarithm. **See Lesson 7-3.**

57. $\log_4 64$ **58.** $\log_2 \frac{1}{32}$ **59.** $\log_5 5\sqrt{5}$ **60.** $\log_{16} 8$

Solve. Check for extraneous solutions. **See Lesson 6-5.**

61. $\sqrt{x} - 3 = 4$ **62.** $\sqrt{x + 1} - 5 = 8$ **63.** $\sqrt{5x - 3} = \sqrt{2x + 3}$

Get Ready! **To prepare for Lesson 8-5, do Exercises 64–67.**

Add or Subtract. **See p. 675.**

64. $\frac{5}{19} + \frac{7}{38}$ **65.** $\frac{2}{15} + \frac{3}{25}$ **66.** $\frac{7}{24} - \frac{5}{36}$ **67.** $\frac{11}{12} - \frac{7}{45}$

8-5 Adding and Subtracting Rational Expressions

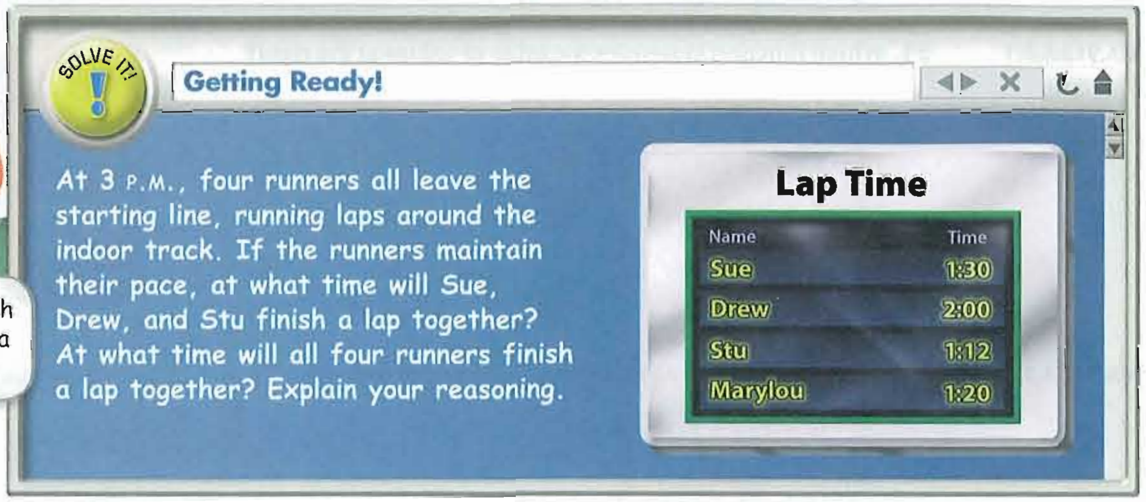
Sunshine State Standards

MA.912.A.5.2 Add, subtract, multiply, and divide rational expressions.

MA.912.A.5.3 Simplify complex fractions.

Objective To add and subtract rational expressions

The runners finish a lap together in a common time.



At 3 P.M., four runners all leave the starting line, running laps around the indoor track. If the runners maintain their pace, at what time will Sue, Drew, and Stu finish a lap together? At what time will all four runners finish a lap together? Explain your reasoning.

Lap Time

Name	Time
Sue	1:30
Drew	2:00
Stu	1:12
Marylou	1:20

Lesson Vocabulary
- complex fraction

You use common multiples of polynomials to add and subtract rational expressions, just as you use common multiples of numbers to add and subtract fractions.

Essential Understanding To operate with rational expressions, you can use much of what you know about operating with fractions.

To add or subtract rational expressions, you first find a common denominator—preferably the least common multiple (LCM) of the denominators.

To find the LCM of several expressions, factor the expressions (numbers or polynomials) completely. The LCM is the product of the prime factors, each raised to the greatest power that occurs in any of the expressions.

Problem 1 Finding the Least Common Multiple

Plan

How do you determine the exponent of each factor for the LCM? Use the exponent from the expression that has that factor to the greatest power.

What is the LCM of $12x^2y(x^2 + 2x + 1)$ and $18xy^3(x^2 + 5x + 4)$?

Step 1 Find the prime factors of each expression.

$$12x^2y(x^2 + 2x + 1) = 2^2 \cdot 3x^2y(x + 1)^2$$

$$18xy^3(x^2 + 5x + 4) = 2 \cdot 3^2xy^3(x + 1)(x + 4)$$

Step 2 Write the product of the prime factors, each raised to the greatest power that occurs in either expression.

$$2^2 \cdot 3^2x^2y^3(x + 1)^2(x + 4)$$

The LCM is $2^2 \cdot 3^2x^2y^3(x + 1)^2(x + 4)$, or $36x^2y^3(x + 1)^2(x + 4)$.

Got It? **1.** What is the LCM of the expressions?

a. $2x + 4$ and $x^2 - x - 6$

b. $x^2 + 3x - 4$, $x^2 + 2x - 8$, and $x^2 - 4x + 4$

The LCM of the denominators of two rational expressions is also the Least Common Denominator (LCD) of the rational expressions. You can use the LCD to add or subtract the expressions.

Recall how you used the LCD to add fractions.

$$\frac{1}{8} + \frac{1}{10} = \frac{1}{2^3} + \frac{1}{2 \cdot 5} = \frac{1}{2^3}\left(\frac{5}{5}\right) + \frac{1}{2 \cdot 5}\left(\frac{2^2}{2^2}\right) = \frac{5}{40} + \frac{4}{40} = \frac{9}{40}$$

Problem 2 Adding Rational Expressions

What is the sum of the two rational expressions in simplest form? State any restrictions on the variable. $\frac{x}{x-1} + \frac{2x-1}{x^2 - 3x + 2}$

Plan

How does the LCD help you simplify this sum?

The LCD is $(x - 1)(x - 2)$. Multiply the first expression by $\frac{x-2}{x-2}$ to get a common denominator.

$$\frac{x}{x-1} + \frac{2x-1}{x^2 - 3x + 2} = \frac{x}{x-1} + \frac{2x-1}{(x-1)(x-2)}$$

Factor the denominators.

$$= \frac{x}{x-1} \cdot \frac{x-2}{x-2} + \frac{2x-1}{(x-1)(x-2)}$$

Rewrite each expression with the LCD.

$$= \frac{x^2 - 2x}{(x-1)(x-2)} + \frac{2x-1}{(x-1)(x-2)}$$

$$= \frac{x^2 - 2x + 2x - 1}{(x-1)(x-2)}$$

Add the numerators. Combine like terms.

$$= \frac{x^2 - 1}{(x-1)(x-2)}$$

$$= \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)}$$

Factor the numerator and divide out the common factors.

$$= \frac{x+1}{x-2}, x \neq 1$$

The sum of the expressions is $\frac{x+1}{x-2}$ for $x \neq 1$ and $x \neq 2$.

Got It? **2.** What is the sum of the two rational expressions in simplest form? State any restrictions on the variable.

a. $\frac{x+1}{x-1} + \frac{-2}{x^2 - x}$

b. $\frac{x}{x^2 - 4} + \frac{1}{x+2}$

c. **Reasoning** Is it possible to add the rational expressions in Problem 2 by finding a common denominator, but not the *least* common denominator? Explain.

Problem 3 Subtracting Rational Expressions

What is the difference of the two rational expressions in simplest form? State any restrictions on the variable. $\frac{x+2}{x^2-2x} - \frac{x+2}{2x-4}$

Plan

How is this problem similar to Problem 2?
The method is the same except you subtract the rational expressions instead of adding them.

$$\frac{x+2}{x^2-2x} - \frac{x+2}{2x-4} = \frac{x+2}{x(x-2)} - \frac{x+2}{2(x-2)}$$ Factor the denominators.

The LCD is $2x(x-2)$.

$$= \frac{x+2}{x(x-2)} \cdot \frac{2}{2} - \frac{x+2}{2(x-2)} \cdot \frac{x}{x}$$ Rewrite each expression with the LCD.

$$= \frac{2(x+2)}{2x(x-2)} - \frac{x(x+2)}{2x(x-2)}$$

$$= \frac{2x+4}{2x(x-2)} - \frac{x^2+2x}{2x(x-2)}$$ Simplify the numerators.

$$= \frac{2x+4-(x^2+2x)}{2x(x-2)}$$ Subtract the numerators.

$$= \frac{-x^2+4}{2x(x-2)}$$ Combine like terms.

$$= \frac{-(x^2-4)}{2x(x-2)}$$ Factor -1 from the numerator.

$$= \frac{-\cancel{(x-2)}(x+2)}{2x\cancel{(x-2)}}$$ Factor $x^2 - 4$ and divide out the common factors.

$$= \frac{-(x+2)}{2x}$$

The difference is $\frac{-(x+2)}{2x}$ for $x \neq 2$ and $x \neq 0$.

Got It? 3. What is the difference of the two rational expressions in simplest form? State any restrictions on the variable.

a. $\frac{x+3}{x-2} - \frac{6x-7}{x^2-3x+2}$

b. $\frac{x-1}{x+5} - \frac{x+3}{x^2+6x+5}$

A **complex fraction** is a rational expression that has at least one fraction in its numerator or denominator or both. Here are some examples.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{xy}} \qquad \frac{\frac{x+3}{2}}{\frac{2}{x-4}} \qquad \frac{\frac{x+3}{x^2-2x+1}+\frac{x}{x^2-3x+2}}{\frac{x}{x^2-4x+4}-\frac{2}{x^2-4}}$$

Sometimes you can simplify a complex fraction by multiplying the numerator and the denominator by the LCD of all the rational expressions. You can also simplify them by combining the fractions in the numerator and those in the denominator. Then multiply the new numerator by the reciprocal of the new denominator.

Problem 4 Simplifying a Complex Fraction

What is a simpler form of the complex fraction?

$$\frac{\frac{1}{x}+\frac{x}{y}}{\frac{1}{y}+1}$$

Think

What is the LCD of $\frac{1}{x}$, $\frac{x}{y}$, and $\frac{1}{y}$?
The LCD of the rational expressions is *xy*.

Method 1 Multiply both the numerator and the denominator by the LCD of all the rational expressions and simplify the result.

$$\frac{\frac{1}{x}+\frac{x}{y}}{\frac{1}{y}+1}=\frac{\left(\frac{1}{x}+\frac{x}{y}\right)\cdot xy}{\left(\frac{1}{y}+1\right)\cdot xy}$$ Multiply the numerator and the denominator by *xy*.

$$=\frac{\frac{1}{x}\cdot xy+\frac{x}{y}\cdot xy}{\frac{1}{y}\cdot xy+1\cdot xy}$$ Use the Distributive Property.

$$=\frac{y+x^2}{x+xy}$$ Simplify.

Method 2 Combine the expressions in the numerator and those in the denominator. Then multiply the new numerator by the reciprocal of the new denominator.

$$\frac{\frac{1}{x}+\frac{x}{y}}{\frac{1}{y}+1}=\frac{\frac{1}{x}\cdot\frac{y}{y}+\frac{x}{y}\cdot\frac{x}{x}}{\frac{1}{y}+1\cdot\frac{y}{y}}$$ Write equivalent expressions with common denominators.

$$=\frac{\frac{y}{xy}+\frac{x^2}{xy}}{\frac{1}{y}+\frac{y}{y}}$$ Multiply.

$$=\frac{\frac{y+x^2}{xy}}{\frac{1+y}{y}}$$ Add.

Think

How do you divide a fraction by a fraction?
Multiply the numerator by the reciprocal of the denominator.

$$=\frac{y+x^2}{xy}\div\frac{1+y}{y}$$ Divide the numerator fraction by the denominator fraction.

$$=\frac{y+x^2}{xy}\cdot\frac{y}{1+y}$$ Multiply by the reciprocal.

$$=\frac{y+x^2}{x+xy}$$ Divide out the common factor, *y*.

Got It? 4. What is a simpler form of the complex fraction?

a. $\dfrac{x}{\frac{1}{x}+\frac{1}{y}}$

b. $\dfrac{\frac{x-2}{x}+\frac{2}{x+1}}{\frac{3}{x-1}-\frac{1}{x+1}}$

Problem 5 Using Rational Expressions to Solve a Problem

Fuel Economy A woman drives an SUV that gets 10 mi/gal (mpg). Her husband drives a hybrid that gets 60 mpg. Every week, they travel the same number of miles. They want to improve their combined mpg. They have two options on how they can improve it.

Option 1: They can tune the SUV and increase its mileage by 1 mpg and keep the hybrid as it is.

Option 2: They can buy a new hybrid that gets 80 mpg and keep the SUV as it is.

Which option will give them a better combined mpg?

Think	Write	
The combined gas mileage is total miles divided by total gallons.	combined mpg $= \frac{\text{SUV miles} + \text{Hybrid miles}}{\text{SUV gallons} + \text{Hybrid gallons}}$ Let x = number of miles each drives in a week.	
Define a variable and describe each option.	Option 1 Tuned SUV gets 11 mpg. Hybrid gets 60 mpg.	Option 2 SUV gets 10 mpg. New hybrid gets 80 mpg.
The gallons used by each vehicle are $\frac{\text{miles}}{\text{mpg}}$. Write the variable expressions for each option's combined mpg.	$\frac{x + x}{\frac{x}{11} + \frac{x}{60}}$	$\frac{x + x}{\frac{x}{10} + \frac{x}{80}}$
Find the LCD of the fractions in each expression. Multiply the numerator and denominator by the LCD.	$\left(\frac{2x}{\frac{x}{11} + \frac{x}{60}}\right) \cdot \left(\frac{660}{660}\right)$	$\left(\frac{2x}{\frac{x}{10} + \frac{x}{80}}\right) \cdot \left(\frac{80}{80}\right)$
Distribute and simplify.	$\frac{(2x)(660)}{\left(\frac{x}{11}\right)(660) + \left(\frac{x}{60}\right)(660)}$ $= \frac{1320x}{60x + 11x}$ $= \frac{1320x}{71x}$	$\frac{(2x)(80)}{\left(\frac{x}{10}\right)(80) + \left(\frac{x}{80}\right)(80)}$ $= \frac{160x}{8x + x}$ $= \frac{160x}{9x}$
Round the ratios and compare them.	$\approx$ 18.6 mpg Option 1 gives the better combined mpg.	$\approx$ 17.8 mpg

Got It? **5.** Suppose Option 3 is to buy a new hybrid that will get double the mileage of the present hybrid. The SUV mileage stays the same. Which of the three options will give the best combined mpg?

Lesson Check

Do you know HOW?

Simplify each sum or difference. State any restrictions on the variables.

1. $\frac{a+11}{3a-5} + \frac{a-21}{3a-5}$

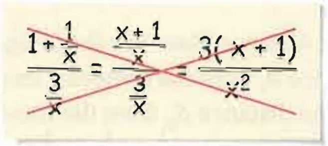
2. $\frac{1}{x^2-4} + \frac{6}{x+2}$

3. $\frac{m}{3m+6} - \frac{4m}{m+2}$

4. $\frac{b-4}{b^2+2b-8} - \frac{b+2}{b^2-16}$

Do you UNDERSTAND?

5. **Error Analysis** Describe and correct the error made in simplifying the complex fraction.



$$\cancel{\frac{1+\frac{1}{x}}{\frac{3}{x}} = \frac{\frac{x+1}{x}}{\frac{3}{x}} = \frac{3(x+1)}{x^2}}$$

6. **Open-Ended** Write two rational expressions that simplify to $\frac{x+1}{x-5}$.

Practice and Problem-Solving Exercises

A Practice

Find the least common multiple of each pair of polynomials. See Problem 1.

7. $9(x+2)(2x-1)$ and $3(x+2)$

8. x^2-1 and x^2+2x+1

9. $5y^2-80$ and $y+4$

10. $x^2-32x-10$ and $2x+10$

Simplify each sum or difference. State any restrictions on the variables. See Problems 2 and 3.

11. $\frac{1}{2x} + \frac{1}{2x}$

12. $\frac{d-3}{2d+1} + \frac{d-1}{2d+1}$

13. $\frac{-2}{x} - \frac{1}{x}$

14. $\frac{-5y}{2y-1} - \frac{y+3}{2y-1}$

15. $\frac{5y+2}{xy^2} + \frac{2x-4}{4xy}$

16. $\frac{5x}{x^2-9} + \frac{2}{x+4}$

17. $\frac{y}{2y+4} - \frac{3}{y+2}$

18. $\frac{x}{3x+9} - \frac{8}{x^2+3x}$

19. $\frac{-3x}{x^2-9} + \frac{4}{2x-6}$

20. $\frac{5x}{x^2-x-6} + \frac{4}{x^2+4x+4}$

21. $\frac{2x}{x^2-x-2} - \frac{4x}{x^2-3x+2}$

Simplify each complex fraction. See Problem 4.

22. $\frac{\frac{1}{x}}{\frac{2}{y}}$

23. $\frac{1-\frac{1}{4}}{2-\frac{3}{5}}$

24. $\frac{\frac{2}{x+y}}{3}$

25. $\frac{\frac{1}{3}}{\frac{3}{b}}$

26. $\frac{1}{1+\frac{x}{y}}$

27. $\frac{3}{\frac{2}{x}+y}$

28. $\frac{\frac{2}{x+y}}{\frac{5}{x+y}}$

29. $\frac{\frac{3}{x-4}}{1-\frac{2}{x-4}}$

30. Your car gets 25 mi/gal around town and 30 mi/gal on the highway. See Problem 5.

a. If 50% of the miles you drive are on the highway and 50% are around town, what is your overall average miles per gallon?

b. If 60% of the miles you drive are on the highway and 40% are around town, what is your overall average miles per gallon?

B Apply

Add or subtract. Simplify where possible. State any restrictions on the variables.

31. $\frac{3}{4x} - \frac{2}{x^2}$

32. $\frac{3}{x+1} + \frac{x}{x-1}$

33. $\frac{4}{x^2-9} + \frac{7}{x+3}$

34. $\frac{5x}{x^2-x-6} - \frac{4}{x^2+4x+4}$

35. $3x + \frac{x^2+5x}{x^2-2}$

36. $\frac{5y}{y^2-7y} - \frac{4}{2y-14} + \frac{9}{y}$

37. Think About a Plan For the image of the overhead projector to be in focus, the distance d_i from the projector lens to the image, the projector lens focal length f, and the distance d_o from the transparency to the projector lens must satisfy the thin-lens equation $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$. What is the focal length of the projector lens if the transparency placed 4 in. from the projector lens is in focus on the screen located 8 ft from the projector lens?

- Can you write the equation for the unknown variable?
- What units would you use for the focal length of the lens?

38. Optics To read small font, you use a magnifying lens with the focal length 3 in. How far from the magnifying lens should you place the page if you want to hold the lens at 1 foot from your eyes? Use the thin-lens equation from Exercise 37.

39. Open-Ended Write two complex fractions that simplify to $\frac{x-2}{x+4}$.

40. Writing Explain how factoring is used when adding or subtracting rational expressions. Include an example in your explanation.

Simplify each complex fraction.

41. $\dfrac{\frac{2}{x} + \frac{3}{y}}{\frac{-5}{x} + \frac{7}{y}}$

42. $\dfrac{1 + \frac{2}{x}}{2 + \frac{3}{2x}}$

43. $\dfrac{\frac{1}{xy} - \frac{1}{y^2}}{\frac{1}{x^2y} - \frac{1}{xy^2}}$

44. $\dfrac{\frac{2}{x+4} + 2}{1 + \frac{3}{x+4}}$

45. Harmony The harmonic mean of two numbers a and b equals $\dfrac{2}{\frac{1}{a} + \frac{1}{b}}$. As you vary the length of a violin or guitar string, its pitch changes. If a full-length string is 1 unit long, then many lengths that are simple fractions produce pitches that harmonize, or sound pleasing together. The harmonic mean relates two lengths that produce harmonious sounds. Find the harmonic mean for each pair of string lengths.

a. 1 and $\frac{1}{2}$ **b.** $\frac{3}{4}$ and $\frac{1}{2}$ **c.** $\frac{3}{4}$ and $\frac{3}{5}$ **d.** $\frac{1}{2}$ and $\frac{1}{4}$

C Challenge

46. Electricity The resistance of a parallel circuit with 3 bulbs is $\dfrac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$.

a. Find the resistance of a parallel circuit with 3 bulbs that have resistances 5 ohms, 4 ohms, 2.5 ohms.

b. The resistance of a parallel circuit with 3 bulbs is 1.5 ohms. Find the resistances of the bulbs, if two of them have equal resistances, while the resistance of the 3rd is 3 ohms less.

47. Show that the sum of the reciprocals of three different positive integers is greater than 6 times the reciprocal of their product.

Sunshine State Standards Practice

MA.912.A.5.2 **48.** Which expression equals $\frac{5x}{x^2 - 9} - \frac{4x}{x^2 + 5x + 6}$?

Ⓐ $\frac{7x}{(x - 3)(x + 3)(x + 2)}$ Ⓒ $\frac{x^2 + 22x}{(x - 3)(x + 3)(x + 2)}$

Ⓑ $\frac{x^2 - 2x}{(x - 3)(x + 3)(x + 2)}$ Ⓓ $\frac{9x^2 - 2x}{(x - 3)(x + 3)(x + 2)}$

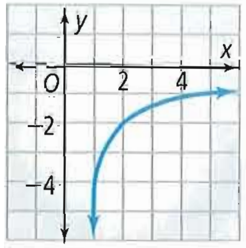
MA.912.A.8.3 **49.** Which of the relationships is represented by the graph at the right?

Ⓕ $y = \log_4(x - 1) + 5$

Ⓖ $y = \log_4(x - 1) - 2$

Ⓗ $y = \log_4(x + 2) - 2$

Ⓘ $y = \log_4(x - 1) - 1$



MA.912.A.5.3 **50.** What is a simpler form of $\frac{\frac{2}{x} - 5}{\frac{6}{x} - 3}$?

Ⓐ $\frac{2 - 5x}{6 - 3x}$ Ⓑ $\frac{2 + 5x}{6 - 3x}$ Ⓒ $\frac{2x - 5}{6x + 3}$ Ⓓ $\frac{6 + 3x}{2 - 5x}$

MA.912.A.10.3 **51.** What word makes the statement "The domain and range of a(n) ____________ function is the set of all real numbers" *sometimes* true?

Ⓕ polynomial Ⓗ exponential

Ⓖ logarithmic Ⓘ quadratic

MA.912.A.5.2 **52. Short Response** What is the least common denominator for the rational expressions $\frac{1}{x^2 - 5x - 6}$ and $\frac{1}{x^2 - 12x + 36}$? Show your work.

Mixed Review

Divide. State any restrictions on the variable. See Lesson 8-4.

53. $\frac{3x^2 - 9x}{x - 2} \div \frac{x^2 - 9}{4x - 8}$ **54.** $\frac{3x - 6}{12x - 24} \div \frac{x^2 - 5x + 6}{3x^2 - 12}$ **55.** $\frac{5x + 15}{10x - 10} \div \frac{x^2 + 6x + 9}{3x^2 - 3}$

Write each logarithmic expression as a single logarithm. See Lesson 7-4.

56. $\log_3 y + 4\log_3 t$ **57.** $7\log p + 2\log q$ **58.** $\log_5 x - \frac{1}{5}\log_5 y$

Let $f(x) = x^2 + 1$ and $g(x) = 3x$. Evaluate each expression. See Lesson 6-6.

59. $(g \circ f)(-3)$ **60.** $(f \circ g)(-3)$ **61.** $(g \circ f)\left(\frac{1}{2}\right)$ **62.** $(f \circ f)(3)$

Get Ready! **To prepare for Lesson 8-6, do Exercises 63–65.**

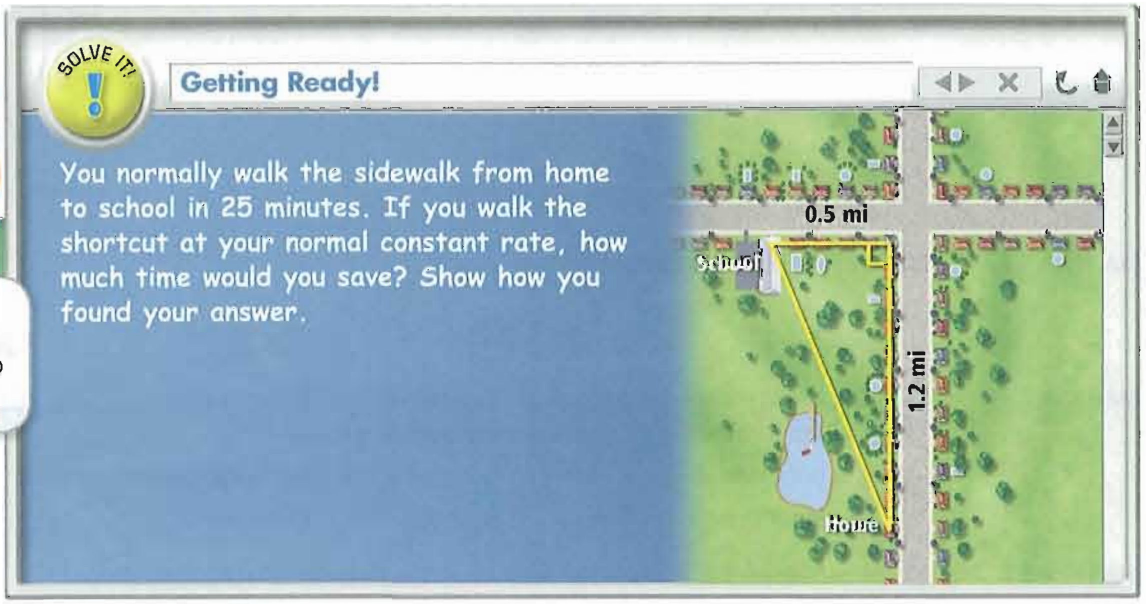
Solve each equation. Check your answers. See Lesson 1-4.

63. $-3(x - 4) = 2(x + 8)$ **64.** $0.2(x + 8) - 3.4 = 2.4$ **65.** $\frac{x}{2} + \frac{x}{3} = 15$

8-6 Solving Rational Equations

Sunshine State Standard
MA.912.A.5.5 Solve rational equations.

Objectives To solve rational equations
To use rational equations to solve problems



You normally walk the sidewalk from home to school in 25 minutes. If you walk the shortcut at your normal constant rate, how much time would you save? Show how you found your answer.

Lesson Vocabulary
- rational equation

Sometimes you can solve a problem using a proportion—an equation involving two rational expressions set equal to each other.

Essential Understanding To solve an equation containing rational expressions, first multiply each side by the least common denominator of the rational expressions. Doing this, however, can introduce extraneous solutions.

A **rational equation** contains at least one rational expression. You can simplify solving a rational equation if you first clear the equation of denominators. You can do this by multiplying by the LCD of the rational expressions in the equation.

Rational Equation	Not a Rational Equation
$\frac{x}{x+1} + \frac{x}{x-1} = \frac{2}{x^2-1}$	$x + \frac{1}{2} = \frac{2}{3}$

Any time you multiply each side of an equation by an algebraic expression, it is possible to introduce an extraneous solution. Recall that an extraneous solution is a solution of the derived equation, but not a solution of the original equation. You must check all solutions in the original equation to confirm that they are indeed solutions.

Problem 1 Solving a Rational Equation

What are the solutions of the rational equation?

A $\frac{x}{x-3}+\frac{x}{x+3}=\frac{2}{x^2-9}$

Think	Write
Factor the denominators to find the LCD.	$\frac{x}{x-3}+\frac{x}{x+3}=\frac{2}{x^2-9}$ $\frac{x}{x-3}+\frac{x}{x+3}=\frac{2}{(x-3)(x+3)}$
Multiply each side by the LCD to clear denominators.	$(x-3)(x+3)\left[\frac{x}{x-3}+\frac{x}{x+3}\right]=(x-3)(x+3)\frac{2}{(x-3)(x+3)}$
Now distribute, simplify, and solve.	$x(x+3)+x(x-3)=2$ $x^2+3x+x^2-3x=2$ $2x^2=2$ $x^2=1$, so $x=\pm1$
Check whether $x = 1$ or $x = -1$ is extraneous. Use the original equation.	$\frac{1}{1-3}+\frac{1}{1+3}\overset{?}{=}\frac{2}{(1)^2-9}$ $\quad$ $\frac{-1}{-1-3}+\frac{-1}{-1+3}\overset{?}{=}\frac{2}{(-1)^2-9}$ $-\frac{1}{2}+\frac{1}{4}=-\frac{1}{4}$ ✓ $\quad$ $\frac{1}{4}+-\frac{1}{2}=-\frac{1}{4}$ ✓
Write the solutions.	The solutions are $x = 1$ and $x = -1$.

B $\frac{x-1}{x^2+3x+2}+\frac{2x}{x+2}=\frac{x-1}{x+1}$

$$\frac{x-1}{x^2+3x+2}+\frac{2x}{x+2}=\frac{x-1}{x+1}$$

The LCD is $(x + 2)(x + 1)$.

$$(x+2)(x+1)\left[\frac{x-1}{(x+2)(x+1)}+\frac{2x}{x+2}\right]=(x+2)(x+1)\left(\frac{x-1}{x+1}\right)$$

$$(x-1)+2x(x+1)=(x+2)(x-1)$$

$$2x^2+3x-1=x^2+x-2$$

$$x^2+2x+1=0$$

$$(x+1)(x+1)=0\text{, so } x=-1$$

The solution $x = -1$ is extraneous because the original equation restricts x so that $x \neq -2$ and $x \neq -1$. There is no solution to this equation.

Think

How is this rational equation related to a quadratic equation? Multiplying each side of the equation by the LCD of the rational expressions turns it into a quadratic equation.

Got It? **1.** What are the solutions of the rational equation?

a. $\frac{x-1}{x+2}=\frac{x^2+2x-3}{x+2}$

b. $\frac{x}{x+1}+\frac{3}{x+4}=\frac{x+3}{x+4}$

Problem 2 Using Rational Equations

Flight A flight across the U.S. takes longer east to west than it does west to east. Assume that winds are constant in the eastward direction. When flying westward, the headwind decreases the airplane's speed. When flying eastward, the tailwind increases its speed. The time for a round trip shown at the right is $7\frac{3}{4}$h. If the airplane cruises at 480 mi/h, what is the speed of the wind?

Let x = the wind speed.

Rate × Time = Distance, so Time $= \frac{\text{Distance}}{\text{Rate}}$

	Distance	Rate	Time
Going west to east	1850	$480 + x$	$\frac{1850}{480 + x}$
Going east to west	1850	$480 - x$	$\frac{1850}{480 - x}$

Total time = Time west to east + Time east to west

$$7.75 = \frac{1850}{480 + x} + \frac{1850}{480 - x}$$

Multiply both sides by the LCD, $(480 + x)(480 - x)$.

$$(480 + x)(480 - x)\,7.75 = (480 + x)(480 - x)\frac{1850}{480 + x} + (480 + x)(480 - x)\frac{1850}{480 - x}$$

$$7.75(480 + x)(480 - x) = 1850(480 - x) + 1850(480 + x)$$

$$1{,}785{,}600 - 7.75x^2 = 888{,}000 - 1850x + 888{,}000 + 1850x$$

$$-7.75x^2 = -9600$$

$$x^2 = \frac{-9600}{-7.75}$$

$$x \approx \pm 35$$

Wind speed is positive, so $x \approx 35$. The west-to-east wind speed is about 35 mi/h.

Check $7.75 = \frac{1850}{480 + x} + \frac{1850}{480 - x}$

$$7.75 \stackrel{?}{=} \frac{1850}{480 + 35} + \frac{1850}{480 - 35}$$

$$7.75 \approx 3.6 + 4.2 ✓$$

Think

If you substitute 35 for x will the equation check exactly?

No; since 35 is an approximation it is likely that the values will be nearly equal, but probably not equal.

Got It? **2. a.** You ride your bike to a store, 4 mi away, to pick up things for dinner. When there is no wind, you ride at 10 mi/h. Today your trip to the store and back took 1 hour. What was the speed of the wind today?

b. Reasoning Explain why there is no difference between the travel time to and from the store when there is no wind.

You can also use a graphing calculator to solve a rational equation.

Problem 3 Using a Graphing Calculator to Solve a Rational Equation

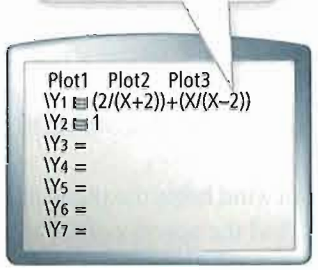
What are the solutions of the rational equation? Use a graphing calculator to solve.

$$\frac{2}{x+2} + \frac{x}{x-2} = 1$$

Plan

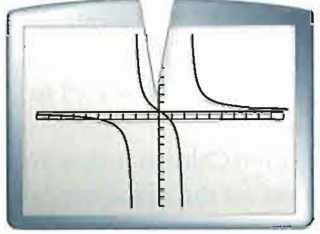
How do the graphs of the two sides of the equation help you solve the equation? The *x*-values of the points of intersection are the solutions to the equation.

Enter one side of the equation as Y_1. Enter the other side as Y_2.

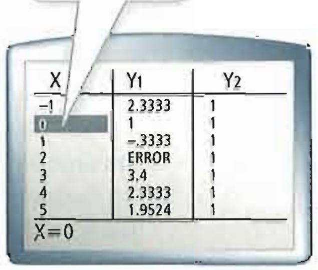


```
Plot1  Plot2  Plot3
\Y1 ■ (2/(X+2))+(X/(X−2))
\Y2 ■ 1
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

There appears to be only one intersection point, at $x = 0$.



$Y_1 = Y_2$ when $x = 0$.



X	Y1	Y2
−1	2.3333	1
0	1	1
1	−.3333	1
2	ERROR	1
3	3.4	1
4	2.3333	1
5	1.9524	1

X=0

The solution is $x = 0$.

Check $\frac{2}{x+2} + \frac{x}{x-2} = 1$

$$\frac{2}{0+2} + \frac{0}{0-2} \stackrel{?}{=} 1$$

$$1 + 0 = 1 \checkmark$$

Got It? **3.** What are the solutions of the rational equation $\frac{x+2}{1-2x} = 5$? Use a graphing calculator to solve.

Lesson Check

Do you know HOW?

Solve each equation. Check each solution.

1. $\frac{4}{x-2} = \frac{x-1}{x-2}$

2. $\frac{2a+1}{6} + \frac{a}{2} = \frac{a-1}{3}$

3. $\frac{2}{n} + \frac{n+2}{n+1} = \frac{-2}{n^2+n}$

4. **Flight** If the speed of an airplane is 350 mi/h with a tail wind of 40 mi/h, what is the speed of the plane in still air?

Do you UNDERSTAND?

5. **Error Analysis** Describe and correct the error made in solving the equation.

$$\frac{5}{x} + \frac{9}{7} = \frac{28}{x}$$
$$\frac{14}{x+7} = \frac{28}{x}$$
$$14x = 28(x+7)$$
$$14x = 28x + 196$$
$$-196 = 14x$$
$$-14 = x$$

6. **Open-Ended** Write a rational equation using expressions that have $x^2 - 9$ as their LCD.

7. **Reasoning** Describe two methods you can use to check whether a solution is extraneous.

Practice and Problem-Solving Exercises

A Practice

Solve each equation. Check each solution. **See Problem 1.**

8. $\frac{1}{4} - x = \frac{x}{8}$

9. $\frac{y}{5} + \frac{y}{2} = 7$

10. $\frac{2x}{3} - \frac{1}{2} = \frac{2x + 5}{6}$

11. $\frac{3x - 2}{12} - \frac{1}{6} = \frac{1}{6}$

12. $\frac{1}{x} + \frac{x}{2} = \frac{x + 4}{2x}$

13. $\frac{11}{3x} - \frac{1}{3} = \frac{-4}{x^2}$

14. $\frac{3}{2x} - \frac{5}{3x} = 2$

15. $\frac{5x}{4} - \frac{3}{x} = \frac{1}{4}$

16. $\frac{2}{y} + \frac{1}{2} = \frac{5}{2y}$

17. $x + \frac{6}{x} = -5$

18. $\frac{1}{4x} - \frac{3}{4} = \frac{7}{x}$

19. $\frac{5}{2x} - \frac{2}{3} = \frac{1}{x} + \frac{5}{6}$

20. Transportation The speed s of an airplane is given by $s = \frac{d}{t}$, where d represents the distance and t is the time. **See Problem 2.**

a. A plane flies 700 miles from New York to Chicago at a speed of 360 mi/h. Find the time for the trip.

b. On the return trip from Chicago to New York, a tail wind helps the plane move faster. The total flying time for the round trip is 3.5 h. Find the speed x of the tail wind.

Graphing Calculator **Solve each equation. Check each solution.** **See Problem 3.**

21. $\frac{3}{x} = 5$

22. $\frac{1}{3x} = -2$

23. $\frac{2}{x - 1} = 4$

24. $\frac{4}{x + 3} = 5$

25. $\frac{5x - 2}{x - 4} = -3$

26. $\frac{3x - 1}{x + 2} = 7$

27. $\frac{2}{x} = \frac{x}{2}$

28. $\frac{2}{x + 3} = \frac{x - 3}{2}$

29. $\frac{2}{x - 1} + \frac{3}{x + 1} = 4$

B Apply

Solve each equation for the given variable.

30. $m = \frac{2E}{V^2}$; E

31. $\frac{c}{E} - \frac{1}{mc} = 0$; E

32. $\frac{m}{F} = \frac{1}{a}$; F

33. $\frac{1}{c} - \frac{c}{a^2 - b^2} = 0$; c

34. $\frac{\ell}{T^2} = \frac{g}{4\pi^2}$; T

35. $\frac{q}{m} = \frac{2V}{B^2r^2}$; B

36. Think About a Plan You and a classmate have volunteered to contact every member of your class by phone to inform them of an upcoming event. You can complete the calls in six days if you work alone. Your classmate can complete them in four days. How long will it take to complete the calls working together?

- If N is the total number of calls, what expression represents the number of calls that you can make per day? What expression represents the number of calls your friend can make per day?
- What is the expression for the number of days needed to make N calls if you are working together?

37. Storage One pump can fill a tank with oil in 4 hours. A second pump can fill the same tank in 3 hours. If both pumps are used at the same time, how long will they take to fill the tank?

38. **Teamwork** You can stuff envelopes twice as fast as your friend. Together, you can stuff 6750 envelopes in 4.5 hours. How long would it take each of you working alone to complete the job?

39. **Grades** On the first four tests of the term your average is 84%. You think you can score 96% on each of the remaining tests. How many consecutive test scores of 96% would you need to bring your average up to 90% for the term?

40. **Error Analysis** Describe and correct the error made in solving the equation.

~~$x - \frac{2}{x-2} = \frac{x+1}{x+2}$~~

~~$x - 2(x + 2) = (x + 1)(x - 2)$~~

~~$x - 2x - 4 = x^2 - x - 2$~~

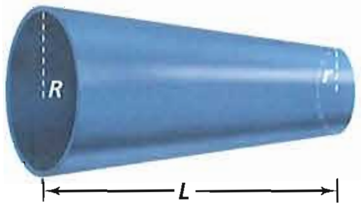
~~$0 = x^2 + 2$~~

~~$-2 = x^2$~~

~~There is no square root of a negative number, so the equation has no solution.~~

41. **Fuel Economy** Suppose you drive an average of 15,000 miles per year, and your car gets 24 miles per gallon. Suppose gasoline costs \$3.60 a gallon.
   - **a.** How much money do you spend each year on gasoline?
   - **b.** You plan to trade in your car for one that gets x more miles per gallon. Write an expression to represent the new yearly cost of gasoline.
   - **c.** Write an expression to represent your total savings on gasoline per year.
   - **d.** Suppose you can save \$600 a year with the new car. How many miles per gallon does the new car get?

42. **Woodworking** A tapered cylinder is made by decreasing the radius of a rod continuously as you move from one end to the other. The rate at which it tapers is the taper per foot. You can calculate the taper per foot using the formula $T = \frac{24(R - r)}{L}$. The lengths R, r, and L are measured in inches.
   - **a.** Solve this equation for L.
   - **b.** What is L for $T = 0.75, 0.85$, and 0.95, if $R = 4$ in.; $r = 3$ in.?



Solve each equation. Check each solution.

43. $\frac{15}{x} + \frac{9x - 7}{x + 2} = 9$

44. $\frac{2}{x + 2} - \frac{1}{x} = \frac{-4}{x(x + 2)}$

45. $\frac{1}{b + 1} + \frac{1}{b - 1} = \frac{2}{b^2 - 1}$

46. $c - \frac{c}{3} + \frac{c}{5} = 26$

47. $\frac{1}{x - 5} = \frac{x}{x^2 - 25}$

48. $\frac{k}{k + 1} + \frac{k}{k - 2} = 2$

49. $\frac{5}{x^2 - 7x + 12} - \frac{2}{3 - x} = \frac{5}{x - 4}$

50. $\frac{10}{2y + 8} - \frac{7y + 8}{y^2 - 16} = \frac{-8}{2y - 8}$

51. $\frac{7x + 3}{x^2 - 8x + 15} + \frac{3x}{x - 5} = \frac{1}{3 - x}$

52. $\frac{2}{x + 3} - \frac{3}{4 - x} = \frac{2x - 2}{x^2 - x - 12}$

53. **Writing** Write and solve a problem that can be modeled by a rational equation.

Challenge

54. **Industry** The average hourly wage $H(x)$ of workers in an industry is modeled by the function $H(x) = \frac{16.24x}{0.062x + 39.42}$, where x represents the number of years since 1970. In what year does the model predict that wages will be \$25/h?

55. **Open-Ended** Write a rational equation that has the following.

a. one solution **b.** two solutions **c.** no real solution

56. **Sports** An automatic pitching machine can pitch all its baseballs in $1\frac{1}{4}$ hours. One attendant can retrieve all the baseballs pitched by one machine in $3\frac{1}{2}$ hours. At least how many attendants working at the same rate should be hired so that the baseballs from 10 machines are all retrieved in less than 8 hours?

Sunshine State Standards Practice

MA.912.A.5.5 **57.** What is the solution of $x + \frac{1}{x} = -2$?

Ⓐ 1, −1 Ⓑ 0 only Ⓒ $-\frac{1}{2}$ only Ⓓ −1 only

MA.912.A.6.2 **58.** Which of the following is equivalent to $\frac{6\sqrt{24}}{2\sqrt{3}}$?

Ⓕ $2\sqrt{2}$ Ⓖ $3\sqrt{2}$ Ⓗ $5\sqrt{2}$ Ⓘ $6\sqrt{2}$

MA.912.A.8.7 **59.** An investment of \$750 will be worth \$1500 after 12 years of continuous compounding at a fixed interest rate. What is that interest rate?

Ⓐ 2.00% Ⓑ 5.78% Ⓒ 6.93% Ⓓ 200%

MA.912.A.3.15 **60.** **Extended Response** A librarian orders 48 fiction and nonfiction books for the school library. A fiction book costs \$15 and a nonfiction book costs \$20. The total cost of the order was \$900. How many nonfiction books did the librarian order? Show your work.

Mixed Review

Simplify each difference. See Lesson 8-5.

61. $\frac{3y + 1}{4y + 4} - \frac{2y + 7}{2y + 2}$ **62.** $\frac{5x}{2y + 4} - \frac{6}{y^2 + 2y}$ **63.** $\frac{x + 1}{2x - 2} - \frac{2x}{x^2 + 2x - 3}$

Solve each equation. See Lesson 7-5.

64. $\log_{10} 0.001 = x$ **65.** $\log_3 27 = 3x + 6$ **66.** $\log_{0.5}(x + 1) = 3$

Find the inverse of each function. Is the inverse a function? See Lesson 6-7.

67. $y = 5 - 2x$ **68.** $y = x^2 + 1$ **69.** $y = x^3 - 4$

Get Ready! **To prepare for Lesson 9-1, do Exercises 70–75.**

Identify the pattern and find the next three terms. See Lesson 1-1.

70. 1, 3, 5, 7, . . . **71.** −2, −4, −6, −8, . . . **72.** 0.2, 1, 5, 25, 125, . . .

73. 50, 45, 40, 35, . . . **74.** 16, 32, 64, . . . **75.** −3, −7, −11, −15, . . .

Concept Byte

For Use With Lesson 8-6

Systems With Rational Equations

Sunshine State Standard
MA.912.A.5.5 Solve rational equations.

You can solve systems with rational equations using some of same methods you used with linear systems.

Activity 1

Follow each direction to solve the system $\begin{cases} y = \dfrac{x}{3x - 1} \\ y = \dfrac{1}{x + 1} \end{cases}$.

1. Set the expressions for y equal to each other.

2. Solve for x.

3. Check your answer by substituting in the original system.

Activity 2

Follow each direction to solve the system $\begin{cases} x - 2 = \dfrac{6}{y} \\ y + 1 = x \end{cases}$.

4. Solve each equation for y.

5. Set the resulting expressions equal to each other.

6. Solve for x.

7. Check your answer by substituting in the original system.

Exercises

Solve each system.

8. $\begin{cases} \dfrac{y}{x^2 - 4x + 3} = -2 \\ x - 2y = 3 \end{cases}$

9. $\begin{cases} y = \frac{1}{x} \\ y = \dfrac{3}{4 - x^2} \end{cases}$

10. $\begin{cases} y = x^2 - 2x - 2 \\ y = \dfrac{x^2 + x - 6}{x + 3} \end{cases}$

11. $\begin{cases} y = \dfrac{x + 2}{x^2 + 3x + 2} + 2 \\ y - 3 = x \end{cases}$

12. Reasoning It is possible for the graph of a system of rational equations to include a point of intersection that is an extraneous solution? Explain.

Concept Byte

For Use With Lesson 8-6

TECHNOLOGY

Rational Inequalities

Sunshine State Standards

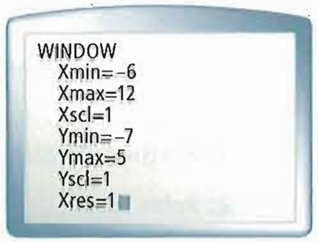
MA.912.A.2.6 Identify and graph rational functions.

MA.912.A.2.7 Perform operations of functions algebraically.

Consider the rational inequality $\frac{x}{4 - x} < 3$.

Activity 1

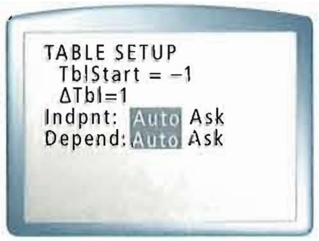
1. Enter $y_1 = \frac{x}{4 - x}$ and $y_2 = 3$ in your graphing calculator. Graph the functions using the settings at the right. Use the calculator's **INTERSECT** feature to find where the two functions are equal. Use the graph to find the solution of the inequality $\frac{x}{4 - x} < 3$.



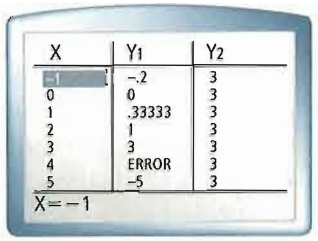
```
WINDOW
  Xmin=-6
  Xmax=12
  Xscl=1
  Ymin=-7
  Ymax=5
  Yscl=1
  Xres=1
```

Activity 2

Using the functions you entered in Activity 1, set up the table as shown.



```
TABLE SETUP
  TblStart = -1
  ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```



```
 X    | Y1     | Y2
-1    | -.2    | 3
 0    | 0      | 3
 1    | .33333 | 3
 2    | 1      | 3
 3    | 3      | 3
 4    | ERROR  | 3
 5    | -5     | 3
X=-1
```

2. Scroll to x-values less than 3. Do they make the inequality true?

3. Scroll to x-values greater than 4. Do they make the inequality true?

4. What happens to the inequality when $x = 3$? When $x = 4$?

5. Change ΔTbl to 0.1. Investigate the inequality between $x = 3$and $x = 4$.

6. Make a conjecture about the solution of the inequality based on your results in Steps 2–5.

Activity 3

Now use algebra to solve the inequality. You can multiply both sides of a rational inequality by the same algebraic expression just as you have done with equations. But you must keep in mind the properties of inequalities. Consider the first step, multiplying each side by $(4 - x)$.

$$\frac{x}{4 - x} < 3$$

$$(4 - x)\frac{x}{4 - x} < (4 - x)3$$

Depending on whether the factor $(4 - x)$ is positive or negative, there are two possible solutions to the inequality.

7. First, consider the case where $4 - x > 0$, or $x < 4$. Justify each step.
Hint: The solution must satisfy both $x < 3$ and $x < 4$.

$$\frac{x}{4-x} < 3$$

a. $(4 - x)\frac{x}{4-x} < (4 - x)3$

b. $x < 12 - 3x$

c. $4x < 12$

d. $x < 3$

8. Combine this result with the given condition $x < 4$. What is the solution for this case?

9. Now consider the case where $4 - x < 0$, or $x > 4$. Justify each step.
Hint: The solution must satisfy both $x > 3$ and $x > 4$.

$$\frac{x}{4-x} < 3$$

a. $(4 - x)\frac{x}{4-x} > (4 - x)3$

b. $x > 12 - 3x$

c. $4x > 12$

d. $x > 3$

10. Combine this result with the given condition $x > 4$. What is the solution for this case?

11. Examine your solutions for Exercises 8 and 10. Now, write the solution of the inequality $\frac{x}{4-x} < 3$.

Exercises

For Exercises 12–17, solve each inequality graphically and algebraically.

12. $\frac{2}{x-1} < x$

13. $x + 1 > \frac{x+5}{x+2}$

14. $\frac{2x}{(x-2)(x+3)} < 1$

15. $\frac{2x+2}{x-1} < x + 1$

16. $\frac{x^2+1}{x} < 2x$

17. $\frac{x-1}{x-2} < \frac{x+3}{x-1}$

18. For Exercises 12–17, check your work by using a table to solve each inequality.

The equation $d = rt$ relates the distance d you travel, the time t it takes to travel that distance, and the rate r at which you travel. So the time it takes to travel a distance d at a rate r is $t = \frac{d}{r}$. If you increase your rate by a to $r + a$, then it takes less time, $t = \frac{d}{r+a}$. In fact, the time you save by going at the faster rate is $T = \frac{d}{r} - \frac{d}{r+a}$.

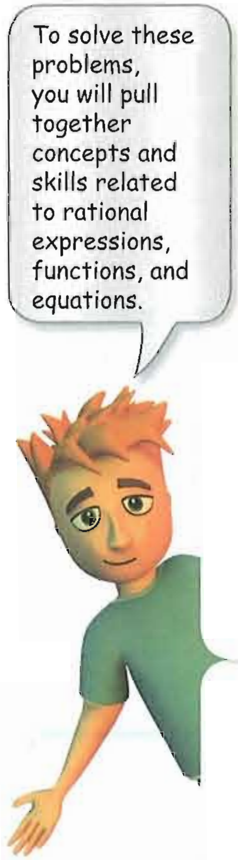
19. a. You normally take a 500-mi trip, averaging 45 mi/h. You want to increase the rate so that you save at least an hour. Write an inequality that describes the situation.

b. Solve your inequality from part (a).

8

Pull It All Together

MA.912.A.2.12 Solve problems using direct, inverse, and joint variations.
MA.912.A.5.5 Solve rational equations.



BIG idea Proportionality

Inverse proportionality involves a relationship in which the products of two quantities remain constant as the corresponding values of the quantities change.

Task 1

Rectangle R has varying length ℓ and width w but a constant perimeter of 4 ft.

- **a.** Express the area A as a function of ℓ. What do you know about this function?
- **b.** For what values of ℓ and w will the area of R be greatest? Give an algebraic argument. Give a geometric argument.

Task 2

Rectangle R has varying length ℓ and width w but a constant area of 4 ft^2.

- **a.** Express the perimeter P as a function of ℓ. What kind of function is P? What is its domain?
- **b.** Describe the asymptotic behavior of P. What can you say about R because of this behavior? Could you have made a similar statement about R in Task 1?
- **c.** For what values of ℓ and w will the perimeter of R be least? Give a calculator-based argument. Give a geometric argument.

BIG idea Function

You can represent functions in a variety of ways (such as graphs, tables, equations, or words). Each representation is particularly useful in certain situations.

Task 3

Describe the discontinuities of $f(x) = \frac{x^2 + x - 6}{x^2 - 5x + 6}$. Find an equivalent form for f that shows how the graph of f is related to the graph of $y = \frac{1}{x}$. Describe the relationship. (You do not have to draw the graphs, but you can if you wish.)

BIG idea Equivalence

You can use symbols to represent an equation in an unlimited number of ways, where all equations have the same solution.

Task 4

Solve $x - 1 = \sqrt{\frac{x^4 - 2x^3}{x^2 - 4}}$.

8 Chapter Review

Connecting BIG ideas and Answering the Essential Questions

1 Proportionality

Quantities x and y are inversely proportional only if growing x by the factor k ($k > 1$) means shrinking y by the factor $\frac{1}{k}$.

2 Function

A rational function may have no asymptotes, one horizontal or oblique asymptote, and any number of vertical asymptotes.

3 Equivalence

$f(x) = \frac{x + a}{x^2 - a^2}$, $x \neq \pm a$, and $g(x) = \frac{1}{x - a}$, $x \neq \pm a$, are equivalent.

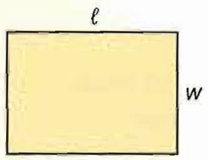
Inverse Variation (Lesson 8-1)

Are ℓ and w inversely proportional?

$A = \ell w$

$P = 2\ell + 2w$

- for a constant area—yes
- for a constant perimeter—no



Rational Functions and Their Graphs (Lesson 8-3)

Asymptotes:

For $y = \frac{2x^2}{x^2 - 9}$

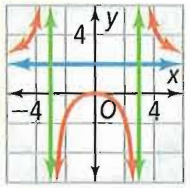
horizontal: $y = 2$

vertical: $x = \pm 3$

For $y = \frac{2x^3 + 6x^2}{x^2 + 1}$

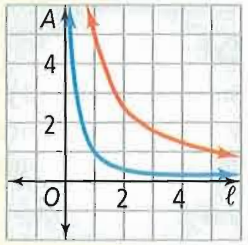
oblique: $y = 2x + 6$.

$y = \frac{x^4 + 5}{x^2 + 1}$ has no asymptotes.



The Reciprocal Function Family (Lesson 8-2)

$A(\ell) = \frac{5}{\ell}$ is a stretch of the graph of $A(\ell) = \frac{1}{\ell}$ by a factor of 5.



Solving Equations Involving Rational Expressions (Lessons 8-4, 8-5, and 8-6)

$$\frac{2x^2}{x^2 - 9} = \frac{x - 6}{x - 3} + \frac{18}{x^2 - 9}$$

$$\frac{2x^2}{x^2 - 9} = \frac{(x - 6)(x + 3)}{(x - 3)(x + 3)} + \frac{18}{x^2 - 9}$$

$$2x^2 = x^2 - 3x - 18 + 18$$

$$x^2 + 3x = 0$$

$$x(x + 3) = 0$$

$$x = 0 ✓ \quad \text{or} \quad x = -3 ✗$$

Chapter Vocabulary

- branch (p. 508)
- combined variation (p. 501)
- complex fraction (p. 536)
- continuous graph (p. 516)
- discontinuous graph (p. 516)
- inverse variation (p. 498)
- joint variation (p. 501)
- non-removable discontinuity (p. 516)
- oblique asymptote (p. 524)
- point of discontinuity (p. 516)
- rational equation (p. 542)
- rational expression (p. 527)
- rational function (p. 515)
- reciprocal function (p. 507)
- removable discontinuity (p. 516)
- simplest form (p. 527)

Choose the correct term to complete each sentence.

1. When the numerator and denominator of a rational expression are polynomials with no common factors, the rational expression is in __?__.

2. If a quantity varies directly with one quantity and inversely with another, it is a(n) __?__.

3. A(n) __?__ has a fraction in its numerator, denominator, or both.

4. If a is a zero of the polynomial denominator of a rational function, the function has a(n) __?__ at $x = a$.

5. A(n) __?__ of the graph of a rational function is one of the continuous pieces of its graph.

8-1 Inverse Variation

Quick Review

An equation in two variables of the form $y = \frac{k}{x}$ or $xy = k$, where $k \neq 0$, is an **inverse variation** with a **constant of variation** k. **Joint variation** describes when one quantity varies directly with two or more other quantities.

Example

Suppose that x and y vary inversely, and $x = 10$ when $y = 15$. Write a function that models the inverse variation. Find y when $x = 6$.

$y = \frac{k}{x}$

$15 = \frac{k}{10}$, so $k = 150$.

The inverse variation is $y = \frac{150}{x}$.

When $x = 6$, $y = \frac{150}{6} = 25$.

Exercises

6. Suppose that x and y vary inversely, and $x = 30$ when $y = 2$. Find y when $x = 5$.

Write a direct or inverse variation equation for each relation.

7.

x	3	4	8
y	24	18	9

8.

x	5	7	9
y	30	42	54

Write the function that models each relationship. Find z when $x = 4$ and $y = 8$.

9. z varies jointly with x and y. When $x = 2$ and $y = 2$, $z = 7$.

10. z varies directly with x and inversely with y. When $x = 5$ and $y = 2$, $z = 10$.

8-2 The Reciprocal Function Family

Quick Review

The graph of a **reciprocal function** has two parts called **branches**. The graph of $y = \frac{k}{x - b} + c$ is a translation of $y = \frac{k}{x}$ by b units horizontally and c units vertically. It has a vertical asymptote at $x = b$ and a horizontal asymptote at $y = c$.

Example

Graph the equation $y = \frac{3}{x - 2} + 1$. Identify the x- and y-intercepts and the asymptotes of the graph.

$b = 2$, so the vertical asymptote is $x = 2$.

$c = 1$, so the horizontal asymptote is $y = 1$.

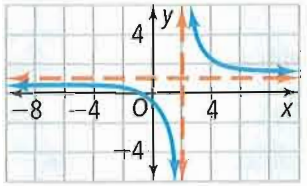
Translate $y = \frac{3}{x}$ two units to the right and one unit up.

When $y = 0$, $x = -1$.

The x-intercept is $(-1, 0)$.

When $x = 0$, $y = -\frac{1}{2}$.

The y-intercept is $(0, -\frac{1}{2})$.



Exercises

Graph each equation. Identify the x- and y-intercepts and the asymptotes of the graph.

11. $y = \frac{1}{x}$

12. $y = \frac{-2}{x^2}$

13. $y = \frac{-1}{x} - 4$

14. $y = \frac{2}{x + 3} - 1$

Write an equation for the translation of $y = \frac{4}{x}$ that has the given asymptotes.

15. $x = 0, y = 3$

16. $x = 2, y = 2$

17. $x = -3, y = -4$

18. $x = 4, y = -3$

8-3 Rational Functions and Their Graphs

Quick Review

The **rational function** $f(x) = \frac{P(x)}{Q(x)}$ has a **point of discontinuity** for each real zero of $Q(x)$.

If $P(x)$ and $Q(x)$ have

- no common factors, then $f(x)$ has a vertical asymptote when $Q(x) = 0$.
- a common real zero a, then there is a hole or a vertical asymptote at $x = a$.
- degree of $P(x) <$ degree of $Q(x)$, then the graph of $f(x)$ has a horizontal asymptote at $y = 0$.
- degree of $P(x) =$ degree of $Q(x)$, then there is a horizontal asymptote at $y = \frac{a}{b}$, where a and b are the coefficients of the terms of greatest degree in $P(x)$ and $Q(x)$, respectively.
- degree of $P(x) >$ degree of $Q(x)$, then there is no horizontal asymptote.

Example

Find any points of discontinuity for the graph of the rational function $y = \frac{2.5}{x + 7}$. Describe any vertical or horizontal asymptotes and any holes.

There is a vertical asymptote at $x = -7$ and a horizontal asymptote at $y = 0$.

Exercises

Find any points of discontinuity for each rational function. Sketch the graph. Describe any vertical or horizontal asymptotes and any holes.

19. $y = \frac{x - 1}{(x + 2)(x - 1)}$

20. $y = \frac{x^3 - 1}{x^2 - 1}$

21. $y = \frac{2x^2 + 3}{x^2 + 2}$

22. The start-up cost of a company is \$150,000. It costs \$.17 to manufacture each headset. Graph the function that represents the average cost of a headset. How many must be manufactured to result in a cost of less than \$5 per headset?

8-4 Rational Expressions

Quick Review

A **rational expression** is in **simplest form** when its numerator and denominator are polynomials that have no common factors.

Example

Simplify the rational expression. State any restrictions on the variable.

$$\frac{2x^2 + 7x + 3}{x - 4} \cdot \frac{x^2 - 16}{x^2 + 8x + 15}$$

$$= \frac{(2x + 1)\cancel{(x + 3)}}{\cancel{x - 4}} \cdot \frac{\cancel{(x - 4)}(x + 4)}{\cancel{(x + 3)}(x + 5)}$$

$$= \frac{(2x + 1)(x + 4)}{x + 5}, x \neq -5, x \neq -3, \text{ and } x \neq 4$$

Exercises

Simplify each rational expression. State any restrictions on the variable.

23. $\frac{x^2 + 10x + 25}{x^2 + 9x + 20}$

24. $\frac{x^2 - 2x - 24}{x^2 + 7x + 12} \cdot \frac{x^2 - 1}{x - 6}$

25. $\frac{4x^2 - 2x}{x^2 + 5x + 4} \div \frac{2x}{x^2 + 2x + 1}$

26. What is the ratio of the volume of a sphere to its surface area?

8-5 Adding and Subtracting Rational Expressions

Quick Review

To add or subtract rational expressions with different denominators, write each expression with the LCD. A fraction that has a fraction in its numerator or denominator or in both is called a **complex fraction**. Sometimes you can simplify a complex fraction by multiplying the numerator and denominator by the LCD of all the rational expressions.

Example

Simplify the complex fraction. $\dfrac{\frac{1}{x}+3}{\frac{5}{y}+4}$

$$\frac{\frac{1}{x}+3}{\frac{5}{y}+4} = \frac{\left(\frac{1}{x}+3\right)\cdot xy}{\left(\frac{5}{x}+4\right)\cdot xy}$$

$$= \frac{\frac{1}{x}\cdot xy + 3\cdot xy}{\frac{5}{x}\cdot xy + 4\cdot xy}$$

$$= \frac{y+3xy}{5y+4xy}$$

Exercises

Simplify the sum or difference. State any restrictions on the variable.

27. $\dfrac{3x}{x^2-4} + \dfrac{6}{x+2}$

28. $\dfrac{1}{x^2-1} - \dfrac{2}{x^2+3x}$

Simplify the complex fraction.

29. $\dfrac{2-\frac{2}{x}}{3-\frac{1}{x}}$

30. $\dfrac{\frac{1}{x+y}}{4}$

8-6 Solving Rational Equations

Quick Review

Solving a **rational equation** often requires multiplying each side by an algebraic expression. This may introduce extraneous solutions—solutions that solve the derived equation but not the original equation. Check all possible solutions in the original equation.

Example

Solve the equation. Check your solution.

$$\frac{1}{2x} - \frac{2}{5x} = \frac{1}{2}$$

$$10x\left(\frac{1}{2x} - \frac{2}{5x}\right) = 10x\left(\frac{1}{2}\right)$$

$$5 - 4 = 5x$$

$$x = \frac{1}{5}$$

Check $\dfrac{1}{2\left(\frac{1}{5}\right)} - \dfrac{2}{5\left(\frac{1}{5}\right)} = \dfrac{5}{2} - 2 = \dfrac{1}{2}$ ✔

Exercises

Solve each equation. Check your solutions.

31. $\dfrac{1}{x} = \dfrac{5}{x-4}$

32. $\dfrac{2}{x+3} - \dfrac{1}{x} = \dfrac{-6}{x(x+3)}$

33. $\dfrac{1}{2} + \dfrac{x}{6} = \dfrac{18}{x}$

34. You travel 10 mi on your bicycle in the same amount of time it takes your friend to travel 8 mi on his bicycle. If your friend rides his bike 2 mi/h slower than you ride your bike, find the rate at which each of you is traveling.

8 Chapter Test

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Do you know HOW?

Write a function that models each variation.

1. $x = 2$ when $y = -8$, and y varies inversely with x.

2. $x = 0.2$ and $y = 3$ when $z = 2$, and z varies jointly with x and y.

3. $x = \frac{1}{3}$, $y = \frac{1}{5}$, and $r = 3$ when $z = \frac{1}{2}$, and z varies directly with x and inversely with the product of r^2 and y.

Is the relationship between the values in each table a *direct variation*, an *inverse variation*, or *neither*? Write equations to model any direct or inverse variations.

4.

x	y
3	6
5	8
7	10
9	12

5.

x	y
4	32
8	16
16	8
32	4

Write and graph an equation of the translation of $y = \frac{7}{x}$ that has the given asymptotes.

6. $x = 1;\ y = 2$

7. $x = -3;\ y = -2$

For each rational function, identify any holes or horizontal or vertical asymptotes of the graph.

8. $y = \frac{x + 1}{x - 1}$

9. $y = \frac{x + 3}{x + 3}$

10. $y = \frac{x - 2}{(x + 1)(x - 2)}$

11. $y = \frac{2x^2}{x^2 - 4x}$

12. $y = \frac{1}{x + 2} - 3$

13. $y = \frac{x^2 + 5}{x - 5}$

Simplify each complex fraction.

14. $\dfrac{\frac{2}{x}}{1 - \frac{1}{y}}$

15. $\dfrac{3 - \frac{3}{x}}{\frac{1}{2} - \frac{1}{x}}$

Simplify each rational expression. State any restrictions on the variable.

16. $\frac{x^2 + 7x + 12}{x^2 - 9}$

17. $\frac{(x + 3)(2x - 1)}{x(x + 4)} \div \frac{(-x - 3)(2x + 1)}{x}$

18. $\frac{x^2 - 1}{x^2 + 2x - 3} - \frac{x + 1}{x + 3}$

19. $\frac{x(x + 4)}{x - 2} + \frac{x - 1}{x^2 - 4}$

Solve each equation. Check your solutions.

20. $\frac{x}{2} = \frac{x + 1}{4}$

21. $\frac{3}{x - 1} = \frac{4}{3x + 2}$

22. $\frac{3x}{x + 1} = 0$

23. $\frac{3}{x + 1} = \frac{1}{x^2 - 1}$

24. $\frac{1}{x} + \frac{1}{3} = \frac{6}{x^2}$

25. $\frac{1}{x} + \frac{x}{x + 2} = 1$

26. Your neighbor can seal your driveway in 4 hours. Working together, you and your neighbor can seal it in 2.3 hours. How long would it take you to seal it working alone?

Do you UNDERSTAND?

27. **Vocabulary** Describe a situation that represents an inverse variation.

28. **Compare and Contrast** How is simplifying rational expressions similar to simplifying fractions? How is it different?

29. **Writing** When does a discontinuity result in a vertical asymptote? When does it result in a hole in the graph?

30. **Open-Ended** Write a function whose graph has a hole, a vertical asymptote, and a horizontal asymptote.

31. **Reasoning** State any restrictions on the variable in the complex fraction. $\dfrac{\frac{x - 3}{x + 4}}{\frac{x^2 - 1}{x}}$

8 Sunshine State Standards Practice

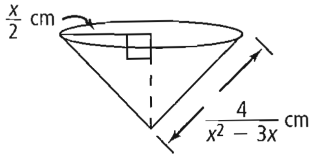
TIPS FOR SUCCESS

Some problems ask you to find the lateral area or the (total) surface area of a three-dimensional figure. Read the sample question at the right. Then follow the tips to answer the question.

TIP 1

Use the formula for the lateral area of a cone: $S = \pi r \ell$.

What is the approximate lateral area of the cone shown below?



Ⓐ $\frac{3}{x + 3}$

Ⓑ $\frac{6}{x - 3}$

Ⓒ $\frac{3}{x - 3}$

Ⓓ $\frac{6}{x + 3}$

TIP 2

Use the information from the diagram for the values you need in the formula.

Think It Through

radius: $r = \frac{x}{2}$ cm

slant height: $\ell = \frac{4}{x^2 - 3x}$ cm

$$S = \pi r \ell$$
$$= \pi\left(\frac{x}{2}\right)\left(\frac{4}{x^2 - 3x}\right)$$
$$= \frac{2\pi}{x - 3} \text{ cm}$$

Since $\pi \approx 3$, the correct answer is B.

Vocabulary Builder

As you solve test items, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. joint variation

B. branch

C. point of discontinuity

D. inverse variation

E. reciprocal function

I. a point where the graph of a function breaks into branches

II. each piece of a discontinuous graph

III. a relation represented by an equation of the form $y = \frac{k}{x}$ or $xy = k$, where $k \neq 0$

IV. a function that can be written in the form $f(x) = \frac{a}{x - h} + k$, where $a \neq 0$

V. one variable varies directly with two or more other variables

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

1. Which expression equals $\frac{5x}{x^2 - 9} - \frac{4x}{x^2 + 5x + 6}$?

Ⓐ $\frac{7x}{(x - 3)(x + 3)(x + 2)}$

Ⓑ $\frac{x^2 - 2x}{(x - 3)(x + 3)(x + 2)}$

Ⓒ $\frac{x^2 + 22x}{(x - 3)(x + 3)(x + 2)}$

Ⓓ $\frac{9x^2 - 2x}{(x - 3)(x + 3)(x + 2)}$

2. If x is a real number, for what values of x is the equation $\frac{2x - 8}{4x^{-1}} = \frac{x^2 - 4x}{2}$ true?

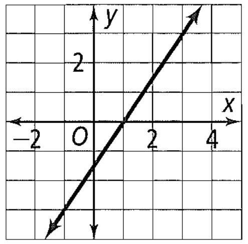
Ⓕ all values of x

Ⓖ some values of x

Ⓗ no values of x

Ⓘ impossible to determine

3. What is the equation of the line that goes through the point $(-3, 2)$ and is parallel to the line shown in the graph below?



(A) $y + 3 = 1.5(x - 2)$

(B) $y + 2 = 1.5(x - 3)$

(C) $y - 3 = 1.5(x - 2)$

(D) $y - 2 = 1.5(x + 3)$

4. Which expression is a simpler form of the complex fraction $\frac{\frac{1}{x} + \frac{3}{y}}{\frac{2}{xy}}$?

(F) $\frac{3xy}{2}$

(H) $\frac{3}{2}$

(G) $\frac{3x + y}{2xy}$

(I) $\frac{3x + y}{2}$

5. Which is the first *incorrect* step in simplifying $\log_9 243$?

Step 1: $\log_9 243 = x$

Step 2: $9^x = 243$

Step 3: $x = 243 \div 9$

Step 4: $= 27$

(A) Step 1

(C) Step 3

(B) Step 2

(D) Step 4

6. Which is/are the solution(s) of the equation $\sqrt{2x + 2} = 2x - 4$?

(F) $x = 3.5$ and $x = 1$

(G) $x = 3.5$ and $x = -1$

(H) $x = 3.5$

(I) $x = 1$

7. Which is the simplest form of the expression?
$4\sqrt{18x^4} - 3\sqrt{72x^4}$

(A) $-6x^2\sqrt{2}$

(C) -6

(B) $-6x^2$

(D) none of the above

8. Ana and Matthew each worked out the same problem, as shown below. Which statement is true of their solutions?

Ana's Work

$$\begin{aligned} \frac{a^{\frac{2}{3}}b^{\frac{4}{3}}}{\sqrt[3]{a^5b}} &= \frac{a^{\frac{2}{3}}b^{\frac{4}{3}}}{a^{\frac{3}{5}}b} \\ &= a^{\frac{2}{3}-\frac{3}{5}}b^{\frac{4}{3}-1} \\ &= a^{\frac{10}{15}-\frac{9}{15}}b^{\frac{4}{3}-\frac{3}{3}} \\ &= a^{\frac{1}{15}}b^{\frac{1}{3}} \end{aligned}$$

Matthew's work

$$\begin{aligned} \frac{a^{\frac{2}{3}}b^{\frac{4}{3}}}{\sqrt[3]{a^5b}} &= \frac{a^{\frac{2}{3}}b^{\frac{4}{3}}}{a^{\frac{5}{3}}b^{\frac{1}{3}}} \\ &= a^{\frac{2}{3}-\frac{5}{3}}b^{\frac{4}{3}-\frac{1}{3}} \\ &= a^{-1}b^{1} \\ &= \frac{b}{a} \end{aligned}$$

(F) Ana is correct.

(G) Matthew is correct.

(H) Both answers are incorrect.

(I) Both answers are correct.

9. If $i = \sqrt{-1}$, what is the value of $-i^4$?

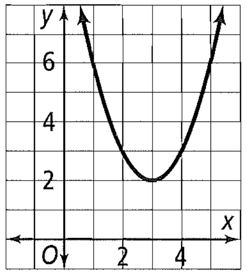
(A) i

(C) 1

(B) $-i$

(D) -1

10. What is the range of the graph shown below?



(F) $x \geq 3$

(H) $x \geq 2$

(G) $y \geq 3$

(I) $y \geq 2$

11. Given the equation $y = \log_x n$ where $n > 0$ and $y < 0$, which statement is valid for real values of x?

(A) $x < 0$

(C) $x \geq 0$

(B) $x \leq 0$

(D) $x > 0$

12. Which expression represents the solution to $4^x = 13$?

Ⓕ $\dfrac{\log 13}{\log 4}$

Ⓖ $\log_4 + \log_{13}$

Ⓗ $\dfrac{\log 4}{\log 13}$

Ⓘ $\log_{13}4$

13. If $f(x) = x^2$ and $g(x) = x - 1$, which statement is true?

Ⓐ $(f \circ g)(x) \geq (g \circ f)(x)$ for all values of x.

Ⓑ $(f \circ g)(x) \leq (g \circ f)(x)$ for all values of x.

Ⓒ $(f \circ g)(x) = (g \circ f)(x)$ only for $x = 1$.

Ⓓ $(f \circ g)(x) \neq (g \circ f)(x)$ for any value of x.

14. If $g(x) = x^2 - 4$ and $h(x) = 4x - 6$, which expression is equal to $\left(\frac{g}{h}\right)(x)$?

Ⓕ $\dfrac{4x - 6}{x^2 - 4}$

Ⓖ $\dfrac{x^2 - 2}{4x - 3}$

Ⓗ $x^2 - 4 - (4x - 6)$

Ⓘ $\dfrac{(x + 2)(x - 2)}{2(2x - 3)}$

15. Which equation represents the translation of $y = \frac{3}{x}$ so that it has asymptotes at $x = -13$ and $y = 5$?

Ⓐ $y = \dfrac{3}{x + 13} + 5$

Ⓑ $y = \dfrac{3}{x + 13} - 5$

Ⓒ $y = \dfrac{3}{x - 13} + 5$

Ⓓ $y = \dfrac{3}{x - 13} - 5$

16. Which equation represents the inverse of $y = x^2 + 15$?

Ⓕ $y = \pm\sqrt{x} - 15$

Ⓖ $y = -15 \pm \sqrt{x}$

Ⓗ $y = \pm\sqrt{x - 15}$

Ⓘ $y = 15 \pm \sqrt{x}$

17. A third-degree polynomial equation with rational coefficients has roots -4 and $-4i$. If the leading coefficient of the equation is $\frac{3}{2}$, what is the equation?

Ⓐ $y = \frac{3}{2}x^3 - 6x^2 + 24x - 96$

Ⓑ $y = \frac{3}{2}x^3 - 4x^2 + 16x - 64$

Ⓒ $y = \frac{3}{2}x^3 + 6x^2 + 24x + 96$

Ⓓ $y = x^3 - 4x^2 + 16x - 64$

18. Which is the solution of $|2x - 5| < 9$?

Ⓕ $2 < x < 7$

Ⓖ $-2 < x < 7$

Ⓗ $x < 2$ or $x > 7$

Ⓘ $x < -2$ or $x > 7$

19. Solve the equation $\frac{2g^2}{d} - c = 3x$ for g.

Ⓐ $g = \pm\sqrt{\dfrac{3x + cd}{2}}$

Ⓑ $g = \pm\sqrt{\dfrac{d(3x + c)}{2}}$

Ⓒ $g = \pm\sqrt{2(3x + cd)}$

Ⓓ $g = \pm\sqrt{\dfrac{2}{d(3x + c)}}$

20. Which of the following is the linear factorization of $x^3 - 12x^2 + 35x$?

Ⓕ $x(x + 5)(x + 7)$

Ⓖ $(x + 5)(x + 7)$

Ⓗ $x(x - 5)(x - 7)$

Ⓘ $(x - 5)(x - 7)$

GRIDDED RESPONSE

21. What is the positive solution of $2x^2 - 11 = 12x$? Round your answer to the nearest hundredth.

22. Suppose that x and y vary inversely. What is the constant of variation if $x = 12$ when $y = 4$?

23. Solve for x: $\frac{5}{2x - 2} = \frac{15}{x^2 - 1}$.

24. Solve for x: $\log(x - 3) = 3$.

25. What is the value of the x-coordinate of the solution of the system of equations?

$$\begin{cases} 2x + y = 6 \\ y - 3 = x \end{cases}$$

26. What is the remainder when $x^4 - 3x^2 + 7x + 3$ is divided by $x - 2$?

27. The product of three consecutive even integers is -2688. What is the value of the largest integer?

28. What is the smallest zero of $f(x) = 2x^5 - 4x^2 + 3x + 7$? Round your answer to the nearest hundredth.

29. What number do you add to each side of the equation when you solve $x^2 + 5x = 4$ by completing the square?

30. How many real roots does $y = x^2 - 3x + 7$ have?

CHAPTER 9

Get Ready!

Lesson 2-1

Evaluating Functions

For each function, find $f(1), f(2), f(3)$, and $f(4)$.

1. $f(x) = 2x + 7$

2. $f(x) = 5x - 4$

3. $f(x) = 0.2x + 0.7$

4. $f(x) = -5x + 3$

5. $f(x) = 4x - \frac{2}{3}$

6. $f(x) = -3x - 9$

Lesson 1-1

Identifying Mathematical Patterns

Identify a pattern and find the next three numbers in the pattern.

7. 9, 4, −1, −6, . . .

8. 1, 2, 4, 8, . . .

9. 18, 9, 10, 1, 2, . . .

10. 7, 10, 13, 16, . . .

Lesson 8-5

Simplifying Complex Fractions

Simplify each complex fraction.

11. $\dfrac{1 - \frac{1}{3}}{\frac{1}{2}}$

12. $\dfrac{\frac{1}{3} + \frac{1}{6}}{\frac{2}{3}}$

13. $\dfrac{1}{1 - \frac{2}{5}}$

14. $\dfrac{1 - \frac{3}{8}}{2 + \frac{1}{4}}$

Looking Ahead Vocabulary

15. Think of a function and evaluate the function for the input numbers 1, 2, 3, 4, and 5. List the five outputs in order. This list is a *sequence* of numbers. The sequence can be infinitely long.

16. Use a linear function to generate a sequence of five numbers. Beginning with the second number, subtract the number that precedes it. Continue doing this until you have found all four differences. Are the results the same? If so, you have discovered that your sequence has a *common difference.*

17. Now use an exponential function to define your sequence. Instead of subtracting, divide each number by the number that precedes it. Do this until you find all four quotients. Are these four results the same? If so, you have discovered that your sequence has a *common ratio.*

CHAPTER 9

Sequences and Series

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Arithmetic and geometric sequences are types of patterns. What patterns do you see in these terraced rice fields? You will learn about all kinds of sequences in this chapter.

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
arithmetic sequence, *p. 572*	progresión aritmética
arithmetic series, *p. 587*	serie aritmética
common difference, *p. 572*	diferencia común
common ratio, *p. 580*	razón común
converge, *p. 598*	convergir
diverge, *p. 598*	divergir
explicit formula, *p. 565*	fórmula explícita
geometric sequence, *p. 580*	progresión geométrica
geometric series, *p. 595*	serie geométrica
limits, *p. 589*	límites
recursive formula, *p. 565*	formula recursiva

BIG ideas

1 **Variable**
Essential Questions How can you represent the terms of a sequence explicitly? How can you represent them recursively?

2 **Equivalence**
Essential Question What are equivalent explicit and recursive definitions for an arithmetic sequence?

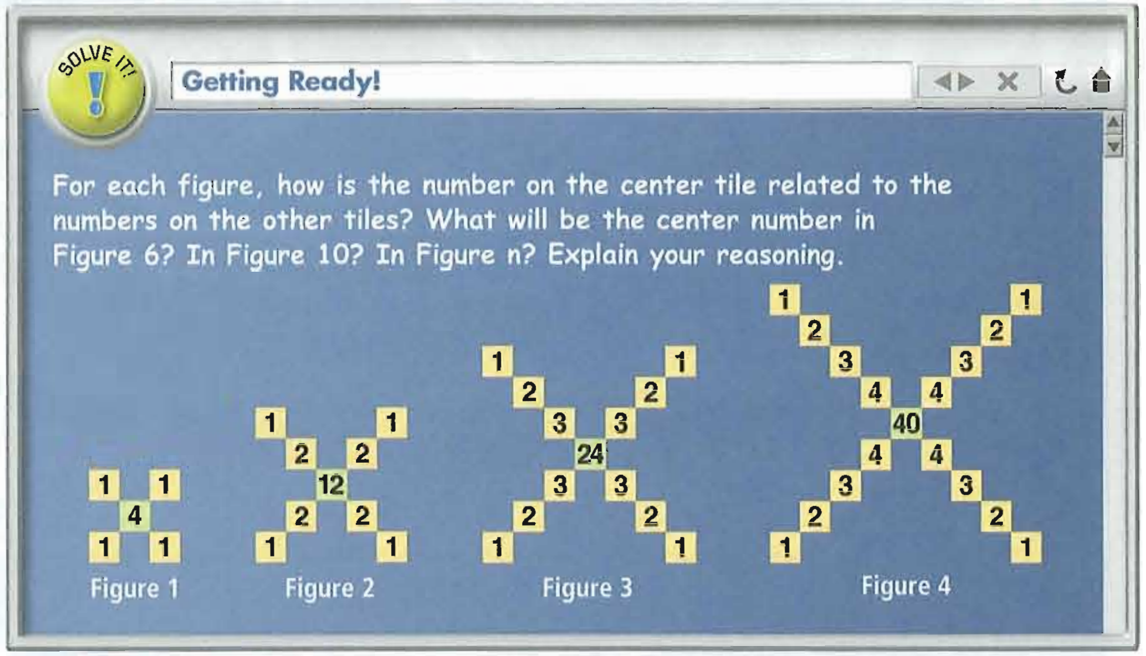
3 **Modeling**
Essential Questions How can you model a geometric sequence? How can you model its sum?

Chapter Preview

9-1 Mathematical Patterns

Sunshine State Standard
Prepares for MA.912.D.11.1 Define arithmetic and geometric sequences and series.

Objectives To identify mathematical patterns found in a sequence
To use a formula to find the *n*th term of a sequence



For each figure, how is the number on the center tile related to the numbers on the other tiles? What will be the center number in Figure 6? In Figure 10? In Figure n? Explain your reasoning.

Lesson Vocabulary
- sequence
- term of a sequence
- explicit formula
- recursive formula

Sometimes you can state a rule to describe a pattern. At other times, you have to do a bit of work to find a rule.

Essential Understanding If the numbers in a list follow a pattern, you may be able to relate each number in the list to its numerical position in the list with a rule.

A **sequence** is an ordered list of numbers. Each number in a sequence is a **term of a sequence**. You can represent a term of a sequence by using a variable with a subscript number to indicate its position in the sequence. For example, a_5 is the fifth term in the sequence $a_1, a_2, a_3, a_4, \ldots$.

The subscripts of sequence terms are often positive integers starting with 1. If so, you can generalize a term as a_n, the *n*th term in the sequence.

1st term	2nd term	3rd term	...	$n - 1$ term	nth term	$n + 1$ term	...
↓	↓	↓		↓	↓	↓	
a_1	a_2	a_3	...	a_{n-1}	a_n	a_{n+1}	...

An **explicit formula** describes the *n*th term of a sequence using the number n.

For example, in the sequence 2, 4, 6, 8, 10, . . . , the *n*th term is twice the value of n. You write this as $a_n = 2n$. The table shows how to find a_n by substituting the value of n into the explicit formula.

n	nth term
1	$a_1 = 2(1) = 2$
2	$a_2 = 2(2) = 4$
3	$a_3 = 2(3) = 6$
4	$a_4 = 2(4) = 8$

Problem 1 Generating a Sequence Using an Explicit Formula

A sequence has an explicit formula $a_n = 3n - 2$. What are the first 10 terms of this sequence?

Plan

How does the explicit formula help you find the value of a term? Replace *n* in the formula with the number of the term. Simplify to find the value of the term.

$a_n = 3n - 2$ Write the formula.

$a_1 = 3(1) - 2 = 1$ Substitute 1 for *n* and simplify.

$a_2 = 3(2) - 2 = 4$ Substitute 2 for *n* and simplify.

You can use a table to organize your work for the remaining terms.

n	a_n
3	$a_3 = 3(3) - 2 = 7$
4	$a_4 = 3(4) - 2 = 10$
5	$a_5 = 3(5) - 2 = 13$
6	$a_6 = 3(6) - 2 = 16$
7	$a_7 = 3(7) - 2 = 19$
8	$a_8 = 3(8) - 2 = 22$
9	$a_9 = 3(9) - 2 = 25$
10	$a_{10} = 3(10) - 2 = 28$

Substitute 3 for *n* and simplify.

And so on.

The first ten terms are 1, 4, 7, 10, 13, 16, 19, 22, 25, 28.

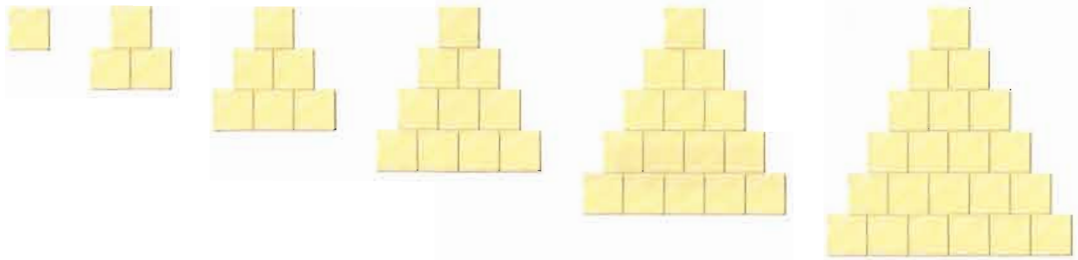
Got It? **1.** A sequence has an explicit formula $a_n = 12n + 3$. What is term a_{12} in the sequence?

Sometimes you can see the pattern in a sequence by comparing each term to the one that came before it. For example, in the sequence 133, 130, 127, 124, . . . , each term after the first term is equal to three less than the previous term.

A recursive definition for this sequence contains two parts.
(a) an initial condition (the value of the first term): $a_1 = 133$
(b) a **recursive formula** (relates each term after the first term to the one before it):
$a_n = a_{n-1} - 3$, for $n > 1$

Problem 2 Writing a Recursive Definition for a Sequence

The number of blocks in a two-dimensional pyramid is a sequence that follows a recursive formula. What is a recursive definition for the sequence?



Think	Write
Count the number of blocks in each pyramid.	1, 3, 6, 10, 15, 21
Subtract consecutive terms to find out what happens from one term to the next.	$a_2 - a_1 = 3 - 1 = 2$ $a_3 - a_2 = 6 - 3 = 3$ $a_4 - a_3 = 10 - 6 = 4$ $a_5 - a_4 = 15 - 10 = 5$ $a_6 - a_5 = 21 - 15 = 6$
Use *n* to express the relationship between successive terms.	$a_n - a_{n-1} = n$
To write a recursive definition, state the initial condition and the recursive formula.	$a_1 = 1$ and $a_n = a_{n-1} + n$.

Got It? **2.** What is a recursive definition for each sequence? (*Hint:* Look for simple addition or multiplication patterns to relate consecutive terms.)

a. 1, 2, 6, 24, 120, 720, . . .

b. 1, 5, 14, 30, 55, . . .

Recursive definitions can be very helpful when you look at a small section of a sequence. However, if you want to know both a_3 and a_{5000} of a sequence, an explicit formula often works better.

Problem 3 Writing an Explicit Formula for a Sequence

What is the 100th term of the pyramid sequence in Problem 2?

Plan

Why do you use the explicit formula to find a_{100}?
Because starting with a_1, it would take 99 iterations of the formula to get a_{100} using the recursive formula.

To find an explicit formula, expand the first few terms of the pyramid sequence.

a_1	a_2	a_3	a_4	a_5	...	a_n
1	1 + 2	1 + 2 + 3	1 + 2 + 3 + 4	1 + 2 + 3 + 4 + 5	...	1 + 2 + ... + n
1	3	6	10	15	...	■

Therefore,

$$a_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n,$$

which you can write as

$$a_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.$$

Adding the two previous equations gives the following result:

$$\begin{array}{rcl} a_n &=& 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ +\ a_n &=& n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ \hline 2a_n &=& (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \\ 2a_n &=& n(n+1) \\ a_n &=& \frac{1}{2}n(n+1) \end{array}$$

The explicit formula for this sequence is $a_n = \frac{1}{2}n(n+1)$.

Substitute 100 into the explicit formula to find the 100th term.

$$\begin{aligned} a_{100} &= \tfrac{1}{2}(100)(100+1) \\ &= \tfrac{1}{2}(100)(101) \\ &= 5050 \end{aligned}$$

The 100th term is 5050.

Got It? **3. a.** What is an explicit formula for the sequence 0, 3, 8, 15, 24, . . . ? What is the 20th term?

b. Reasoning Why is using an explicit formula often more efficient than using a recursive definition?

Problem 4 Using Formulas to Find Terms of a Sequence

Finance **Pierre began the year with an unpaid balance of \$300 on his credit card. Because he had not read the credit card agreement, he did not realize that the company charged 1.8% interest each month on his unpaid balance, in addition to a \$29 penalty in any month he might fail to make a minimum payment. Pierre ignored his credit card bill for 4 consecutive months before finally deciding to pay off the balance. What did he owe after 4 months of non-payment?**

Think

Why is it helpful to change FLOAT to 2?

This problem involves money, so, real-world solutions will only have 2 decimal places.

Step 1 Write a recursive definition.

Initial condition: $a_0 = 300$ (Use a_0 so that a_1 represents the balance after 1 month.)

Recursive formula: $a_n = 1.018 \cdot a_{n-1} + 29$, for $n > 1$

Step 2 Use a calculator. In the **MODE** menu, change the digit display from **FLOAT** to 2.

```
Normal Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

Enter 300 in the home screen. Enter the recursive formula **1.018ANS+29**. Press enter for the balance after one month.

Press enter three more times until the calculator shows the balance after 4 months.

```
300
                300.00
1.018Ans+29
                334.40
```

```
                369.42
                405.07
                441.36
```

After 4 months, Pierre owes \$441.36.

Got It? **4.** If the credit card company were to allow Pierre to continue making no payments, after how many months would his balance exceed \$1000?

Lesson Check

Do you know HOW?

Find the first five terms of each sequence.

1. $a_n = 5n - 3$

2. $a_n = n^2 - 2n$

3. What is a recursive definition for the sequence 3, 6, 12, 24, . . . ?

4. What is an explicit formula for the sequence 5, 8, 11, 14, . . . ?

Do you UNDERSTAND?

5. **Vocabulary** Explain the difference between an explicit formula and a recursive definition. Give an example of each.

6. **Error Analysis** A student writes that $a_n = 3n + 1$ is an explicit formula for the sequence 1, 4, 7, 10, ... Explain the student's error and write a correct explicit formula for the sequence.

Practice and Problem-Solving Exercises

A Practice

Find the first six terms of each sequence.

See Problem 1.

7. $a_n = 3n + 2$

8. $a_n = -5n + 1$

9. $a_n = \frac{1}{2}n$

10. $a_n = n^2 + 1$

11. $a_n = 3n^2 - n$

12. $a_n = 2^n - 1$

13. $a_n = \frac{1}{2}n^3 - 1$

14. $a_n = (-3)^n$

Write a recursive definition for each sequence.

See Problem 2.

15. 80, 77, 74, 71, 68, . . .

16. 4, 8, 16, 32, 64, . . .

17. 0, 3, 7, 12, 18, . . .

18. 1, 4, 7, 10, 13, . . .

19. 100, 10, 1, 0.1, 0.01, . . .

20. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

21. 4, −8, 16, −32, 64, . . .

22. 1, 2, 6, 24, 120, . . .

23. 1, 5, 14, 30, . . .

Write an explicit formula for each sequence. Find the tenth term.

See Problem 3.

24. 4, 5, 6, 7, 8, . . .

25. 4, 7, 10, 13, 16, . . .

26. 3, 7, 11, 15, 19, . . .

27. $-2\frac{1}{2}, -2, -1\frac{1}{2}, -1, \ldots$

28. 1, 4, 9, 16, . . .

29. 2, 5, 10, 17, 26, . . .

30. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

31. 1, 3, 9, 27, . . .

32. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$

Find the eighth term of each sequence.

33. −2, −1, 0, 1, 2, . . .

34. 43, 41, 39, 37, 35, . . .

35. $40, 20, 10, 5, \frac{5}{2}, \ldots$

36. 6, 1, −4, −9, . . .

37. $144, 36, 9, \frac{9}{4}, \ldots$

38. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

39. 2, 1, −2, −7, −14, . . .

40. $\frac{3}{4}, -\frac{3}{2}, 3, -6, \ldots$

41. $2, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, \ldots$

42. **Exercise** You walk 1 mile the first day of your training, 1.2 miles the second day, 1.6 miles the third day, and 2.4 miles the fourth day. If you continue this pattern, how many miles do you walk the seventh day?

See Problem 4.

B Apply

Determine whether each formula is *explicit* or *recursive*. Then find the first five terms of each sequence.

43. $a_n = 2a_{n-1} + 3$, where $a_1 = 3$

44. $a_n = \frac{1}{2}(n)(n - 1)$

45. $a_n = (n - 5)(n + 5)$

46. $a_n = -3a_{n-1}$, where $a_1 = -2$

47. $a_n = -4n^2 - 2$

48. $a_n = 2n^2 + 1$

Use the given rule to write the 4th, 5th, 6th, and 7th terms of each sequence.

49. $a_n = (n + 1)^2$

50. $a_n = 2(n - 1)^3$

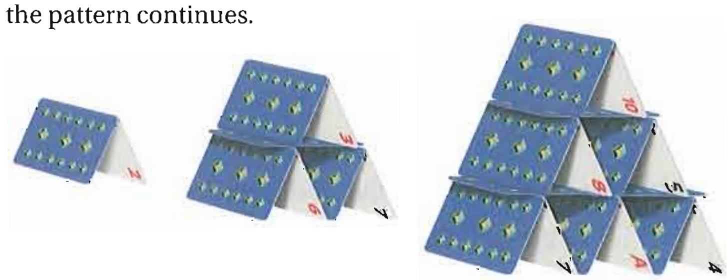
51. $a_n = \frac{n^2}{n + 1}$

52. $a_n = \frac{n + 1}{n + 2}$

53. **Think About a Plan** You invested money in a company and each month you receive a payment for your investment. Over the first four months, you received \$50, \$52, \$56, and \$62. If this pattern continues, how much do you receive in the tenth month?

- What pattern do you see between consecutive terms?
- Can you write a recursive or explicit formula to describe the pattern?
- How can you use your formula to find the amount you receive in the tenth month?

54. **Entertainment** Suppose you are building towers of cards with levels as displayed below. Copy and complete the table, assuming the pattern continues.



Number of Levels	Cards Needed
1	2
2	7
3	■
4	■
5	■

Find the next two terms in each sequence. Write a formula for the *n*th term. Identify each formula as *explicit* or *recursive*.

55. 5, 8, 11, 14, 17, . . .

56. 3, 6, 12, 24, 48, . . .

57. 1, 8, 27, 64, 125, . . .

58. 4, 16, 64, 256, 1024, . . .

59. 49, 64, 81, 100, 121, . . .

60. −1, 1, −1, 1, −1, 1, . . .

61. −16, −8, −4, −2, . . .

62. −75, −68, −61, −54, . . .

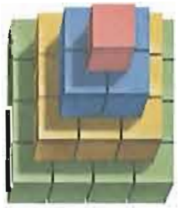
63. 21, 13, 5, −3, . . .

64. **a.** **Open-Ended** Write four terms of a sequence of numbers that you can describe both recursively and explicitly.

b. Write a recursive definition and an explicit formula for your sequence.

c. Find the 20th term of the sequence by evaluating one of your formulas. Use the other formula to check your work.

65. **Geometry** Suppose you are stacking boxes in levels that form squares. The numbers of boxes in successive levels form a sequence. The figure at the right shows the top four levels as viewed from above.



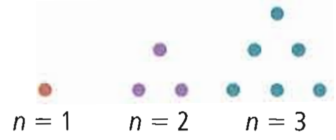
a. How many boxes of equal size would you need for the next lower level?

b. How many boxes of equal size would you need to add three levels?

c. Suppose you are stacking a total of 285 boxes. How many levels will you have?

Challenge

66. **Geometry** The triangular numbers form a sequence. The diagram represents the first three triangular numbers: 1, 3, 6.



a. Find the fourth and fifth triangular numbers.

b. Write a recursive formula for the *n*th triangular number.

c. Is the explicit formula $a_n = \frac{1}{2}(n^2 + n)$ the correct formula for this sequence? Explain.

Use each recursive definition to write an explicit formula for the sequence.

67. $a_1 = 10, a_n = 2a_{n-1}$

68. $a_1 = -5, a_n = a_{n-1} - 1$

69. $a_1 = 1, a_n = a_{n-1} + 4$

70. Finance Use the information in the ad.

- **a.** Suppose you start a savings account. Write a recursive definition and an explicit formula for the amount of money you would have in the bank at the end of any week.
- **b.** How much money would you have in the bank after four weeks?
- **c.** Assume the bank pays interest every four weeks. To calculate your interest, multiply the balance at the end of the four weeks by 0.005. Then, add that amount to your account on the last day of the four-week period. Write a recursive formula for the amount of money you have after each interest payment.
- **d. Reasoning** What is the bank's annual interest rate?

MUN-E-BANK

is offering a
GREAT deal!

Start a savings
club account by
depositing only
$25 today and
$5 a week
starting next week.

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.A.6.2 **71.** Scientists determine an object is moving at the rate of $(5 - \sqrt{2})$ ft/s. How many seconds will it take the object to travel 125 ft? Round the answer to the nearest tenth of a second.

MA.912.A.6.5 **72.** What is the solution of $\sqrt{4x - 23} - 3 = 2$?

MA.912.A.8.5 **73.** Using a calculator, what is the solution of $1080 = 15^{3x-4}$? Round the answer to the nearest hundredth.

MA.912.A.8.6 **74.** Using the change of base formula, what is the solution of $\log_5 x = \log_3 20$? Round the answer to the nearest tenth.

MA.912.A.8.7 **75.** The battery power available to operate a deep space probe is given by the formula $P = 42e^{-0.005t}$, where P is power in watts and t is time in years. For how many years can the probe run if it requires 35 watts? Round the answer to the nearest tenth year.

Mixed Review

Solve each equation. Check the solution. See Lesson 8-6.

76. $\frac{y}{y+1} = \frac{2}{3}$

77. $\frac{4}{2a} = \frac{5}{a+6}$

78. $\frac{3}{b+2} = \frac{6}{b-1}$

Find the slope of the line that passes through the two points. See Lesson 2-3.

79. (4, 5) and (1, 8)

80. (−3, −3) and (2, 2)

81. (1, 3) and (4, 9)

Get Ready! **To prepare for Lesson 9-2, do Exercises 82–84.**

Identify the pattern and find the next three terms. See Lesson 1-1.

82. 10, 8, 6, 4, 2, 0, . . .

83. 100, 117, 134, 151, 168, . . .

84. $\frac{5}{7}, \frac{8}{7}, \frac{11}{7}, 2, \ldots$

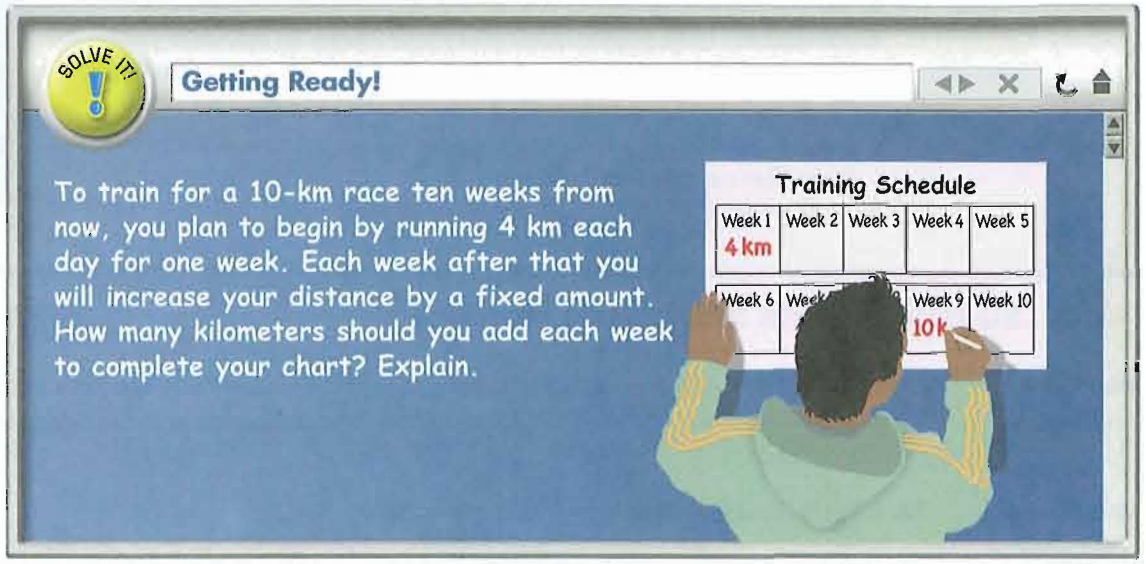
9-2 Arithmetic Sequences

Sunshine State Standards

MA.912.D.11.1 Define arithmetic and geometric sequences and series.

MA.912.D.11.3 Find specified terms of arithmetic and geometric sequences.

Objective To define, identify, and apply arithmetic sequences



Getting Ready!

To train for a 10-km race ten weeks from now, you plan to begin by running 4 km each day for one week. Each week after that you will increase your distance by a fixed amount. How many kilometers should you add each week to complete your chart? Explain.

Dynamic Activity
Arithmetic Sequences

Lesson Vocabulary
- arithmetic sequence
- common difference
- arithmetic mean

It sometimes is helpful to represent a situation with a sequence of numbers. There are different types of numerical sequences.

Essential Understanding In an *arithmetic sequence*, the difference between any two consecutive terms is always the same number. You can build an arithmetic sequence by adding the same number to each term.

An **arithmetic sequence** is a sequence where the difference between consecutive terms is constant. This difference is the **common difference**.

take note

Key Concept Arithmetic Sequence

An arithmetic sequence with a starting value a and common difference d is a sequence of the form

$$a, a + d, a + 2d, a + 3d, \dots.$$

A recursive definition for this sequence has two parts:

$a_1 = a$	initial condition
$a_n = a_{n-1} + d$, for $n > 1$	recursive formula

An explicit definition for this sequence is a single formula:

$$a_n = a + (n - 1)d, \text{ for } n \geq 1$$

Problem 1 Identifying Arithmetic Sequences

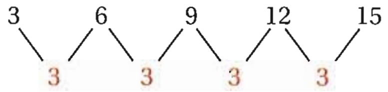
Is the sequence an arithmetic sequence?

Plan

How do you know whether a sequence is arithmetic? The differences between consecutive terms are the same in an arithmetic sequence.

A **3, 6, 9, 12, 15, . . .**

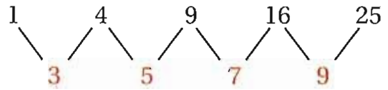
Find the differences between consecutive terms.



Each difference is 3. The sequence has a common difference. The sequence is an arithmetic sequence.

B **1, 4, 9, 16, 25, . . .**

Find the difference between consecutive terms.



There is no common difference. The sequence is not an arithmetic sequence.

Got It? **1.** Is the sequence an arithmetic sequence?

a. 2, 4, 8, 16, . . .

b. 1, 5, 9, 13, 17, . . .

Problem 2 Analyzing Arithmetic Sequences

A **What is the 100th term of the arithmetic sequence that begins 6, 11, . . .?**

The first term a is 6. The common difference d is $11 - 6 = 5$.

$a_n = a + (n - 1)d$	Use the explicit formula.
$a_{100} = 6 + (100 - 1)5$	Substitute 100 for n, 6 for a, and 5 for d.
$a_{100} = 501$	Simplify.

The 100th term is 501.

Plan

What do you need to find the second term given the first term? You need to know the common difference of the arithmetic sequence.

B **What are the second and third terms of the arithmetic sequence 100, ■, ■, 82, . . .?**

The first term a is 100. The fourth term a_4 is 82. There are 3 common differences between 100 and 82.

$82 = 100 + 3d$	Add $3d$ to move from 100 to 82.
$-18 = 3d$	Solve for d.
$-6 = d$	

The common difference is −6. The terms are 100, 94, 88, 82, ...

The second and third terms are 94 and 88.

Got It? **2. a.** What is the 46th term of the arithmetic sequence that begins 3, 5, 7, . . .?

b. What are the second and third terms of this arithmetic sequence?
80, ■, ■, 125, . . .

The **arithmetic mean**, or average, of two numbers x and y is $\frac{x+y}{2}$.

In an arithmetic sequence, the middle term of any three consecutive terms is the arithmetic mean of the other two terms.

Think

To use the formula for arithmetic mean, what are x and y?
x is 15 and y is 59.

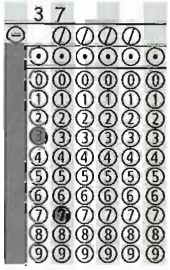
Problem 3 Using the Arithmetic Mean

GRIDDED RESPONSE

What is the missing term of the arithmetic sequence . . . , 15, ▒, 59, . . . ?

$$\text{arithmetic mean} = \frac{15 + 59}{2} = 37$$

The missing term is 37.

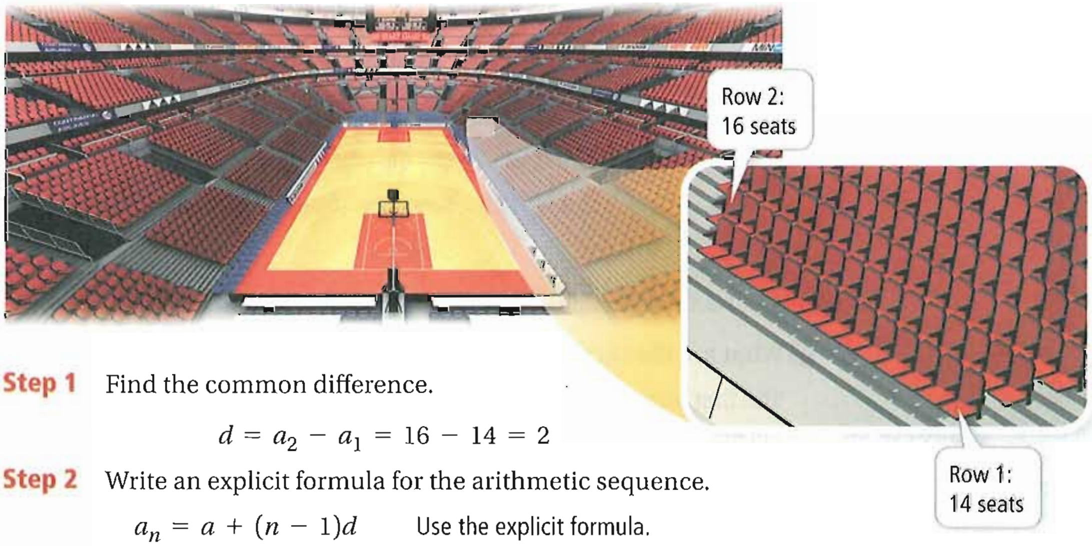


Got It? **3. a.** The 9th and 11th terms of an arithmetic sequence are 132 and 98. What is the 10th term?

b. Reasoning If you know the 5th and 6th terms of an arithmetic sequence, how can you find term 7 using the arithmetic mean?

Problem 4 Using an Explicit Formula for an Arithmetic Sequence

Sports Arena The numbers of seats in the first 13 rows in a section of an arena form an arithmetic sequence. Rows 1 and 2 are shown in the diagram below. How many seats are in Row 13?



Plan

What two terms should you use to find the common difference?
Use the consecutive terms given.

Step 1 Find the common difference.

$$d = a_2 - a_1 = 16 - 14 = 2$$

Step 2 Write an explicit formula for the arithmetic sequence.

$a_n = a + (n - 1)d$	Use the explicit formula.
$a_{13} = 14 + (13 - 1)2$	Substitute 13 for n, 14 for a, and 2 for d.
$= 38$	Simplify.

There are 38 seats in Row 13.

Got It? **4.** The numbers of seats in the first 16 rows in a curved section of another arena form an arithmetic sequence. If there are 20 seats in Row 1 and 23 seats in Row 2, how many seats are in Row 16?

Lesson Check

Do you know HOW?

Find the tenth term of each arithmetic sequence.

1. 2, 8, 14, 20, . . .

2. 15, 23, 31, . . .

Find the missing term of each arithmetic sequence.

3. . . . 4, ■, 22, . . .

4. . . . 25, ■, 53, . . .

Do you UNDERSTAND?

5. **Vocabulary** Explain what it means for a sequence to be an arithmetic sequence.

6. **Open-Ended** Give an example of a sequence that is not an arithmetic sequence.

Practice and Problem-Solving Exercises

A Practice

Determine whether each sequence is arithmetic. If so, identify the common difference.

See Problem 1.

7. 10, 20, 30, 40, . . .

8. 1, 1, 2, 3, 5, 8, . . .

9. −21, −18, −15, −12, . . .

10. 97, 86, 75, 64, . . .

11. 3, 7, 11, 15, . . .

12. 100, 10, 1, 0.1, . . .

Find the 32nd term of each sequence.

See Problem 2.

13. 34, 37, 40, 43, . . .

14. −9, −8.7, −8.4, . . .

15. 23, 30, 37, 44, . . .

16. 9, 4, −1, −6, −11, . . .

17. 0.1, 0.5, 0.9, 1.3, . . .

18. 101, 105, 109, 113, . . .

Find the missing term of each arithmetic sequence.

See Problem 3.

19. −15, ■, 1, . . .

20. 14, ■, 28, . . .

21. . . . 5, ■, 21, . . .

22. . . . 98, ■, 66, . . .

23. 25, ■, −10, . . .

24. . . . 65, ■, −60, . . .

See Problem 4.

25. **Savings** A student deposits the same amount of money into her bank account each week. At the end of the second week she has $30 in her account. At the end of the third week she has $45 in her account. How much will she have in her bank account at the end of the ninth week?

B Apply

Find the 17th term of each sequence.

26. $a_{16} = 18, d = 5$

27. $a_{16} = 21, d = -3$

28. $a_{18} = -5, d = 12$

29. $a_{18} = 32, d = -4$

30. $a_{16} = \frac{1}{5}, d = \frac{1}{2}$

31. $a_{18} = -9, d = -11$

32. **Think About a Plan** The arithmetic mean of the monthly salaries of two employees is $3210. One employee earns $3470 per month. What is the monthly salary of the other employee?

- What is the given information and what is the unknown?
- What equation can you use to find the other monthly salary?

33. **Error Analysis** A student claims that the next term of the arithmetic sequence 0, 2, 4, . . . is 8. Explain and correct the student's error.

Find the arithmetic mean a_n of the given terms.

34. $a_{n-1} = 7, a_{n+1} = 1$

35. $a_{n-1} = 100, a_{n+1} = 140$

36. $a_{n-1} = 4, a_{n+1} = -3$

37. $a_{n-1} = 0.3, a_{n+1} = 1.9$

38. $a_{n-1} = r, a_{n+1} = s$

39. $a_{n-1} = -2x, a_{n+1} = 2x$

40. a. Graphing Calculator Use your calculator to generate an arithmetic sequence with a common difference of −7. How could you use a calculator to find the 6th term? The 8th term? The 20th term?

b. Reasoning Explain how your answer to part (a) relates to the explicit formula $a_n = a + (n - 1)d$.

Write an explicit and a recursive formula for each sequence.

41. $2, 4, 6, 8, 10, \dots$

42. $0, 6, 12, 18, 24, \dots$

43. $-5, -4, -3, -2, -1, \dots$

44. $-4, -8, -12, -16, -20, \dots$

45. $-5, -3.5, -2, -0.5, 1, \dots$

46. $-32, -20, -8, 4, 16, \dots$

47. $1, 1\frac{1}{3}, 1\frac{2}{3}, 2, \dots$

48. $0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \dots$

49. $27, 15, 3, -9, -21, \dots$

50. Reasoning What information do you need to find a term of a sequence using an explicit formula?

51. Writing Describe some advantages and some disadvantages of a recursive formula and an explicit formula. When is it appropriate to use each formula?

52. Transportation Suppose a trolley stops at a certain intersection every 14 min. The first trolley of the day gets to the stop at 6:43 A.M. How long do you have to wait for a trolley if you get to the stop at 8:15 A.M.? At 3:20 P.M.?

Find the missing terms of each arithmetic sequence. (*Hint:* The arithmetic mean of the first and fifth terms is the third term.)

53. $2, a_2, a_3, a_4, -22, \dots$

54. $10, a_2, a_3, a_4, -11.6, \dots$

55. $1, a_2, a_3, a_4, -35, \dots$

56. $\dots \frac{13}{5}, a_6, a_7, a_8, \frac{37}{5}, \dots$

57. $17, a_2, a_3, a_4, 17, \dots$

58. $660, a_2, a_3, a_4, 744, \dots$

59. $\dots -17, a_4, a_5, a_6, 1, \dots$

60. $\dots a + 1, a_3, a_4, a_5, a + 17, \dots$

61. Income The arithmetic mean of the monthly salaries of two people is \$4475. One person earns \$3895 per month. What is the monthly salary of the other person?

62. Reasoning Suppose you turn the water on in an empty bathtub with vertical sides. After 20 s, the water has reached a level of 1.15 in. You then leave the room. You want to turn the water off when the level in the bathtub is 8.5 in. How many minutes later should you return? (*Hint:* Begin by identifying two terms of an arithmetic sequence.)

Challenge

63. In an arithmetic sequence with $a_1 = 4$ and $d = 9$, which term is 184?

64. In an arithmetic sequence with $a_1 = 2$ and $d = -2$, which term is −82?

65. The arithmetic mean of two terms in an arithmetic sequence is 42. One term is 30. Find the other term.

66. The arithmetic mean of two terms in an arithmetic sequence is −6. One term is −20. Find the other term.

Given two terms of each arithmetic sequence, find a_1 and d.

67. $a_3 = 5$ and $a_5 = 11$

68. $a_4 = 8$ and $a_7 = 20$

69. $a_3 = 32$ and $a_7 = -8$

70. $a_{10} = 17$ and $a_{14} = 34$

71. $a_4 = -34.5$ and $a_5 = -12.5$

72. $a_4 = -2.4$ and $a_6 = 2$

Find the indicated term of each arithmetic sequence.

73. $a_1 = k, d = k + 4; a_9$

74. $a_1 = k + 7, d = 2k - 5; a_{11}$

Sunshine State Standards Practice

MA.912.A.4.10

75. The equation $X(t) = t^4 - 5t^2 + 6$ gives the position of a comet relative to a fixed point, measured in millions of miles, at time t, measured in days. Solve the equation $X(t) = 0$. At what times is the position zero?

Ⓐ 2, 3

Ⓑ −2, −3

Ⓒ ±2, ±3

Ⓓ $\pm\sqrt{2}, \pm\sqrt{3}$

MA.912.A.5.3

76. Simplify $\dfrac{3 - \frac{1}{x}}{\frac{1}{2x} - 5}$.

Ⓕ $\dfrac{6x - 2}{1 - 10x}$

Ⓖ $\dfrac{3x - 1}{1 - 10x}$

Ⓗ $\dfrac{4}{1 - 10x}$

Ⓘ $\dfrac{3x - 1}{1 - 5x}$

MA.912.A.5.5

77. Extended Response What are all the solutions of $\frac{3}{x^2 - 1} + \frac{4x}{x + 1} = \frac{1.5}{x - 1}$? Show your work.

Mixed Review

Determine whether each formula is *explicit* or *recursive*. Then find the first five terms of each sequence. See Lesson 9-1.

78. $a_1 = -2, a_n = a_{n-1} - 5$

79. $a_n = 3n(n + 1)$

80. $a_n = n^2 - 1$

81. $a_1 = -121, a_n = a_{n-1} + 13$

Write an equation in point-slope form for each pair of points. See Lesson 2-4.

82. (0, 3) and (3, 11)

83. (4, 6) and (10, 30)

84. (1, 10) and (5, 42)

85. **Geometry** The formula for volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$. Find the radius of a sphere as a function of its volume. Rationalize the denominator. See Lesson 6-2.

Get Ready! **To prepare for Lesson 9-3, do Exercises 86–88.**

Find the next term in each sequence. See Lesson 9-1.

86. 2, 4, 8, 16, . . .

87. 1, 5, 25, 125, . . .

88. −1, −3, −9, −27, . . .

Concept Byte

For Use With Lesson 9-2

EXTENSION

The Fibonacci Sequence

Sunshine State Standard

Prepares for MA.912.D.11.1 Define arithmetic and geometric sequences.

One famous mathematical sequence is the Fibonacci sequence. You can find each term of the sequence using addition, but the sequence is not arithmetic.

Example

The recursive formula for the Fibonacci sequence is $F_n = F_{n-2} + F_{n-1}$, with $F_1 = 1$ and $F_2 = 1$. Using the formula, what are the first five terms of the sequence?

$F_1 = 1$

$F_2 = 1$

$F_3 = F_1 + F_2 = 1 + 1 = 2$

$F_4 = F_2 + F_3 = 1 + 2 = 3$

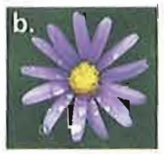
$F_5 = F_3 + F_4 = 2 + 3 = 5$

The first five terms of the Fibonacci sequence are 1, 1, 2, 3, 5.

Exercises

1. **Nature** The numbers of the Fibonacci sequence are often found in other areas, especially nature. Which term of the Fibonacci sequence does each picture represent?


a.


b.


c.


d.

2. Find "diagonals" in Pascal's triangle at the right by starting with the first 1 in each row and moving one row up and one number to the right. For example, the diagonal starting in the fifth row is 1, 3, 1. The diagonal starting in the sixth row is 1, 4, 3. For each diagonal, write the sum of its entries. What pattern do the sums form?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

3. **a.** Generate the first ten terms of the Fibonacci sequence.
  b. Find the sum of the first ten terms of the Fibonacci sequence. Divide the sum by 11. What do you notice?
  c. **Open-Ended** Choose two numbers other than 1 and 1. Generate a Fibonacci-like sequence from them. Write the first ten terms of your sequence, find the sum, and divide the sum by 11. What do you notice?
  d. **Make a Conjecture** What is the sum of the first ten terms of any Fibonacci-like sequence?

4. **a.** Study the pattern at the right. Write the next line.
  b. Without calculating, use the pattern to predict the sum of the squares of the first ten terms of the Fibonacci sequence.
  c. Verify the prediction you made in part (b).

$$1^2 + 1^2 = 2 = 1 \cdot 2$$
$$1^2 + 1^2 + 2^2 = 6 = 2 \cdot 3$$
$$1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \cdot 5$$
$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40 = 5 \cdot 8$$

9 Mid-Chapter Quiz

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

Find the first five terms of each sequence.

1. $a_n = 3n + 1$

2. $a_n = -2n - 1$

3. $a_n = n^2 + 2n$

4. $a_n = 3a_{n-1}$, where $a_1 = 2$

5. $a_n = 5 - a_{n-1}$, where $a_1 = 1$

6. $a_n = a_{n-1} + 2n$, where $a_1 = 1$

Write a recursive definition for each sequence.

7. 2, −4, 8, −16, . . .

8. 1, 4, 7, 10, . . .

9. 4, 2, 5, 1, 6, . . .

Write an explicit formula for each sequence.

10. 2, 4, 8, 16, . . .

11. 5, 2, −1, −4, . . .

12. 2, 5, 10, 17, . . .

Find the ninth and tenth terms of each arithmetic sequence.

13. 1, 8, 15, 22, . . .

14. 4, 10, 16, 22, . . .

15. 6, 3, 0, −3, . . .

Determine whether each sequence is arithmetic. If so, identify the common difference.

16. 1, 3, 9, 27, . . .

17. 11, 22, 33, 44, . . .

18. 1, −1, −3, −5, −7, . . .

19. 0, 2, 5, 9, 14, . . .

Find the missing term of each arithmetic sequence.

20. . . . , 3, ■, 17, . . .

21. . . . , 25, ■, −15, . . .

22. . . . , −3, ■, 8, . . .

23. . . . , 66, ■, 48, . . .

Find the missing terms of each arithmetic sequence.

24. 4, a_2, a_3, a_4, 32, . . .

25. 10, a_2, a_3, a_4, −20, . . .

26. 5, a_2, a_3, a_4, 35, . . .

Do you UNDERSTAND?

27. **Open-Ended** Write the first four terms of an arithmetic sequence with a common difference of 3 and a third term of 10. Then write both a recursive definition and an explicit formula for this sequence.

28. **Investments** You invested money in a fund and each month you receive a payment for your investment. Over the first four months, you received \$50, \$52, \$55, and \$59. If this pattern continues, how much will you receive in the tenth month?

- **a.** Write a formula to describe this sequence.
- **b.** Identify your formula as explicit or recursive.
- **c.** **Writing** Explain the difference between an explicit formula and a recursive formula. Use your formula from part (a) as part of your explanation.

29. **Open-Ended** Write the first five terms of a sequence that is not an arithmetic sequence. Then give both an explicit and recursive formula to describe this sequence.

30. **Sports** A tennis club charges players a \$20 court fee plus a \$10 hourly charge with a 5-hour maximum. A posted list of the total charges for 1, 2, 3, 4, or 5 hours forms an arithmetic sequence. What is the first term and what is the common difference?

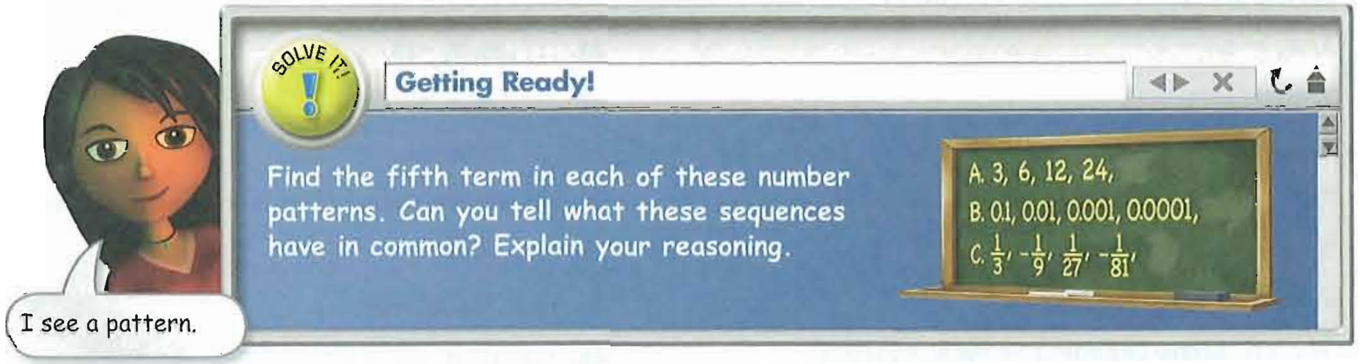
9-3 Geometric Sequences

Sunshine State Standards

MA.912.D.11.1 Define arithmetic and geometric sequences and series.

MA.912.D.11.3 Find specified terms of arithmetic and geometric sequences.

Objective To define, identify, and apply geometric sequences



Find the fifth term in each of these number patterns. Can you tell what these sequences have in common? Explain your reasoning.

A. 3, 6, 12, 24,

B. 0.1, 0.01, 0.001, 0.0001,

C. $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81},$

Dynamic Activity
Geometric Sequences

Lesson Vocabulary
- geometric sequence
- common ratio
- geometric mean

You build a *geometric sequence* by multiplying each term by a constant.

Essential Understanding In a *geometric sequence*, the ratio of any term to its preceding term is a constant value.

take note

Key Concept Geometric Sequence

A **geometric sequence** with a starting value a and a **common ratio** r is a sequence of the form

$$a, ar, ar^2, ar^3, \ldots$$

A recursive definition for the sequence has two parts:

$a_1 = a$ initial condition

$a_n = a_{n-1} \cdot r$, for $n > 1$ recursive formula

An explicit definition for this sequence is a single formula:

$a_n = a_1 \cdot r^{n-1}$, for $n \geq 1$

Problem 1 Identifying Geometric Sequences

Is the sequence geometric? If it is, what are a_1 and r?

A **3, 6, 12, 24, 48, . . .**

Find the ratios between consecutive terms.

$$3 \quad 6 \quad 12 \quad 24 \quad 48$$

$$\frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \frac{48}{24} = 2$$

The common ratio is 2. The sequence is geometric with $a_1 = 3$ and $r = 2$.

Think

How do I find the ratios between consecutive terms?
Divide the second term by the first term, then the third term by the second term, and so on.

B **3, 6, 9, 12, 15, . . .**

Find the ratio between consecutive terms.

$$\begin{array}{ccccccccc} 3 & & 6 & & 9 & & 12 & & 15 \\ & \frac{6}{3} & \neq & \frac{9}{6} & \neq & \frac{12}{9} & \neq & \frac{15}{12} & \end{array}$$

The ratios are different. With no common ratio, the sequence is not geometric.

C $\mathbf{3^5, 3^{10}, 3^{15}, 3^{20}, \ldots}$

Use the properties of exponents to simplify the ratios of successive terms.

$$\begin{array}{ccccccc} 3^5 & & 3^{10} & & 3^{15} & & 3^{20} \\ & \frac{3^{10}}{3^5} & = & \frac{3^{15}}{3^{10}} & = & \frac{3^{20}}{3^{15}} & = 3^5 \end{array}$$

The common ratio is 3^5. The sequence is geometric with $a_1 = 3^5$ and $r = 3^5$.

Got It? **1.** Is the sequence geometric? If it is, what are a_1 and r?

a. 2, 4, 8, 16, . . . **b.** 1, 5, 9, 13, 17, . . . **c.** $2^3, 2^7, 2^{11}, 2^{15}, \ldots$

Problem 2 Analyzing Geometric Sequences

What are the indicated terms of the geometric sequence?

A **the 10th term of the geometric sequence 4, 12, 36, . . .**

The first term a_1 is 4. The common ratio r is $12 \div 4 = 3$.

$a_n = a_1 r^{n-1}$	Use the explicit formula.
$a_{10} = 4 \cdot 3^{10-1}$	Substitute 10 for n, 4 for a_1, and 3 for r.
$a_{10} = 78{,}732$	Simplify.

The 10th term is 78,732.

Plan

What do you need to find the second term given the first term?

You need the common ratio.

B **the second and third terms of the geometric sequence 2, ■, ■, −54, . . .**

The first term a_1 is 2. The fourth term a_4 is −54.

$a_n = a_1 r^{n-1}$	Use the explicit formula.
$a_4 = 2r^{4-1}$	Substitute 2 for a_1 and 4 for n.
$-54 = 2r^3$	Substitute −54 for a_4. Simplify.
$-27 = r^3$	Solve for r.
$-3 = r$	

The common ratio is −3. Begin with 2 and multiply by −3.

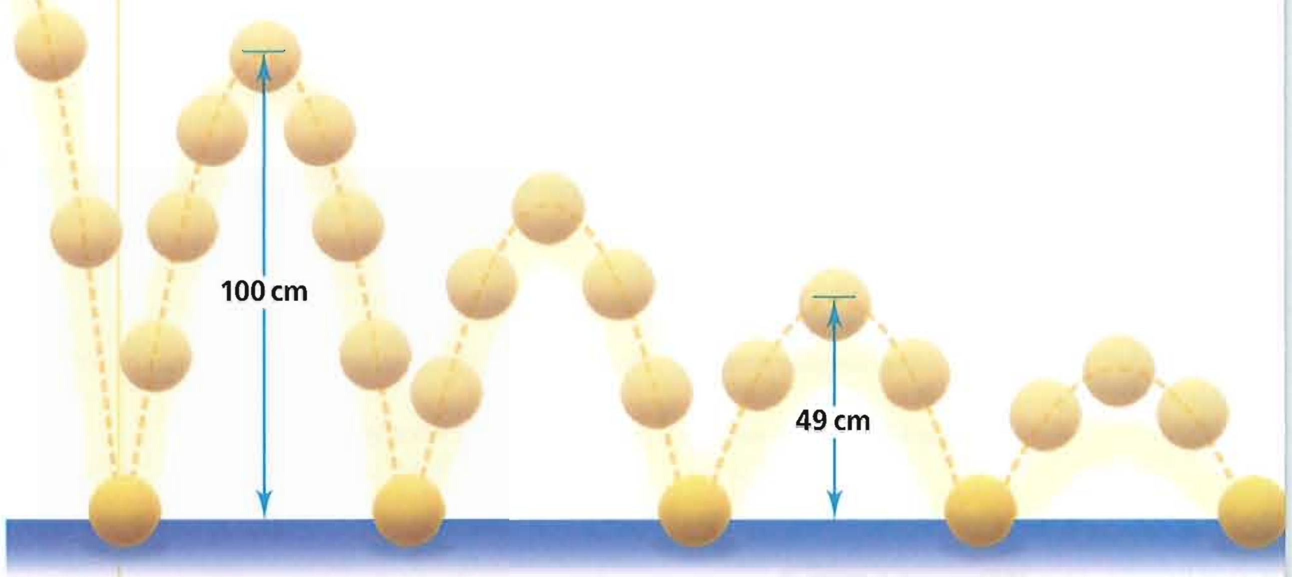
2, −6, 18, −54, . . .

The second and third terms are −6 and 18.

Got It? **2.** What is the 2nd term of the geometric sequence 3, ■, 12, . . . ?

Problem 3 Using a Geometric Sequence

Physics When a ball bounces, the heights of consecutive bounces form a geometric sequence. What are the heights of the 4th and 5th bounces?



Think	Write
The heights of the first and third bounces are given in the picture.	$a_1 = 100$ $a_3 = 49$
Use the explicit formula to relate a_1 to a_3, and to find r.	$a_n = a_1 r^{n-1}$ $a_3 = a_1 r^{3-1}$ $49 = 100r^2$
Solve for r. (r must be positive since the bounces are above the floor.)	$100r^2 = 49$ $r = \sqrt{\frac{49}{100}} = \frac{7}{10}$
To find a_4 and a_5, build the sequence recursively, starting from a_3.	$a_n = a_{n-1} \cdot r$ $a_4 = a_3 \cdot r = 49 \cdot \frac{7}{10} = 34.3$ $a_5 = a_4 \cdot r = 34.3 \cdot \frac{7}{10} \approx 24$
Write the answer.	The heights are 34.3 cm and 24 cm.

Got It? **3. a.** **Reasoning** To find the height of the 10th bounce, would you use the recursive or the explicit formula? Explain.

b. What are the heights of the 6th and 10th bounces?

In a geometric sequence, the square of the middle term of any three consecutive terms is equal to the product of the other two terms. For example, examine the sequence 2, −6, 18, −54, ...

$$(-6)^2 = 2 \cdot 18 = 36$$

$$2, -6, 18, -54, \ldots$$

$$18^2 = (-6)(-54) = 324$$

In an arithmetic sequence, recall that the middle term of any three consecutive terms is the arithmetic mean of the other two terms.

The **geometric mean** of two positive numbers x and y is $\sqrt{xy}$.

Note that the geometric mean is positive by definition. While there are two possible values for the missing term in the geometric sequence 3, ■, 12, . . . , there is only one geometric mean. The geometric mean is one possible value to fill in the geometric sequence. The opposite of the geometric mean is the other.

Problem 4 Using the Geometric Mean

Multiple Choice **What are the possible values of the missing term of the geometric sequence?**

48, ■, 3, . . .

Ⓐ ±4 Ⓑ ±9 Ⓒ ±12 Ⓓ ±20

Find the geometric mean of 48 and 3.

$$\sqrt{48 \cdot 3} = \sqrt{144}$$
$$= 12$$

The possible values for the missing term are ±12. The correct answer choice is C.

Think

Why would this question ask for "possible values" rather than "the value"?

The geometric mean and its opposite are both possible values.

Got It? **4.** The 9th and 11th terms of a geometric sequence are 45 and 80. What are possible values for the 10th term?

Lesson Check

Do you know HOW?

Determine whether each sequence is geometric. If so, find the common ratio.

1. 5, 10, 15, . . .

2. 10, 20, 40, . . .

Find the seventh term of each geometric sequence.

3. 1, −3, 9, . . .

4. 100, 20, 4, . . .

Do you UNDERSTAND?

5. **Error Analysis** To find the third term of the geometric sequence 5, 10, ■, ■, 80, your friend says that there are two possible answers—the geometric mean of 5 and 80, and its opposite. Explain your friend's error.

6. **Compare and Contrast** How is finding a missing term of a geometric sequence using the geometric mean similar to finding a missing term of an arithmetic sequence using the arithmetic mean? How is it different?

Practice and Problem-Solving Exercises

A Practice

Determine whether each sequence is geometric. If so, find the common ratio. **See Problem 1.**

7. 1, 2, 4, 8, . . .
8. 1, 2, 3, 4, . . .
9. 1, −2, 4, −8, . . .
10. −1, 1, −1, 1, . . .
11. 10, 4, 1.6, 0.64, . . .
12. 7, 0.7, 0.07, 0.007, . . .
13. 18, −6, 2, $-\frac{2}{3}$, . . .
14. 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . .
15. 10, 15, 22.5, 33.75, . . .
16. 2, −10, 50, −250, . . .
17. −1, −6, −36, −216, . . .
18. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, . . .

Find the eighth term of each geometric sequence. **See Problem 2.**

19. 3, 9, 27, . . .
20. −3, 6, −12, . . .
21. 10, 5, 2.5, . . .
22. $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, . . .
23. 24, −6, $\frac{3}{2}$, . . .
24. −30, 7.5, −1.875, . . .

25. **Science** When radioactive substances decay, the amount remaining will form a geometric sequence when measured over constant intervals of time. The table shows the amount of Np-240, a radioactive isotope of Neptunium, initially and after 2 hours. What are the amounts left after 1 hour, 3 hours, and 4 hours? **See Problem 3.**

Hours Elapsed	0	1	2	3	4
Grams of Np-240	1244	■	346	■	■

Find the missing term of each geometric sequence. It could be the geometric mean or its opposite. **See Problem 4.**

26. 5, ■, 911.25, . . .
27. 9180, ■, 255, . . .
28. $\frac{2}{5}$, ■, $\frac{8}{45}$, . . .
29. 3, ■, 0.75, . . .
30. 5, ■, 2.8125, . . .
31. 12, ■, 3, . . .

B Apply

Write an explicit formula for each sequence. Then generate the first five terms.

32. $a_1 = 1, r = 0.5$
33. $a_1 = 100, r = -20$
34. $a_1 = 7, r = 1$
35. $a_1 = 1024, r = 0.5$
36. $a_1 = 4, r = 0.1$
37. $a_1 = 10, r = -1$

Identify each sequence as *arithmetic, geometric,* or *neither*. Then find the next two terms.

38. 45, 90, 180, 360, . . .
39. 25, 50, 75, 100, . . .
40. 3, −3, 3, −3, . . .
41. −5, 10, −20, 40, . . .
42. 2, 1, 0.5, 0.25, . . .
43. 1, 4, 9, 16, . . .

Find the missing terms of each geometric sequence. (*Hint:* The geometric mean of the first and fifth terms is the third term. Some terms might be negative.)

44. 972, ■, ■, ■, 12, . . .
45. 2.5, ■, ■, ■, 202.5, . . .
46. 12.5, ■, ■, ■, 5.12, . . .
47. −4, ■, ■, ■, $-30\frac{3}{8}$, . . .

48. **Think About a Plan** Suppose a balloon is filled with 5000 cm^3 of helium. It then loses one fourth of its helium each day. How much helium will be left in the balloon at the start of the tenth day?

- How can you write a sequence of numbers to represent this situation?
- Is the sequence arithmetic, geometric, or neither?
- How can you write a formula for this sequence?

49. **Athletics** During your first week of training for a marathon, you run a total of 10 miles. You increase the distance you run each week by twenty percent. How many miles do you run during your twelfth week of training?

50. **a.** **Open-Ended** Choose two positive numbers. Find their geometric mean.
b. Find the common ratio for a geometric sequence that includes the terms from part (a) as its first three terms.
c. Find the 9th term of the geometric sequence from part (b).
d. Find the geometric mean of the term from part (c) and the first term of your sequence. What term of the sequence have you just found?

For the geometric sequence 3, 12, 48, 192, . . . , find the indicated term.

51. 5th term **52.** 17th term **53.** 20th term **54.** nth term

Find the 10th term of each geometric sequence.

55. $a_9 = 8, r = \frac{1}{2}$

56. $a_9 = -5, r = -\frac{1}{2}$

57. $a_{11} = -5, r = -\frac{1}{2}$

58. $a_9 = -\frac{1}{3}, r = \frac{1}{2}$

59. **Writing** Describe the similarities and differences between a common difference and a common ratio.

Challenge

60. **Banking** Copy and complete the table below. Use the geometric mean. Assume compound interest is earned and no withdrawals are made.

Period 1	Period 2	Period 3
$140.00	■	$145.64
$600.00	■	$627.49
$25.00	■	$32.76
$57.50	■	$60.37
$100.00	■	$111.98
$250.00	■	$276.55

Find a_1 for a geometric sequence with the given terms.

61. $a_5 = 112$ and $a_7 = 448$

62. $a_9 = \frac{1}{2}$ and $a_{12} = \frac{1}{16}$

Sunshine State Standards Practice

MA.912.D.11.1

63. What is the common ratio in the geometric sequence 4, 10, 25, 62.5, . . . ?

Ⓐ 0.4
Ⓑ 2.5
Ⓒ 15
Ⓓ 25

MA.912.D.11.3

64. The first term of a geometric sequence is 1 and its common ratio is 6. What is the sixth term?

Ⓕ 31
Ⓖ 3176
Ⓗ 7776
Ⓘ 46,656

MA.912.A.4.5

65. Determine by inspection the end behavior of the graph of $y = -2x^3 + 5x - 4$.

Ⓐ falls to the left, falls to the right: (↙, ↘)
Ⓑ falls to the left, rises to the right: (↙, ↗)
Ⓒ rises to the left, falls to the right: (↖, ↘)
Ⓓ rises to the left, rises to the right: (↖, ↗)

MA.912.A.5.6

66. What are the asymptotes of the graph of $y = \frac{10}{x - 5}$?

Ⓕ $x = 0, y = 5$
Ⓖ $x = 5, y = 0$
Ⓗ $x = 5, y = 10$
Ⓘ $x = 10, y = 5$

MA.912.A.5.6

67. Short Response What are the points of discontinuity of $y = \frac{x(2x - 1)(x + 1)}{(x + 5)(x + 1)}$?

Mixed Review

Write an explicit and a recursive formula for each arithmetic sequence. See Lesson 9-2.

68. −3, 0, 3, 6, . . .

69. 17, 8, −1, . . .

70. −2, −13, −24, . . .

Simplify each expression. Rationalize all denominators. Assume that all variables are positive. See Lesson 6-2.

71. $(\sqrt{7})(\sqrt{98})$

72. $\frac{3\sqrt{6}}{7\sqrt{2x}}$

73. $\frac{\sqrt{6x^4y}}{\sqrt{6x^2y^3}}$

74. $(\sqrt[3]{5})(\sqrt[3]{150})$

Find the vertical asymptotes and holes for the graph of each rational function. See Lesson 8-3.

75. $y = \frac{x - 3}{x + 3}$

76. $y = \frac{x - 3}{x + 1}$

77. $y = \frac{x - 3}{x(x - 1)}$

78. $y = \frac{x(x + 3)}{(x - 3)(x + 3)}$

Get Ready! **To prepare for Lesson 9-4, do Exercises 79–81.**

Write a recursive formula for each sequence. See Lesson 9-1.

79. 1, 3, 6, 10, . . .

80. 1, 4, 9, 16, . . .

81. 1, 5, 14, 30, . . .

9-4 Arithmetic Series

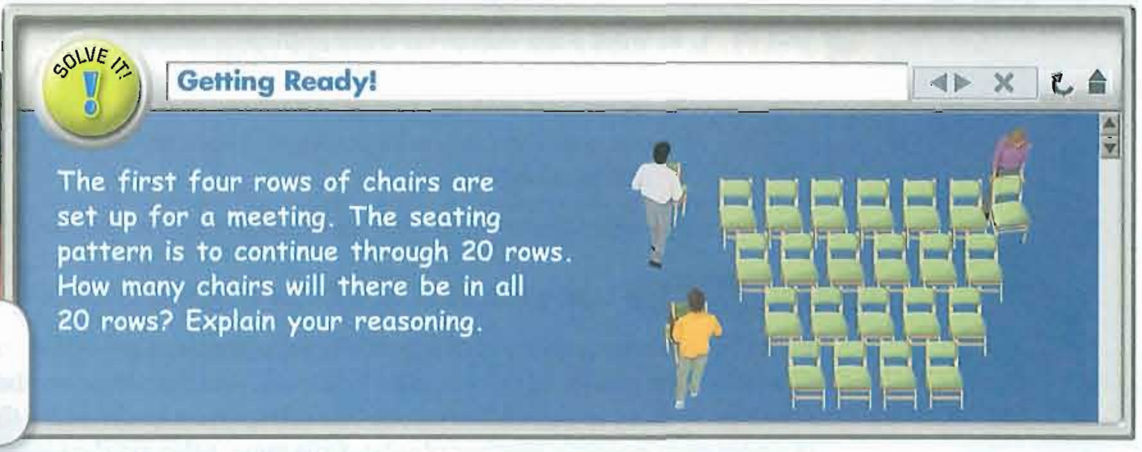
Sunshine State Standards

MA.912.D.11.1 Define arithmetic and geometric sequences and series.

MA.912.D.11.2 Use sigma notation to describe series.

MA.912.D.11.4 Find partial sums of arithmetic and geometric series.

Objective To define arithmetic series and find their sums



The first four rows of chairs are set up for a meeting. The seating pattern is to continue through 20 rows. How many chairs will there be in all 20 rows? Explain your reasoning.

Lesson Vocabulary
- series
- finite series
- infinite series
- arithmetic series
- limits

Just as you found formulas for terms of sequences, you can find formulas for the sums of the terms of sequences.

Essential Understanding When you know two terms and the number of terms in a finite arithmetic sequence, you can find the sum of the terms.

A **series** is the indicated sum of the terms of a sequence. A **finite series**, like a finite sequence, has a first term and a last term, while an **infinite series** continues without end.

Finite sequence	**Finite series**
6, 9, 12, 15, 18	$6 + 9 + 12 + 15 + 18$ (The sum is 60.)
Infinite sequence	**Infinite series**
3, 7, 11, 15, . . .	$3 + 7 + 11 + 15 + \dots$

An **arithmetic series** is a series whose terms form an arithmetic sequence (as shown above). When a series has a finite number of terms, you can use a formula involving the first and last term to evaluate the sum.

take note

Property **Sum of a Finite Arithmetic Series**

The sum S_n of a finite arithmetic series $a_1 + a_2 + a_3 + \cdots + a_n$ is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

where a_1 is the first term, a_n is the nth term, and n is the number of terms.

Problem 1 Finding the Sum of a Finite Arithmetic Series

Think

How many even integers are there from 2 to 100?
Double 1, 2, 3, . . . , 50, and you get 2, 4, 6, . . . , 100, the even integers from 2 to 100. There are 50 even integers.

What is the sum of the even integers from 2 to 100?

The series $2 + 4 + 6 + \cdots + 100$ is arithmetic with first term 2, last (and 50th) term 100, and common difference 2. The sum is

$$S_{50} = \frac{50}{2}(2 + 100) = 25(102) = 2550.$$

Got It? **1. a.** What is the sum of the finite arithmetic series
$4 + 9 + 14 + 19 + 24 + \cdots + 99$?

b. **Reasoning** Will the sum of a sequence of even numbers always be an even number? Will the sum of a sequence of odd numbers always be an odd number? Explain.

Problem 2 Using the Sum of a Finite Arithmetic Series

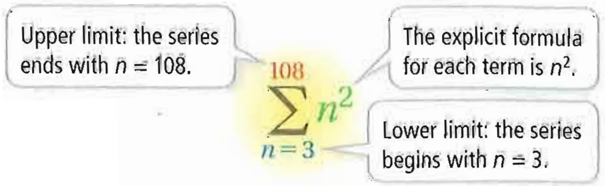
Bonus **A company pays a $10,000 bonus to salespeople at the end of their first 50 weeks if they make 10 sales in their first week, and then improve their sales numbers by two each week thereafter. One salesperson qualified for the bonus with the minimum possible number of sales. How many sales did the salesperson make in week 50? In all 50 weeks?**

Think	Write
The first week sales were 10. Sales increased by 2 each week.	$a_1 = 10$ $d = 2$
The sequence is arithmetic. Use the explicit formula to find the sales in week 50. Substitute 50 for n, 10 for a_1, and 2 for d. Then simplify.	$a_n = a_1 + (n - 1)d$ $a_{50} = 10 + (50 - 1)2$ $= 10 + (49)2$ $= 10 + 98 = 108$
Use the formula for S_n to find the total sales for all 50 weeks. Substitute 50 for n, 10 for a_1, and 108 for a_{50}. Then simplify.	$S_n = \frac{n}{2}(a_1 + a_n)$ $S_{50} = \frac{50}{2}(10 + 108)$ $= 25(118) = 2950$
Write the answers.	The salesperson made 108 sales in week 50 and 2950 sales in all 50 weeks.

Got It? **2.** The company in Problem 2 has an alternative bonus plan. It pays a $5000 bonus if a new salesperson makes 10 sales in the first week and then improves by *one* sale per week each week thereafter. One salesperson qualified for this bonus with the minimum number of sales. How many sales did the salesperson make in week 50? In all 50 weeks?

You can use the Greek capital letter sigma, Σ, to indicate a sum. With it, you use *limits* to indicate how many terms you are adding. **Limits** are the least and greatest values of n in the series. You write the limits below and above the Σ to indicate the first and last terms of the series.

For example, you can write the series $3^2 + 4^2 + 5^2 + \cdots + 108^2$ as $\sum_{n=3}^{108} n^2$.



For an infinite series, summation notation shows ∞ as the upper limit.

To find the number of terms in a series written in Σ form, subtract the lower limit from the upper limit and add 1.

The number of terms in the series above is $108 - 3 + 1 = 106$.

Problem 3 Writing a Series in Summation Notation

Plan

What do you need to write a series in summation notation?
You need an explicit formula for the *n*th term and the lower and upper limits.

Multiple Choice **What is summation notation for the series?**

$7 + 11 + 15 + \cdots + 203 + 207$

Ⓐ $\sum_{n=1}^{51}(4n + 3)$ Ⓑ $\sum_{n=1}^{50}(4n + 3)$ Ⓒ $\sum_{n=1}^{50}(7n)$ Ⓓ $\sum_{n=1}^{51}(7n)$

The sequence 7, 11, 15, . . . , 203, 207 is arithmetic with first term $a = 7$ and common difference $d = 4$.

$a_n = a_1 + (n - 1)d$	Use the explicit formula for an arithmetic sequence.
$a_n = 7 + (n - 1)4$	Substitute 7 for a_1, and 4 for d.
$= 4n + 3$	Simplify.

An explicit formula for the *n*th term is $4n + 3$.

$a_n = 4n + 3$	Use the explicit formula to find the value of n for the term 207.
$207 = 4n + 3$	Substitute 207 for a_n.
$204 = 4n$	Solve for n.
$51 = n$	

The upper limit is 51. You can write the series as $\sum_{n=1}^{51}(4n + 3)$. The correct answer is A.

Got It? **3.** What is summation notation for the series?

a. $-5 + 2 + 9 + 16 + \cdots + 261 + 268$
b. $500 + 490 + 480 + \cdots + 20 + 10$

take note

Key Concept Summation Notation and Linear Functions

If the explicit formula for the *n*th term in summation notation is a *linear* function of n, then the series is arithmetic. The slope of the linear function is the common difference between terms of the series.

Problem 4 Finding the Sum of a Series

What is the sum of the series written in summation notation?

A $\sum_{n=1}^{70}(5n + 3)$

Since the formula is a linear function of n, the series is arithmetic.

$a_1 = 5(1) + 3 = 8$	Find a_1.
$a_{70} = 5(70) + 3 = 353$	Find a_{70}.
$S_n = \frac{n}{2}(a_1 + a_n)$	Use the formula for the sum of a finite arithmetic series.
$S_{70} = \frac{70}{2}(8 + 353) = 12{,}635$	There are 70 terms, so $n = 70$.

Think

Is the function $f(n) = (n - 1)^2$ linear?

No; the function is quadratic.

B $\sum_{n=1}^{7}(n - 1)^2$

This series is not arithmetic. However, you can evaluate by adding the terms.

$$\sum_{n=1}^{7}(n - 1)^2$$

$$= (1 - 1)^2 + (2 - 1)^2 + (3 - 1)^2 + (4 - 1)^2 + (5 - 1)^2 + (6 - 1)^2 + (7 - 1)^2$$

$$= 0 + 1 + 4 + 9 + 16 + 25 + 36 = 91$$

Got It? **4.** What is the sum of each finite series?

a. $\sum_{n=1}^{40}(3n - 8)$

b. $\sum_{n=1}^{4}n^3$

c. $\sum_{n=0}^{100}(-1)^n$

On a graphing calculator, you can find the sum of a finite series by using commands from the **LIST** menu.

Problem 5 Using a Graphing Calculator to Find the Sum of a Series

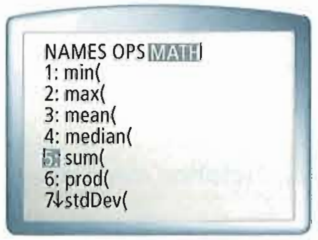
Plan

What calculator commands do you use to find the sum of a series?

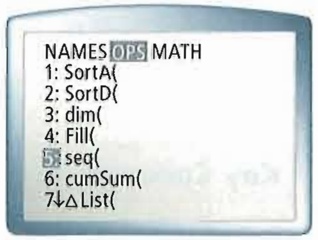
Use the sum and sequence commands.

What is the sum of the series written in summation notation? $\sum_{n=1}^{70}(5n + 3)$

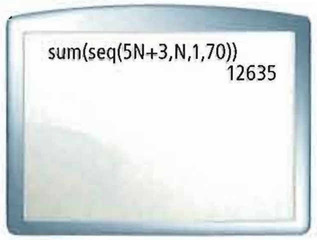
Input the sum command.



Input the sequence command.



Input the formula, N, and the lower and upper limits.



```
sum(seq(5N+3,N,1,70))
                12635
```

The sum is 12,635.

Got It? **5.** Use a graphing calculator. What is $\sum_{n=1}^{50}(n^2 - n)$?

Lesson Check

Do you know HOW?

Find the sum of each finite arithmetic series.

1. $4 + 7 + 10 + 13 + 16 + 19 + 22$

2. $10 + 20 + 30 + \cdots + 110 + 120$

Write each arithmetic series in summation notation.

3. $3 + 6 + 9 + 12 + 15 + 18 + 21$

4. $1 + 5 + 9 + \cdots + 41 + 45$

Do you UNDERSTAND?

5. Vocabulary What is the difference between an arithmetic sequence and an arithmetic series?

6. Error Analysis A student writes the arithmetic series $3 + 8 + 13 + \ ...\ + 43$ in summation notation as $\sum_{n=3}^{8}(3 + 5n)$. Describe and correct the error.

7. Reasoning Is it possible to have more than one arithmetic series with four terms whose sum is 44? Explain.

Practice and Problem-Solving Exercises

A Practice

Find the sum of each finite arithmetic series. **See Problem 1.**

8. $2 + 4 + 6 + 8$

9. $8 + 9 + 10 + \cdots + 15$

10. $5 + 6 + 7 + \cdots + 11$

11. $1 + 4 + 7 + \cdots + 31$

12. $7 + 14 + 21 + \cdots + 105$

13. $(-3) + (-6) + (-9) + \cdots + (-30)$

14. Grades A student has taken three math tests so far this semester. His scores for the first three tests were 75, 79, and 83. **See Problem 2.**

a. Suppose his test scores continue to improve at the same rate. What will be his grade on the sixth (and final) test?

b. What will be his total score for all six tests?

Write each arithmetic series in summation notation. **See Problem 3.**

15. $4 + 8 + 12 + 16 + 20$

16. $7 + 9 + 11 + \cdots + 21$

17. $5 + 8 + 11 + \cdots + 38$

18. $100 + 90 + 80 + \cdots + 10$

19. $(-3) + (-6) + (-9) + \cdots + (-30)$

20. $105 + 97 + 89 + \cdots + (-71)$

Find the sum of each finite series. **See Problem 4.**

21. $\sum_{n=1}^{5}(2n - 1)$

22. $\sum_{n=1}^{10}(3n - 4)$

23. $\sum_{n=1}^{8}(7 - n)$

24. $\sum_{n=1}^{4}2^n$

25. $\sum_{n=1}^{9}(-1)^n \cdot 2$

26. $\sum_{n=5}^{10}(20 - n)$

Use a graphing calculator to find the sum of each series. **See Problem 5.**

27. $\sum_{n=1}^{50}(2n - 3)$

28. $\sum_{n=1}^{26}(n^2 - 3n)$

29. $\sum_{n=1}^{10}(-2)^n$

30. $\sum_{n=1}^{20}(n^3 - 10n^2)$

31. $\sum_{n=5}^{73}(-4n + 32)$

32. $\sum_{n=5}^{25}(n^2 - 14n + 32)$

B Apply

33. **Think About a Plan** A meeting room is set up with 16 rows of seats. The number of seats in a row increases by two with each successive row. The first row has 12 seats. What is the total number of seats?

- How can you find the number of seats in each row using an explicit formula?
- What is the number of seats in the 16th row?
- How can you find the sum of the seats in 16 rows?

Determine whether each list is a *sequence* or a *series* and *finite* or *infinite*.

34. 1, 2, 4, 8, 16, 32, . . .

35. 1, 0.5, 0.25, 0.125, 0.0625

36. $5 + 10 + \cdots + 25$

37. $-0.5 - 0.25 - 0.125 - \dots$

38. $\frac{4}{3}, \frac{7}{3}, \frac{10}{3}, \frac{13}{3}, \frac{16}{3}, \dots$

39. $2.3 + 4.6 + 9.2 + 18.4$

Each sequence has eight terms. Evaluate each related series.

40. $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots, \frac{15}{2}$

41. $1, -1, -3, \dots, -13$

42. 5, 13, 21, . . . , 61

43. $-3.5, -1.25, 1, \dots, 12.25$

44. $1765, 1414, 1063, \dots, -692$

45. $-13, -14.5, -16, \dots, -23.5$

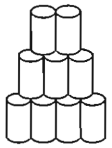
46. **Architecture** In a 20-row theater, the number of seats in a row increases by three with each successive row. The first row has 18 seats.

a. Write an arithmetic series to represent the number of seats in the theater.

b. Find the total seating capacity of the theater.

c. Front-row tickets for a concert cost \$60. After every 5 rows, the ticket price goes down by \$5. What is the total amount of money generated by a full house?

47. **a.** **Grocery** A supermarket displays cans in a triangle. Write an explicit formula for the sequence of the number of cans.



b. Use summation notation to write the related series for a triangle with 10 cans in the bottom row.

c. Suppose the triangle had 17 rows. How many cans would be in the 17th row?

d. **Reasoning** Could the triangle have 110 cans? 140 cans? Explain.

Evaluate each series to the given term.

48. $2 + 4 + 6 + 8 + \dots$; 10th term

49. $-5 - 25 - 45 - \dots$; 9th term

50. **a.** **Open-Ended** Write two explicit formulas for arithmetic sequences.

b. Write the first five terms of each related series.

c. Use summation notation to rewrite each series.

d. Evaluate each series.

C Challenge

Use the values of a_1 and S_n to find the value of a_n.

51. $a_1 = 4$ and $S_{40} = 6080$; a_{40}

52. $a_1 = -6$ and $S_{50} = -5150$; a_{50}

Find a_1 for each arithmetic series.

53. $S_8 = 440$ and $d = 6$

54. $S_{30} = 240$ and $d = -2$

55. Evaluate S_{10} for the series $x + (x + y) + (x + 2y) + \dots$

56. Evaluate S_{15} for the series $3x + (3x - 2y) + (3x - 4y) + \dots$

Sunshine State Standards Practice

MA.912.D.11.2 **57.** Which expression represents the series $14 + 20 + 26 + 32 + 38 + 44 + 50$?

(A) $\sum_{n=2}^{8}(7n - 1)$

(B) $\sum_{n=3}^{9}(6n - 4)$

(C) $\sum_{n=3}^{8}(6n - 4)$

(D) $\sum_{n=8}^{14}(n + 6)$

MA.912.D.11.1 **58.** What is the common ratio in the geometric sequence $\frac{9}{2}, 3, 2, \frac{4}{3}, \dots$?

(F) $\frac{3}{2}$

(G) $\frac{9}{2}$

(H) $\frac{2}{3}$

(I) $\frac{27}{2}$

MA.912.A.6.4 **59.** Which expression is NOT equivalent to $\sqrt[4]{4n^2}$?

(A) $(4n^2)^{\frac{1}{4}}$

(B) $2n^{\frac{1}{2}}$

(C) $(2|n|)^{\frac{1}{2}}$

(D) $\sqrt{2|n|}$

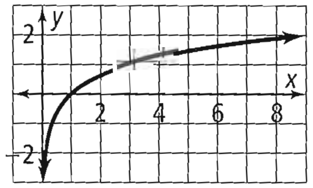
MA.912.A.8.3 **60.** The graph shows the inverse of which function?

(F) $y = 3x$

(G) $y = -3^{2x}$

(H) $y = 3^x$

(I) $y = 2^{3x}$



MA.912.A.7.3 **61. Short Response** Solve the equation $x^2 + 10x + 40 = 5$ by completing the square.

Mixed Review

Write an explicit formula for each geometric sequence. Then find the first three terms. See Lesson 9-3.

62. $a_1 = 1, r = 2$

63. $a_1 = -1, r = -1$

64. $a_1 = 3, r = \frac{3}{2}$

Simplify each rational expression. State any restrictions on the variable. See Lesson 8-4.

65. $\frac{x^2 + 4x + 3}{x^2 - 3x - 4}$

66. $\frac{c^2 - 8c + 12}{c^2 - 11c + 30}$

67. $\frac{3z^4 + 36z^3 + 60z^2}{3z^3 - 3z^2}$

Get Ready! **To prepare for Lesson 9-5, do Exercises 68–70.**

Find the common ratio for each geometric sequence. See Lesson 9-3.

68. 90, −30, 10, . . .

69. 64, 48, 36, . . .

70. −9, 4.5, −2.25, . . .

Concept Byte

For Use With Lesson 9-5

Geometry and Infinite Series

Sunshine State Standard

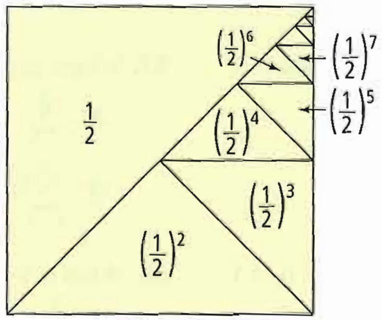
MA.912.D.11.4 Find partial sums of geometric series, and find sums of infinite convergent geometric series.

You can use geometric figures to model some infinite series.

Activity 1

Geometry Draw a geometric figure to model the series.

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n + \ldots$$



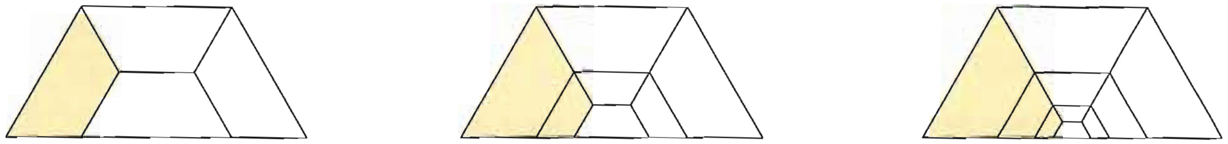
Draw a square. Shade one half of the square. Then shade one half of the remaining unshaded region. Continue until the square is full.

So the series appears to have a sum of 1.

You can write an infinite series from a geometric model.

Activity 2

Geometry Write the series modeled by the trapezoids. Estimate the sum of the series. Explain your reasoning.

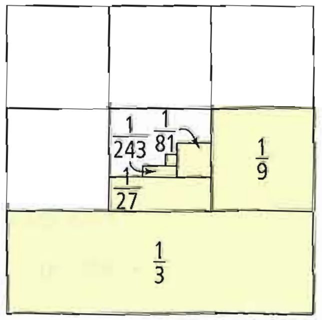


$$\frac{1}{4} \qquad + \qquad \left(\frac{1}{4}\right)^2 \qquad + \qquad \left(\frac{1}{4}\right)^3 + \ldots$$

The shaded region approaches one third of the figure.

So the series $\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots + \left(\frac{1}{4}\right)^n + \ldots$ appears to have a sum of $\frac{1}{3}$.

Exercises



1. **a.** Write the series modeled by the figure at the right.
b. Evaluate the series. Explain your reasoning.

2. Draw a figure to model the series.

$$\frac{1}{5} + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \cdots + \left(\frac{1}{5}\right)^n + \ldots$$

3. **Make a Conjecture** Consider the series.

$$\frac{1}{c} + \left(\frac{1}{c}\right)^2 + \left(\frac{1}{c}\right)^3 + \cdots + \left(\frac{1}{c}\right)^n + \ldots, c > 1$$

What is the sum of the series? Explain your reasoning.

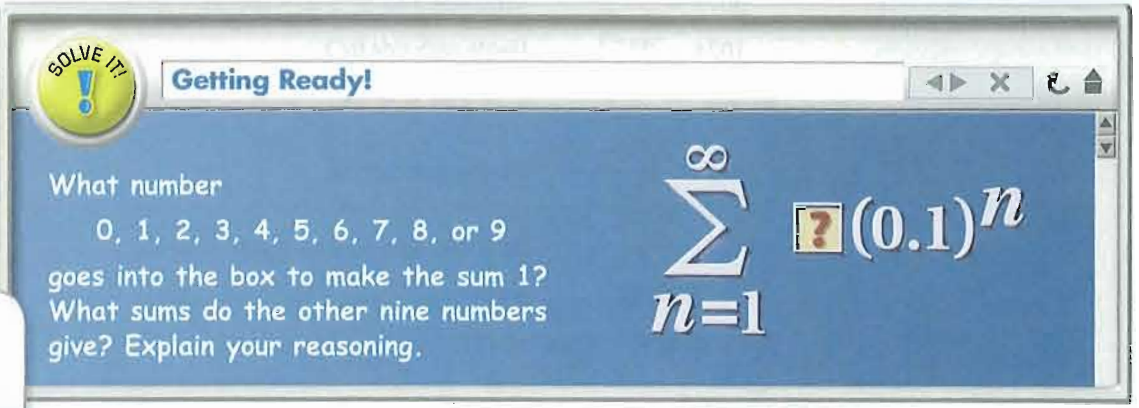
9-5 Geometric Series

Sunshine State Standards

MA.912.D.11.2 Use sigma notation to describe series.

MA.912.D.11.4 Find partial sums of geometric series, and find sums of infinite convergent geometric series. Use sigma notation where applicable.

Objective To define geometric series and find their sums



Getting Ready!

What number

0, 1, 2, 3, 4, 5, 6, 7, 8, or 9

goes into the box to make the sum 1? What sums do the other nine numbers give? Explain your reasoning.

$$\sum_{n=1}^{\infty} \boxed{?}(0.1)^n$$

Lesson Vocabulary
- geometric series
- converge
- diverge

You can write any whole number that has the same digit in every place as the sum of the terms of a geometric sequence. For example,

$$4444 = 4(10)^0 + 4(10)^1 + 4(10)^2 + 4(10)^3$$

You can write any rational number as an infinite repeating decimal. For example, $\frac{47}{90} = 0.5222\ldots$

Therefore, you can write any rational number as a number plus the sum of an infinite geometric sequence.

$$0.5222\ldots = 0.5 + 2(0.1)^2 + 2(0.1)^3 + 2(0.1)^4 + \ldots$$

Essential Understanding Just as with finite arithmetic series, you can find the sum of a finite geometric series using a formula. You need to know the first term, the number of terms, and the common ratio.

A **geometric series** is the sum of the terms of a geometric sequence.

take note

Key Concept Sum of a Finite Geometric Series

The sum S_n of a finite geometric series $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}$, $r \neq 1$, is

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

where a_1 is the first term, r is the common ratio, and n is the number of terms.

Problem 1 Finding the Sums of Finite Geometric Series

What is the sum of the finite geometric series?

Plan

What do you need to find the sum?
You need the first term, the common ratio, and the number of terms in the series.

A $3 + 6 + 12 + 24 + \cdots + 3072$

The first term is 3. The common ratio is $\frac{6}{3} = \frac{12}{6} = \frac{24}{12} = 2$. The nth term is 3072.

$a_n = a_1 r^{n-1}$	Use the explicit formula.
$3072 = 3 \cdot 2^{n-1}$	Substitute 3 for a_1, 2 for r, and 3072 for a_n.
$1024 = 2^{n-1}$	Divide each side by 3.

1024 is 2^{10}, so $n - 1 = 10$ and $n = 11$.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$	Use the sum formula.
$S_{11} = \frac{3(1 - 2^{11})}{1 - 2}$	Substitute 3 for a_1, 2 for r, and 11 for n.
$= 6141$	Simplify.

The sum of the series is 6141.

Think

When the lower limit on n is not 1, how can you find the number of terms?
The number of terms always equals upper limit − lower limit + 1.

B $\sum_{n=0}^{20} 4\left(\frac{1}{2}\right)^n$

The first term is $a_1 = 4\left(\frac{1}{2}\right)^0 = 4$. The common ratio is $r = \frac{1}{2}$. The lower limit is 0 and the upper limit is 20, so the number of terms is $n = 21$.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$	Use the sum formula.
$S_{21} = \frac{4\left(1 - \left(\frac{1}{2}\right)^{21}\right)}{1 - \frac{1}{2}}$	Substitute 4 for a_1, $\frac{1}{2}$ for r, and 21 for n.
≈ 8	Use a calculator.

The sum of the series is approximately 8.

Got It? **1.** What is the sum of the finite geometric series?

a. $-15 + 30 - 60 + 120 - 240 + 480$

b. $\sum_{n=1}^{10} 5 \cdot (-2)^{n-1}$

The Soldier's Reasonable Request A famous story involves a soldier who rescues his king in battle. The king grants him any prize "within reason" from the riches of the kingdom. The soldier asks for a chessboard with a single kernel of wheat on the first square, two kernels of wheat on the second square, then four, then eight, and so on for all 64 squares of the chessboard. The king decides that the request is reasonable.

See Problem 2 for the outcome.

Problem 2 Using the Geometric Series Formula

According to the story on the preceding page, how many total kernels of wheat did the soldier request?

Know	Need	Plan
The amount of wheat in the first 4 squares	The total amount of wheat	Use the sum formula to find the total amount of wheat.

Step 1 Identify the first term, common ratio, and the number of terms.

$a_1 = 1, r = 2, n = 64$

Step 2 Use the sum formula.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$ Write the sum formula.

$$S_{64} = \frac{1(1 - 2^{64})}{1 - 2}$$ Substitute for a_1, r, and n.

$$= 2^{64} - 1 \approx 1.845 \times 10^{19}$$ Simplify.

There will be approximately 1.845×10^{19} kernels of wheat.

Got It? 2. To save money for a vacation, you set aside \$100. For each month thereafter, you plan to set aside 10% more than the previous month. How much money will you save in 12 months?

The Rest of the Story A bushel of wheat contains about a million kernels. The total US output of wheat in a recent year was just over 2.1 billion bushels. How many years of production at that level would it take the United States to produce enough wheat to satisfy the soldier's "reasonable" request?

The terms of a geometric series grow rapidly when the common ratio is greater than 1. Likewise, they diminish rapidly when the common ratio is between 0 and 1. In fact, they diminish so rapidly that an *infinite geometric series* has a finite sum.

take note

Key Concept Infinite Geometric Series

An infinite geometric series with first term a_1 and common ratio $|r| < 1$ has a finite sum

$$S = \frac{a_1}{1 - r}.$$

An infinite geometric series with $|r| \geq 1$ does not have a finite sum.

To say that an infinite series $a_1 + a_2 + a_3 + \dots$, has a sum means that the *sequence of partial sums* $S_1 = a_1, S_2 = a_1 + a_2, S_3 = a_1 + a_2 + a_3, \dots, S_n = a_1 + a_2 + \cdots + a_n, \dots$ **converges** to a number S as n gets very large.

When an infinite series does not converge to a sum, the series **diverges**. An infinite geometric series with $|r| \geq 1$ diverges.

Problem 3 Analyzing Infinite Geometric Series

Does the series *converge* or *diverge*? If it converges, what is the sum?

Think

When does an infinite geometric series converge?
An infinite geometric series converges when the absolute value of the common ratio is less than 1.

A $1 + \frac{1}{2} + \frac{1}{4} + \dots$

$$r = \frac{1}{2} \div 1 = \frac{1}{2}$$

Since $|r| = \left|\frac{1}{2}\right| < 1$, the series converges.

$$S = \frac{a_1}{1 - r} = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

B $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)\left(-\frac{5}{4}\right)^n$

Since $|r| = \left|-\frac{5}{4}\right| = \frac{5}{4} > 1$, the series diverges.

Got It? **3.** Does the infinite series *converge* or *diverge*? If it converges, what is the sum?

a. $\frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \dots$ **b.** $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \dots$ **c.** $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

d. **Reasoning** Will an infinite geometric series either converge or diverge? Explain.

Lesson Check

Do you know HOW?

Evaluate each finite geometric series.

1. $\frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \frac{1}{80}$

2. $9 - 6 + 4 - \frac{8}{3} + \frac{16}{9}$

Determine whether each infinite geometric series *diverges* or *converges*.

3. $1 - \frac{1}{6} + \frac{1}{36} - \frac{1}{216} + \dots$

4. $\frac{1}{64} + \frac{1}{32} + \frac{1}{16} + \dots$

Do you UNDERSTAND?

5. **Error Analysis** A classmate uses the formula for the sum of an infinite geometric series to evaluate $1 + 1.1 + 1.21 + 1.331 + \dots$ and gets -10. What error did your classmate make?

6. **Writing** Explain how you can determine whether an infinite geometric series has a sum.

7. **Compare and Contrast** How are the formulas for the sum of a finite arithmetic series and the sum of a finite geometric series similar? How are they different?

Practice and Problem-Solving Exercises

A Practice

Evaluate the sum of the finite geometric series.

See Problem 1.

8. $1 + 2 + 4 + 8 + \cdots + 128$

9. $4 + 12 + 36 + 108 + \cdots + 972$

10. $3 + 6 + 12 + 24 + 48 + \cdots + 768$

11. $-5 - 10 - 20 - 40 - \cdots - 2560$

12. $\sum_{n=1}^{5} 3^n$

13. $\sum_{n=1}^{4} \left(\frac{1}{2}\right)^{n+1}$

14. $\sum_{n=1}^{4} \left(\frac{2}{3}\right)^{n-1}$

15. $\sum_{n=1}^{5} \left(\frac{1}{3}\right)^{n-1}$

16. Financial Planning In March, a family starts saving for a vacation they are planning for the end of August. The family expects the vacation to cost $1375. They start with $125. Each month they plan to deposit 20% more than the previous month. Will they have enough money for their trip? If not, how much more do they need?

See Problem 2.

Determine whether each infinite geometric series *diverges* or *converges*. If the series converges, state the sum.

See Problem 3.

17. $1 + \frac{1}{4} + \frac{1}{16} + \ldots$

18. $1 - \frac{1}{2} + \frac{1}{4} - \ldots$

19. $4 + 2 + 1 + \ldots$

20. $1 + 2 + 4 + \ldots$

21. $6 + 18 + 54 + \ldots$

22. $-54 - 18 - 6 - \ldots$

23. $1 - 1 + 1 - \ldots$

24. $1 + \frac{1}{5} + \frac{1}{25} + \ldots$

25. $\frac{1}{4} + \frac{1}{2} + 1 + 2 + \ldots$

Evaluate each infinite geometric series.

26. $1.1 + 0.11 + 0.011 + \ldots$

27. $1.1 - 0.11 + 0.011 - \ldots$

28. $1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \ldots$

29. $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

30. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \ldots$

31. $3 - 2 + \frac{4}{3} - \frac{8}{9} + \ldots$

B Apply

Determine whether each series is *arithmetic* or *geometric*. Then evaluate the finite series for the specified number of terms.

32. $2 + 4 + 8 + 16 + \ldots; n = 10$

33. $2 + 4 + 6 + 8 + \ldots; n = 20$

34. $-5 + 25 - 125 + 625 - \ldots; n = 9$

35. $6.4 + 8 + 10 + 12.5 + \ldots; n = 7$

36. $1 + 2 + 3 + 4 + \ldots; n = 1000$

37. $81 + 27 + 9 + 3 + \ldots; n = 200$

38. Think About a Plan The height a ball bounces is less than the height of the previous bounce due to friction. The heights of the bounces form a geometric sequence. Suppose a ball is dropped from one meter and rebounds to 95% of the height of the previous bounce. What is the total distance traveled by the ball when it comes to rest?

- Does the problem give you enough information to solve the problem?
- How can you write the general term of the sequence?
- What formula should you use to calculate the total distance?

39. Communications Many companies use a telephone chain to notify employees of a closing due to bad weather. Suppose a company's CEO (Chief Executive Officer) calls four people. Then each of these people calls four others, and so on.

a. Make a diagram to show the first three stages in the telephone chain. How many calls are made at each stage?

b. Write the series that represents the total number of calls made through the first six stages.

c. How many employees have been notified after stage six?

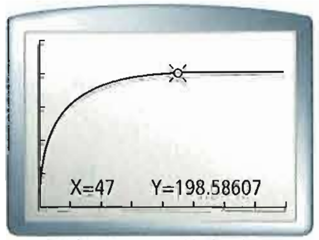
40. Graphing Calculator The graph models the sum of the first n terms in the geometric series with $a_1 = 20$ and $r = 0.9$.

a. Write the first four sums of the series.

b. Use the graph to evaluate the series to the 47th term.

c. Write and evaluate the formula for the sum of the series.

d. Graph the formula using the window values shown. Use the graph to verify your answer to part (b).



Xmin=0 Ymin=0
Xmax=94 Ymax=250
Xscl=10 Yscl=50

Evaluate each infinite series that has a sum.

41. $\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^{n-1}$ **42.** $\sum_{n=1}^{\infty}3\left(\frac{1}{4}\right)^{n-1}$ **43.** $\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^{n-1}$ **44.** $\sum_{n=1}^{\infty}7(2)^{n-1}$ **45.** $\sum_{n=1}^{\infty}(-0.2)^{n-1}$

46. Open-Ended Write an infinite geometric series that converges to 3. Use the formula to evaluate the series.

47. Reasoning Find the specified value for each infinite geometric series.

a. $a_1 = 12$, $S = 96$; find r **b.** $S = 12$, $r = \frac{1}{6}$; find a_1

48. Writing Suppose you are to receive an allowance each week for the next 26 weeks. Would you rather receive (a) \$1000 per week or (b) \$.02 the first week, \$.04 the second week, \$.08 the third week, and so on for the 26 weeks? Justify your answer.

49. The sum of an infinite geometric series is twice its first term.

a. Error Analysis A student says the common ratio of the series is $\frac{3}{2}$. What is the student's error?

b. Find the common ratio of the series.

50. Physics Because of friction and air resistance, each swing of a pendulum is a little shorter than the previous one. The lengths of the swings form a geometric sequence. Suppose the first swing of a pendulum has a length of 100 cm and the return swing is 99 cm.

a. On which swing will the arc first have a length less than 50 cm?

b. What is the total distance traveled by the pendulum when it comes to rest?

51. Where did the formula for summing finite geometric series come from? Suppose the geometric series has first term a_1 and constant ratio r, so that $S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}$.

a. Show that $rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^n$.

b. Use part (a) to show that $S_n - rS_n = a_1 - a_1r^n$.

c. Use part (b) to show that $S_n = \frac{a_1 - a_1r^n}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}$.

Challenge

52. The function $S(n) = \frac{10(1 - 0.8^n)}{0.2}$ represents the sum of the first n terms of an infinite geometric series.

a. What is the domain of the function?

b. Find $S(n)$ for $n = 1, 2, 3, \dots, 10$. Sketch the graph of the function.

c. Find the sum S of the infinite geometric series.

53. Use the formula for the sum of an infinite geometric series to show that $0.\overline{9} = 1$. (*Hint:* $0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} = \dots$)

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.D.11.2 **54.** What is the value of $\sum_{n=1}^{5}(2n - 3)$?

MA.912.D.11.4 **55.** Evaluate the infinite geometric series $\frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \dots$. Enter your answer as a fraction.

MA.912.A.8.2 **56.** Use $\log_5 2 \approx 0.43$ and $\log_5 7 \approx 1.21$ and the properties of logarithms to approximate $\log_5 \sqrt{14}$ without using a calculator.

MA.912.A.8.5 **57.** Use a calculator to solve the equation $7^{2x} = 75$. Round the answer to the nearest hundredth.

MA.912.A.8.6 **58.** Use the Change of Base Formula and a calculator to solve $\log_9 x = \log_6 15$. Round the answer to the nearest tenth.

Mixed Review

Evaluate each series to the given term. See Lesson 9-4.

59. $12.5 + 15 + 17.5 + 20 + 22.5 + \dots$; 7th term

60. $-100 - 95 - 90 - 85 - \dots$; 11th term

Add or subtract. Simplify where possible. See Lesson 8-5.

61. $\frac{7}{2c} - \frac{2}{c^2}$

62. $\frac{5}{y + 3} + \frac{15}{y - 3}$

63. $\frac{4}{x^2 - 36} + \frac{x}{x - 6}$

Use the properties of logarithms to evaluate each expression. See Lesson 7-4.

64. $\log_2 \frac{1}{8} + \log_2 8$

65. $\log_{15} 25 + \log_{15} 9$

66. $3 \log_9 3 - \frac{1}{4} \log_9 81$

Get Ready! **To prepare for Lesson 10-1, do Exercises 67–69.**

Graph each function. See Lesson 4-2.

67. $y = x^2 - 4$

68. $y = x^2 - 6x - 9$

69. $y = -4x^2 + 1$

9 Pull It All Together

BIG idea Variable

You can represent quantities using variables and algebraic expressions.

Task 1

The sum S_n of a finite arithmetic series of n terms is $S_n = \frac{n}{2}(a_1 + a_n)$ where a_1 is the first term and a_n is the nth term.

- **a.** Show that $S_n = na_1 + \frac{n(n-1)}{2}d$ by replacing a_n with its value in terms of a_1, n, and d in the above formula.
- **b.** Explain why $S_n = na_1 + \frac{n(n-1)}{2}d$ makes sense by explaining how you can extract each of na_1 and $\frac{n(n-1)}{2}d$ from the sum $a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-1)d)$.

BIG idea Equivalence

You can represent any *function* in an unlimited number of ways, where all representations have the same domain and the same pairing of inputs with outputs.

Task 2

A sequence is a function with domain the natural numbers 1, 2, 3, ... Using function notation, you can write the sequence $a_n = a_1 + (n - 1)d$ as $a(n) = a_1 + (n - 1)d$ and the sequence $a_n = a_1r^{n-1}$ as $a(n) = a_1r^{n-1}$.

- **a.** Is $a(n) = a_1 + (n - 1)d$ a linear function? Explain. If not, how can you adjust its definition so that it is a linear function? What is the slope?
- **b.** What type of function does $a(n) = a_1r^{n-1}$ suggest? To what family of functions does this function belong? Explain how it is related to the parent function of that family. Draw its graph.
- **c.** What type of function is suggested by the sum sequence $S(n) = \frac{a_1(1 - r^n)}{1 - r}$? By $S(n) = \frac{n}{2}(a_1 + a(n))$? Explain each answer.

BIG idea Modeling

You can represent many real-world mathematical problems algebraically. These representations can lead to algebraic solutions.

Task 3

Each of these sequence or series formulas involves four quantities. For each formula, describe the four quantities. Then explain how you can find the fourth quantity if you know the values of the other three.

- **a.** $a_n = a_1 + (n - 1)d$
- **b.** $S_n = \frac{n}{2}(a_1 + a_n)$
- **c.** $a_n = a_1r^{n-1}$
- **d.** $S_n = \frac{a_1(1 - r^n)}{1 - r}$

9 Chapter Review

Connecting BIG ideas and Answering the Essential Questions

1 Variable

You can define a sequence

- by describing its *n*th term with a formula using *n*.
- by stating its first term and a formula that relates the $n - 1$ and *n*th terms.

2 Equivalence

$a_n = a + (n - 1)d$ and $a_1 = a$, $a_n = a_{n-1} + d$ for $n > 1$ define the same arithmetic sequence, $a, a + d, a + 2d, \ldots$

3 Modeling

You can model a geometric sequence explicitly or recursively. The sum of its first *n* terms is $\frac{a_1(1 - r^n)}{1 - r}$.

Mathematical Patterns and Arithmetic Sequences (Lessons 9-1 and 9-2)

$a_n = 2 - \frac{3}{4}(n - 1)$ and $a_1 = 2$, so $a_n = a_{n-1} - \frac{3}{4}$ represents the arithmetic sequence $2, \frac{5}{4}, \frac{2}{4}, -\frac{1}{4}, -1, \ldots$.

Mathematical Patterns and Geometric Sequences (Lessons 9-1 and 9-3)

$a_n = 2\left(-\frac{1}{2}\right)^{n-1}$ and $a_1 = 2$, so $a_n = \left(-\frac{1}{2}\right)a_{n-1}$ represents the geometric sequence $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$.

Arithmetic Series (Lesson 9-4)

The sum $S_n = a_1 + a_2 + \cdots + a_n$ of an arithmetic series is $S_n = \frac{n}{2}(a_1 + a_n)$. The sum

$$S_6 = 2 + \frac{5}{4} + \frac{2}{4} - \frac{1}{4} - 1 - \frac{7}{4}$$
$$= 3\left(2 - \frac{7}{4}\right) = \frac{3}{4}.$$

Geometric Series (Lesson 9-5)

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ is the sum of the first *n* terms of a geometric series. If the series is infinite with $|r| < 1$, the sum is $S = \frac{a_1}{1 - r}$.

For the geometric series, $a_1 = 2$, $a_n = \left(-\frac{1}{2}\right)a_{n-1}$,

$$S_6 = \frac{2\left(1 - \left(-\frac{1}{2}\right)^6\right)}{1 - \left(-\frac{1}{2}\right)}, \text{ or } \frac{21}{16}, \text{ and}$$

$$S = \frac{2}{1 - \left(-\frac{1}{2}\right)} = \frac{4}{3}.$$

Chapter Vocabulary

- arithmetic mean (p. 574)
- arithmetic sequence (p. 572)
- arithmetic series (p. 587)
- common difference (p. 572)
- common ratio (p. 580)
- converge (p. 598)
- diverge (p. 598)
- explicit formula (p. 565)
- finite series (p. 587)
- geometric mean (p. 583)
- geometric sequence (p. 580)
- geometric series (p. 595)
- infinite series (p. 587)
- limits (p. 589)
- recursive formula (p. 565)
- sequence (p. 564)
- series (p. 587)
- term of a sequence (p. 564)

Choose the vocabulary term that correctly completes each sentence.

1. When you use Σ to write a series, you can use _?_ to indicate how many terms you are adding.

2. An ordered list of terms is a _?_.

3. If an infinite geometric series _?_, then it must have a sum.

4. There is a constant _?_ between consecutive terms in a geometric sequence.

5. A formula that expresses the *n*th term of a sequence in terms of *n* is a(n) _?_.

9-1 Mathematical Patterns

Quick Review

A **sequence** is an ordered list of numbers called **terms.**

A **recursive definition** gives the first term and defines the other terms by relating each term after the first term to the one before it.

An **explicit formula** expresses the nth term in a sequence in terms of n, where n is a positive integer.

Example

A sequence has an explicit formula $a_n = n^2$. What are the first three terms of this sequence?

$a_1 = (1)^2 = 1$ Substitute 1 for n and evaluate.

$a_2 = (2)^2 = 4$ Substitute 2 for n and evaluate.

$a_3 = (3)^2 = 9$ Substitute 3 for n and evaluate.

The first three terms are 1, 4, and 9.

Exercises

Find the first five terms of each sequence.

6. $a_n = -2n + 3$

7. $a_n = -n^2 + 2n$

8. $a_n = 2a_{n-1} - 1$, where $a_1 = 2$

9. $a_n = \frac{1}{2}a_{n-1}$, where $a_1 = 20$

Write a recursive definition for each sequence.

10. 5, 22, 39, 56, . . .

11. −2, 7, 16, 25, . . .

Write an explicit formula for each sequence.

12. 1, 4, 7, 10, . . .

13. 4, 1.5, −1,− 3.5, . . .

9-2 Arithmetic Sequences

Quick Review

In an **arithmetic sequence**, the difference between consecutive terms is constant. This difference is the **common difference.**

For an arithmetic sequence, a is the first term, a_n is the nth term, n is the number of the term, and d is the common difference.

An explicit formula is $a_n = a + (n - 1)d$.

A recursive formula is $a_n = a_{n-1} + d$, with $a_1 = a$.

The **arithmetic mean** of two numbers x and y is the average of the two numbers $\frac{x + y}{2}$.

Example

What is the missing term of the arithmetic sequence 11, ■, 27, . . . ?

$$\text{arithmetic mean} = \frac{11 + 27}{2} = \frac{38}{2} = 19$$

The missing term is 19.

Exercises

Determine whether each sequence is arithmetic. If so, identify the common difference and find the 32nd term of the sequence.

14. 2, 4, 7, 10, . . .

15. 3, 18, 33, 48, . . .

16. 7, 10, 13, 16, . . .

17. 2, 5, 9, 14, . . .

Find the missing term(s) of each arithmetic sequence.

18. 1, ■, 9, . . .

19. 104, ■, 99, . . .

20. −1, ■, 11, . . .

21. −4.6, ■, −5.2, . . .

22. −13, ■, ■, ■, −3, . . .

23. 2, ■, ■, ■, −0.4, . . .

Write an explicit formula for each arithmetic sequence.

24. −2, 7, 16, 25, . . .

25. 62, 59, 56, 53, . . .

9-3 Geometric Sequences

Quick Review

In a **geometric sequence**, the ratio of consecutive terms is constant. This ratio is the **common ratio**.

For a geometric sequence, a is the first term, a_n is the nth term, n is the number of the term, and r is the common ratio.

An explicit formula is $a_n = a \cdot r^{n-1}$.

A recursive formula is $a_n = a_{n-1} \cdot r$, with $a_1 = a$.

The geometric mean of two positive numbers x and y is $\sqrt{xy}$.

Example

What is the sixth term of the geometric sequence that begins 2, 6, 18, . . . ?

$a_1 = 2$ and $r = 6 \div 2 = 3$

$a_6 = 2 \cdot 3^{6-1} = 486$ Substitute 6 for n, 2 for a_1, and 3 for r.

The sixth term is 486.

Exercises

Determine whether each sequence is geometric. If so, identify the common ratio and find the next two terms.

26. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

27. 1, 3, 5, 7, . . .

28. 3, 3.6, 4.32, 5.184, ...

Find the missing term(s) of each geometric sequence.

29. 3, ■, 12, . . .

30. 0.004, ■, 0.4, . . .

31. −20, ■, ■, ■, −1.25, . . .

Write an explicit formula for each geometric sequence.

32. 1, 2, 4, 8, . . .

33. $25, 5, 1, \frac{1}{5}, \ldots$

Use an explicit formula to find the 10th term of each geometric sequence.

34. 5, 10, 20, 40, . . .

35. −3, 6, −12, 24, . . .

9-4 Arithmetic Series

Quick Review

A **series** is the expression for the sum of the terms of a sequence.

An **arithmetic series** is the sum of the terms of an arithmetic sequence. The sum S_n of the first n terms of an arithmetic series is $S_n = \frac{n}{2}(a_1 + a_n)$. You can use a summation symbol, Σ, and lower and upper **limits** to write a series. The lower limit is the least value of n and the upper limit is the greatest value of n.

Example

What is the sum of the arithmetic series?

$2 + 5 + 8 + 11 + 14 + 17 + 20$

$a_1 = 2$, $a_7 = 20$, and $n = 7$.

$S_7 = \frac{7}{2}(2 + 20)$ Substitute 7 for n, 2 for a_1, and 20 for a_7.

$= 77$ Evaluate.

Exercises

Use summation notation to write each arithmetic series for the specified number of terms. Then evaluate the sum.

36. $10 + 7 + 4 + \ldots ; n = 5$

37. $50 + 55 + 60 + \ldots ; n = 7$

38. $6 + 7.4 + 8.8 + \ldots ; n = 11$

39. $21 + 19 + 17 + \ldots ; n = 8$

Find the number of terms in each series, the first term, and the last term. Then evaluate the sum.

40. $\sum_{n=1}^{3}(17n - 25)$

41. $\sum_{n=2}^{10}\left(\frac{1}{2}n + 3\right)$

9-5 Geometric Series

Quick Review

A **geometric series** is the sum of the terms of a geometric sequence. The sum S_n of the first n terms of a geometric series is $S_n = \frac{a_1(1 - r^n)}{1 - r}$, $r \neq 1$.

When an infinite series has a finite sum, the series **converges**. When the series does not converge, the series **diverges**.

In a geometric series, when $|r| < 1$, the series converges to $S = \frac{a_1}{1 - r}$. When $|r| \geq 1$, the series diverges.

Example

What is the sum of the geometric series?

$5 + 10 + 20 + 40 + 80 + 160$

$n = 6, a_1 = 5$, and $r = 10 \div 5 = 2$.

$S_6 = \frac{5(1 - 2^6)}{1 - 2}$ Substitute 6 for n, 5 for a_1, and 2 for r.

$= 315$ Evaluate.

The sum is 315.

Exercises

Evaluate each finite series for the specified number of terms.

42. $1 + 2 + 4 + \dots ; n = 5$

43. $80 - 40 + 20 - \dots ; n = 8$

44. $12 + 2 + \frac{1}{3} + \dots ; n = 4$

Determine whether each infinite geometric series *converges* or *diverges*. If the series converges, state the sum.

45. $150 + 30 + 6 + \dots$

46. $2.2 + 2.42 + 2.662 + \dots$

47. $-10 - 20 - 40 - \dots$

48. $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots$

9 Chapter Test

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

Write a recursive definition and an explicit formula for each sequence. Then find a_{12}.

1. 7, 13, 19, 25, 31, . . .

2. 10, 20, 40, 80, 160, . . .

Determine whether each sequence is *arithmetic, geometric*, or *neither*. Then find the tenth term.

3. 23, 27, 31, 35, 39, . . .

4. −12, −5, 2, 9, 16, . . .

5. −5, 15, −45, 135, −405, . . .

6. $\frac{1}{4}$, 1, 4, 16, . . .

Find the missing term of each arithmetic sequence.

7. 4, ■, 12, . . .

8. −11, ■, 23, . . .

Determine whether each sequence is *arithmetic* or *geometric*. Then identify the common difference or common ratio.

9. 1620, 540, 180, 60, 20, . . .

10. 78, 75, 72, 69, 66, 63, 60, . . .

11. $\frac{3}{32}, \frac{3}{16}, \frac{3}{8}, \frac{3}{4}, \frac{3}{2}$, 3, 6, . . .

a_1 is the first term of a sequence, r is a common ratio, and d is a common difference. Write the first five terms.

12. $a_1 = 2, r = -2$

13. $a_1 = 3, d = 7$

14. $a_1 = -100, r = \frac{1}{5}$

15. $a_1 = 19, d = -4$

Find the missing term of each geometric sequence.

16. 2, ■, 0.5, . . .

17. 2, ■, 8, . . .

Find the sum of each infinite geometric series.

18. $0.5 + 0.05 + 0.005 + \dots$

19. $1 - \frac{1}{2} + \frac{1}{4} - \dots$

20. $6 + 5 + \frac{25}{6} + \dots$

Determine whether each series is *arithmetic* or *geometric*. Then evaluate the finite series for the specified term number.

21. $2 + 7 + 12 + \dots; n = 8$

22. $5000 + 1000 + 200 + \dots; n = 5$

23. $1 + 0.01 - 0.98 - \dots; n = 5$

24. $2 + 6 + 18 + \dots; n = 6$

Find the sum of each series.

25. $\sum_{n=1}^{5}(3n + 1)$

26. $\sum_{n=1}^{8}\frac{2n}{3}$

27. $\sum_{n=4}^{10}(0.8n - 0.4)$

28. $\sum_{n=2}^{6}(-2)^{n-1}$

Do you UNDERSTAND?

29. You have saved \$50. Each month you add \$10 more to your savings.

- **a.** Write an explicit formula to model the amount you have saved after n months.
- **b.** How much have you saved after six months?

30. Open-Ended Write an arithmetic sequence. Then write an explicit formula for it.

31. Reasoning How can you tell if a geometric series converges or diverges? Include examples of both types of series. Evaluate the series that converges.

32. A diamond is purchased for \$2500. Suppose its value increases 5% each year.

- **a.** What is the value of diamond after 8 years?
- **b. Writing** Explain how you can write an explicit formula for a geometric sequence to answer the question.

Sunshine State Standards Practice

TIPS FOR SUCCESS

Read the question at the right. Then follow the tips to answer the sample question.

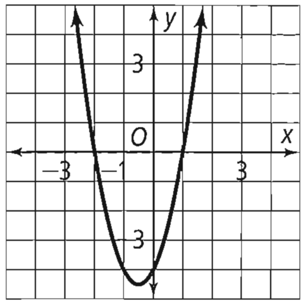
TIP 1

Find where y = 0. These are the x-intercepts of the graph.

TIP 2

Check your solutions in the original equation.

The graph below shows the quadratic function $y = 2x^2 + 2x - 4$. Use the graph to find the solutions of $2x^2 + 2x - 4 = 0$.



Ⓐ −4 and 0 Ⓒ −1 and 2

Ⓑ −2 and 1 Ⓓ 2 and −1

Think It Through

The x-intercepts of the graph are at $x = -2$ and $x = 1$.

Substitute to verify your answers.

$$\begin{aligned} 2(-2)^2 + 2(-2) - 4 &= 2(4) - 4 - 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} 2(1)^2 + 2(1) - 4 &= 2 + 2 - 4 \\ &= 0 \end{aligned}$$

The correct answer is B.

Vocabulary Builder

As you solve test items, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. recursive formula

B. limit

C. explicit formula

D. sequence

I. a formula that expresses the nth term in terms of n

II. an ordered list of numbers

III. the least or greatest integer value of n in a series

IV. a formula that gives the first term in a sequence and defines the other terms by relating each term to the one before it

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

1. What is the solution set of the equation $(2x - 4)(x + 6) = 0$?

Ⓐ $\{-4, 6\}$ Ⓒ $\{2, -6\}$

Ⓑ $\{-4, 6\}$ Ⓓ $\{-2, -6\}$

2. What are the first five terms of the sequence $a_n = 2n - 1$?

Ⓕ 0, 1, 2, 3, 4 Ⓗ 2, 4, 6, 8, 10

Ⓖ 1, 3, 5, 7, 9 Ⓘ 3, 5, 7, 9, 11

3. What is the common ratio in a geometric series if $a_2 = \frac{2}{5}$ and $a_5 = \frac{16}{135}$?

Ⓐ $\frac{2}{5}$ Ⓒ $\frac{6}{65}$

Ⓑ $\frac{2}{3}$ Ⓓ $\frac{8}{27}$

4. The total area of a sheet of paper can be represented by $27x^3 + 64y^3$. Which factors could represent the length times the width?

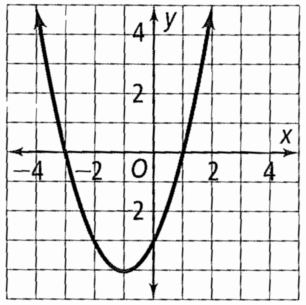
Ⓕ $(3x + 4y)(3x^2 + 4y^2)$

Ⓖ $(3x + 4y)(9x^2 - 12xy + 16y^2)$

Ⓗ $(3x - 4y)(9x^2 - 12xy + 16y^2)$

Ⓘ $(3x + 4y)(3x^2 - 3xy + 4y^2)$

5. The graph below shows the quadratic function $y = x^2 + 2x - 3$. Use the graph to find all the solutions of $x^2 + 2x - 3 = 0$.



Ⓐ −3 Ⓒ −3 and 1

Ⓑ −1 Ⓓ 3 and −1

6. The table shows ordered pairs that satisfy the equation $y = -x^2 + 2x + 15$.

x	−3	0	2	3	5
y	0	15	15	12	0

Based on this table, what is the solution set of the equation $-x^2 + 2x + 15 = 0$?

Ⓕ {0, 15} Ⓗ {5, 0}

Ⓖ {−3, 15} Ⓘ {−3, 5}

7. What is the product of $\frac{x^2 + 5x + 4}{(x - 1)(x + 1)}$ and $\frac{x^2 - 5x + 6}{x - 2}$?

Ⓐ $\frac{x^2 + 7x + 12}{x - 1}$ for $x \neq 1$

Ⓑ $\frac{x^2 + x - 12}{x - 1}$ for $x \neq -1, 1,$ or 2

Ⓒ $\frac{x^2 + x - 12}{x - 1}$ for $x \neq 1$

Ⓓ $\frac{x^2 + 7x + 12}{x - 1}$ for $x \neq -1, 1,$ or 2

8. What is the sum of the infinite geometric series $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$?

Ⓕ $\frac{1}{4}$ Ⓗ $\frac{1}{2}$

Ⓖ $\frac{1}{3}$ Ⓘ 3

9. If $f(x) = 4x^4 - 9$ and $g(x) = 2x^2 + 3$, what is $\left(\frac{f}{g}\right)(x)$?

Ⓐ $2x^2 - 3$

Ⓑ $2x + 3$

Ⓒ $2x - 3$

Ⓓ $2x^2 + 3$

10. Marisol wants to start saving money for college. She sees the advertisement below in her local newspaper.

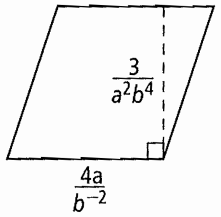
Start a savings account today and earn
4.7% annual interest!
Bonus Offer: After one year, an additional
$50 will be added to your account!

For x dollars in this savings account, $I(x) = 1.047x$ is the value of the account after one year. $B(x) = x + 50$ is the value of the account after the one-year bonus. $(B \circ I)(x)$ models the value of this account after one year of investment time and the one-year bonus. Marisol opens a savings account by depositing \$120. What is the value of the account after one year?

Ⓕ \$175.64 Ⓗ \$226.40

Ⓖ \$182.58 Ⓘ \$249.90

11. What is the area of the parallelogram below?



Ⓐ $\frac{12}{ab^2}$ Ⓒ $\frac{12b^2}{a}$

Ⓑ $\frac{3}{ab^2}$ Ⓓ $\frac{3b^5}{4a^3}$

12. If $\log 5 \approx 0.69897$ and $\log 6 \approx 0.77815$, what is the approximate value of $\log 150$?

Ⓕ 0.34188

Ⓖ 0.38017

Ⓗ 1.20412

Ⓘ 2.17609

13. What is $\frac{4x^2 - 1}{2x^2 - 5x - 3} \cdot \frac{x^2 - 6x + 9}{2x^2 + 5x - 3}$?

Ⓐ 1

Ⓑ $x + 3$

Ⓒ $x - 3$

Ⓓ $\left(\frac{x - 3}{x + 3}\right)$

14. Which is the factored form of $0.81p^2 - 0.09$?

Ⓕ $(0.9p + 0.045)(0.9p - 0.045)$

Ⓖ $(0.9p + 0.3)(0.9p - 0.3)$

Ⓗ $(0.9p + 0.03)(0.9p - 0.03)$

Ⓘ $(0.9p + 0.81)(0.9p - 0.81)$

15. Which arithmetic sequence does NOT include the term 33?

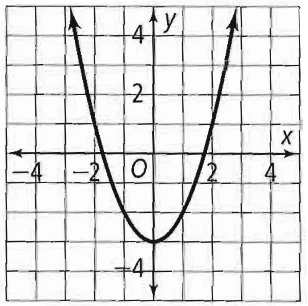
Ⓐ 1, 5, 9, 13, . . .

Ⓑ 3, 9, 15, . . .

Ⓒ 1, 11, 21, . . .

Ⓓ 85, 72, 59, . . .

16. The graph shows a transformation of which parent function?



Ⓕ $y = x$

Ⓖ $y = x^2$

Ⓗ $y = |x|$

Ⓘ $y = \frac{1}{x}$

17. Which is an equation of the line that passes through the points (3, 5) and (7, 1)?

Ⓐ $y - 5 = x - 3$

Ⓑ $y - 5 = x - 1$

Ⓒ $y - 5 = -(x - 3)$

Ⓓ $y - 5 = -(x - 1)$

18. Which is the inverse of $y = \sqrt{x - 5}$?

Ⓕ $y = \sqrt{x + 5}$

Ⓖ $y = x^2 + 5$

Ⓗ $x = \sqrt{y + 5}$

Ⓘ $y = (x + 5)^2$

19. Which expression is NOT equivalent to $\log 45$?

Ⓐ $\log 5 + 2 \log 3$

Ⓑ $\log 90 - \log 45$

Ⓒ $\log 5 + \log 9$

Ⓓ $\log 90 - \log 2$

GRIDDED RESPONSE

20. What is the sum of the following finite geometric series?

$1 + 4 + 16 + 64 + 256$

21. What is the sum of the following finite arithmetic series?

$14 + 20 + 26 + 32 + 38 + 44 + 50$

22. What is the value of $\log_4 256$?

23. Rita works a part-time job at a clothing store and earns \$7 per hour. Juan works at another clothing store and earns \$6 per hour plus a 10% commission on sales. How many sales, in dollars, would Juan have to make in two hours to earn the same amount as Rita in a two-hour shift?

24. Let $f(x) = x + 1$ and $g(x) = x^2$. What is $(g \circ f)(2)$?

25. What is the slope of a line perpendicular to the line $y = -\frac{1}{2}x + 5$?

26. The 30th term of a finite arithmetic series is 4.4. The sum of the first 30 terms is 78. What is the first term of the series?

27. What is the absolute value of $-3 + 4i$?

28. In a geometric sequence, $a_1 = 2$ and $a_5 = 162$. What is a_3?

29. What is a real solution of $27x^3 + 8$? Express your answer as a fraction.

CHAPTER 10

Get Ready!

Lesson 4-1

Graphing Quadratic Functions

Graph each function.

1. $y = -x^2$

2. $y = \frac{1}{3}x^2$

3. $y = 2x^2 + 5$

4. $y = x^2 + 6x + 8$

Lesson 4-3

Identifying Quadratic Functions

Determine whether each function is *linear* or *quadratic*. Identify the quadratic, linear, and constant terms.

5. $y = 6x - x^2 + 1$

6. $f(x) = -2(3 + x)^2 + 2x^2$

7. $y = 2x - y - 13$

8. $y = 4x(7 - 2x)$

9. $g(x) = -2x^2 - 3(x - 2)$

10. $y = x - 2(x + 5)$

Lesson 4-6

Completing the Square

Complete the square.

11. $x^2 + 8x + \blacksquare$

12. $x^2 - 5x + \blacksquare$

13. $x^2 + 14x + \blacksquare$

Rewrite each equation in vertex form. Then graph the function.

14. $y = x^2 + 6x + 7$

15. $y = 2x^2 - 4x + 10$

16. $y = -3x^2 + x$

Lesson 4-1

Graphing Quadratic Functions in Vertex Form

Graph each function.

17. $y = 2(x - 3)^2 + 1$

18. $y = -1(x + 7)^2 - 4$

Lesson 2-7

Graphing Absolute Value Functions

Graph each function.

19. $y = 2|x|$

20. $y = |x| + 2$

Looking Ahead Vocabulary

21. The word *radius* is a Latin word for the spoke of a wheel. It is also the source of the word "radio" because electromagnetic rays radiate from a radio in every direction. Why do you think mathematicians use the term radius to label any line segment from the center of a circle to any point on the circle?

22. In geometry, you learned that a *vertex* is typically a corner or point where two lines intersect. The four corners of a square are called vertices. Using this information, what can you conclude about the vertex of a parabola?

CHAPTER

10

Quadratic Relations and Conic Sections

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VIDEO

Math definitions in English and Spanish

VOCABULARY

The online Solve It will get you in gear for each lesson.

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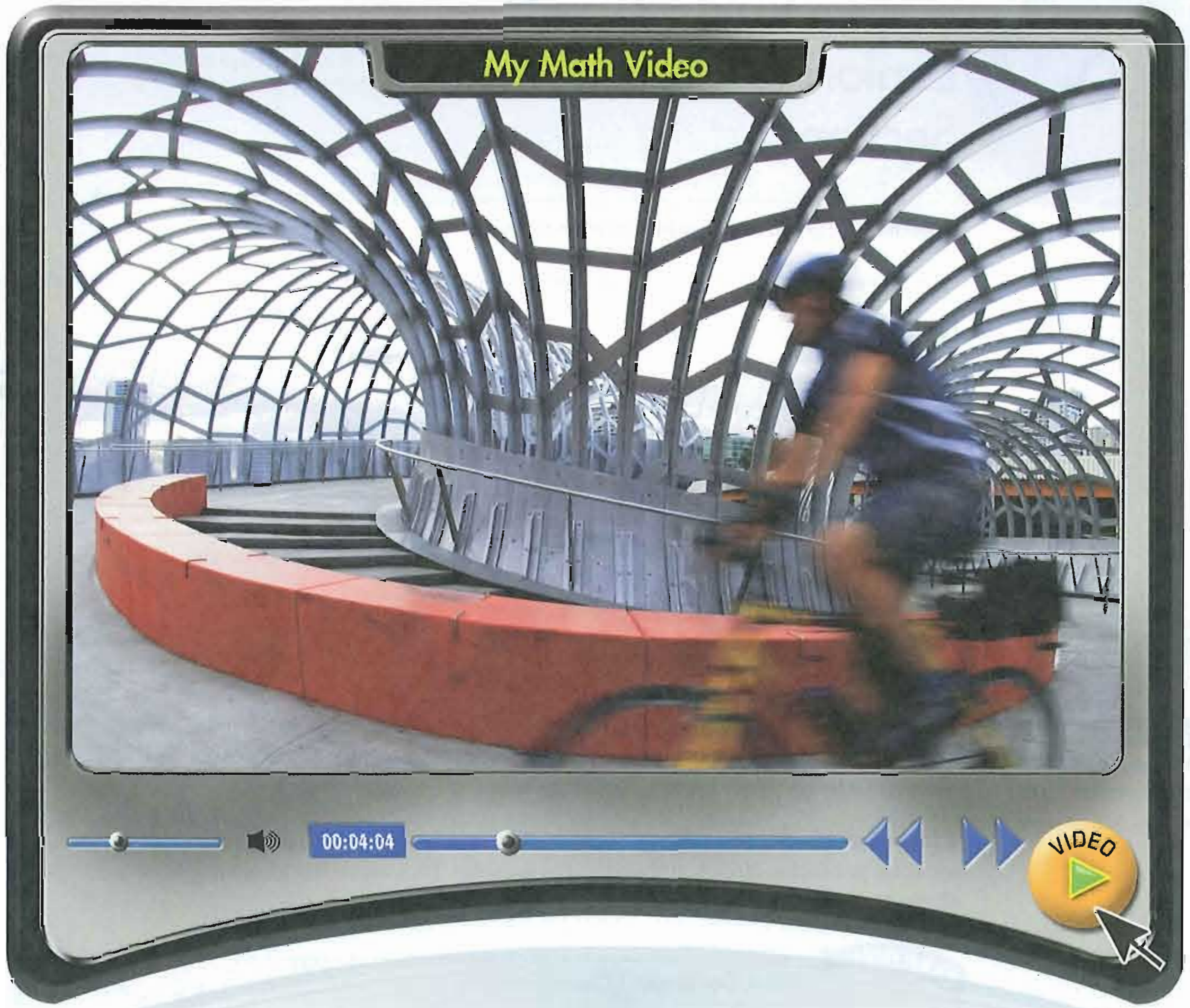
I'm going to help you learn how to work with curves that you can trace along the surface of a cone. These curves are called conic sections.

People can manufacture these curves to make beautiful architecture, as shown on the next page.

Vocabulary

English/Spanish Vocabulary Audio Online:

English	Spanish
center of a circle, *p. 630*	centro de un círculo
circle, *p. 630*	círculo
conic section, *p. 614*	sección cónica
directrix, *p. 622*	directriz
ellipse, *p. 638*	elipse
hyperbola, *p. 645*	hipérbola
radius, *p. 630*	radio
standard form of an equation of a circle, *p. 630*	forma normal de la ecuación de un círculo



BIG ideas

1 **Modeling**

Essential Question What is the intersection of a cone and a plane parallel to a line along the side of the cone?

2 **Equivalence**

Essential Question What is the graph of $\frac{x^2}{9} + \frac{y^2}{9} = 1$?

3 **Coordinate Geometry**

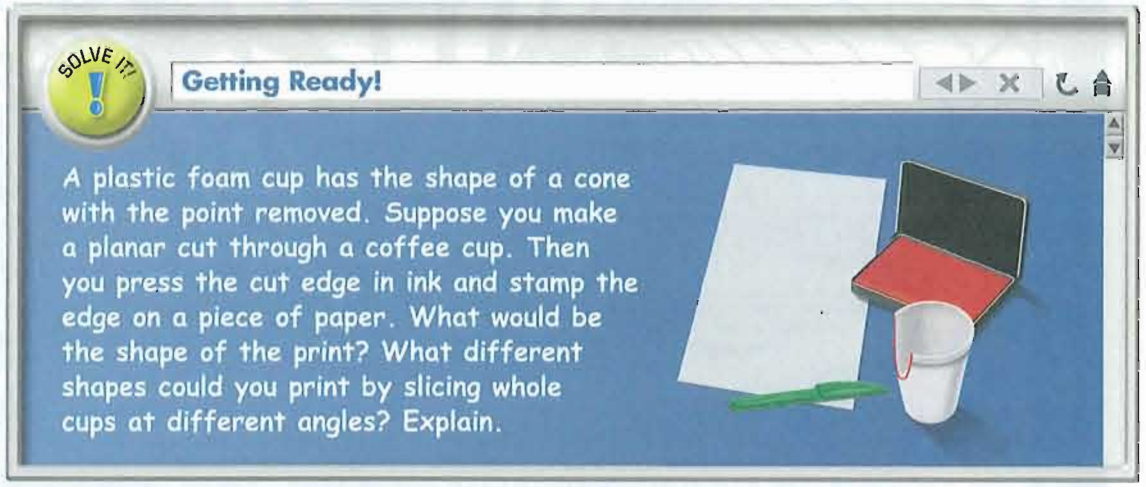
Essential Question What is the difference between the algebraic representations of ellipses and hyperbolas?

Chapter Preview

10-1 Exploring Conic Sections

Sunshine State Standard
MA.912.A.9.2 Graph conic sections with and without using graphing technology.

Objective To graph and identify conic sections



A plastic foam cup has the shape of a cone with the point removed. Suppose you make a planar cut through a coffee cup. Then you press the cut edge in ink and stamp the edge on a piece of paper. What would be the shape of the print? What different shapes could you print by slicing whole cups at different angles? Explain.

Lesson Vocabulary
- conic section

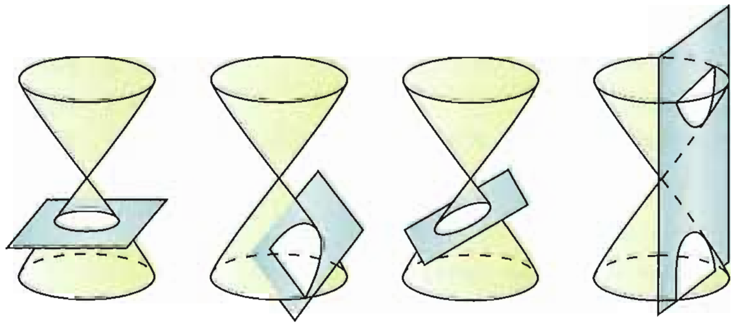
In Chapter 4, you studied parabolas. Geometrically, a parabola has the shape of a cross section of a cone that you cut in a particular way. Parabolas form a family of curves that belong to a larger family known as *conic sections*.

Essential Understanding There are four types of curves known as conic sections: parabolas, circles, ellipses, and hyperbolas. Each curve has its own distinct shape and properties.

take note

Key Concept Conic Sections

A **conic section** is a curve you get by intersecting a plane and a double cone. By changing the inclination of the plane, you can get a circle, a parabola, an ellipse, or a hyperbola.



You can use lines of symmetry to graph a conic section.

Problem 1 Graphing a Circle

What is the graph of $x^2 + y^2 = 25$? What are its lines of symmetry? What are the domain and range?

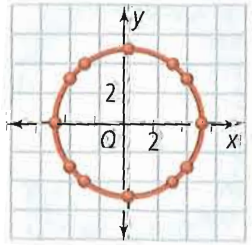
Know	Need	Plan
An equation	The lines of symmetry of the graph, the domain and range of the relation	• Plot points and connect them with a smooth curve. • Look for lines of symmetry on the graph. • Determine the domain and range.

Think

Can you find values of x and y that satisfy the equation?
Yes; find the x- and y-intercepts. Then look for other values of x and y that make the calculations easy.

Make a table of values. Plot the points and connect them with a smooth curve.

x	−5	−4	−3	0	3	4	5
y	0	±3	±4	±5	±4	±3	0



The graph is a circle with radius 5. Its center is the origin. Every line through the center is a line of symmetry.

The domain is the set of real numbers x with $-5 \leq x \leq 5$. The range is the set of real numbers y with $-5 \leq y \leq 5$.

Got It? **1. a.** What is the graph of $x^2 + y^2 = 9$? What are its lines of symmetry? What are the domain and range?

b. **Reasoning** In Problem 1, why is there no point on the graph with x-coordinate 6?

Its many lines of symmetry make a circle a special kind of an *ellipse*. In general, an ellipse has only two lines of symmetry.

Problem 2 Graphing an Ellipse

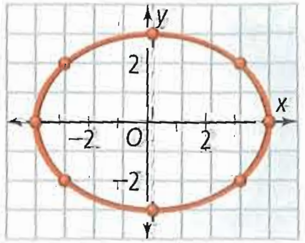
What is the graph of $9x^2 + 16y^2 = 144$? What are its lines of symmetry? What are the domain and range?

Think

What values should you substitute for x?
Substitute both positive and negative values for x.

Make a table of values. Plot the points and connect them with a smooth curve.

x	−4	−3	0	3	4
y	0	±2	±3	±2	0



The graph is an ellipse. The center is the origin. The ellipse has two lines of symmetry, the x-axis and the y-axis.

The domain is the set of real numbers x with $-4 \leq x \leq 4$. The range is the set of real numbers y with $-3 \leq y \leq 3$.

Got It? **2.** What is the graph of $2x^2 + y^2 = 18$? What are its lines of symmetry? What are the domain and range?

Not all conic sections consist of one smooth curve. The hyperbola consists of two separate curves called branches.

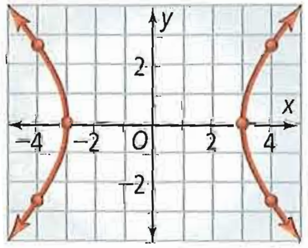
Problem 3 Graphing a Hyperbola

What is the graph of $x^2 - y^2 = 9$? What are its lines of symmetry? What are the domain and range?

Make a table of values.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y	±4	±2.6	0	—	—	—	—	—	0	±2.6	±4

Plot the points and connect them with smooth curves.



The graph is a hyperbola that consists of two branches. Its center is the origin. It has two lines of symmetry, the x-axis and the y-axis.

The domain is the set of real numbers x with $x \leq -3$ or $x \geq 3$. The range is the set of real numbers.

Think

How will you know which points to connect?
Plot enough points so you see a pattern. Only connect points if you know that the points between them satisfy the equation.

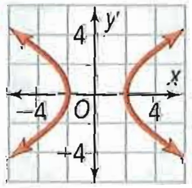
Got It? 3. What is the graph of $x^2 - y^2 = 16$? What are its lines of symmetry? What are the domain and range?

In this chapter, there is a separate lesson for each of the conic sections. You should already be able to identify each curve by its shape and features such as the vertex of a parabola, the center of an ellipse, circle, or hyperbola, and the intercepts of each curve.

Problem 4 Identifying Graphs of Conic Sections

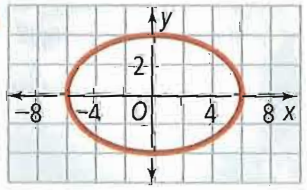
What are the center and intercepts of each conic section? What are the domain and range?

A



The center of the hyperbola is (0, 0). The x-intercepts are (−2, 0) and (2, 0). There are no y-intercepts. The domain is the set of real numbers x with $x \leq -2$ or $x \geq 2$. The range is the set of real numbers.

B

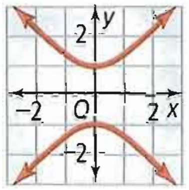


The center of the ellipse is (0, 0). The x-intercepts are (−6, 0) and (6, 0). The y-intercepts are (0, −4) and (0, 4). The domain is the set of real numbers x with $-6 \leq x \leq 6$. The range is the set of real numbers y with $-4 \leq y \leq 4$.

Think

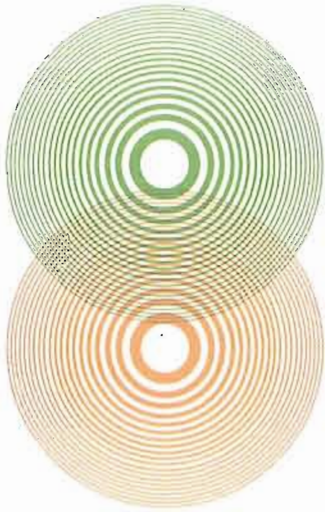
What do you observe from the graph of the hyperbola?
The graph of this hyperbola extends forever but has no x-values between −2 and 2.

Got It? **4.** What are the center and intercepts of the conic section? What are the domain and range?



Problem 5 Using Models

Design **Two patterns, such as arrays of dots or lines, can overlap to form moiré patterns. In the diagram, what pattern do the intersecting ripples form? Which of these equations is a possible model for the pattern: $x^2 - y^2 = 1$, $x^2 + y^2 = 16$, or $25x^2 + 9y^2 = 225$?**



Think

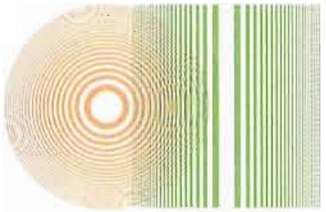
How can you identify a possible model?
Find the type of conic section represented by each equation.

The intersecting ripples form a circle.

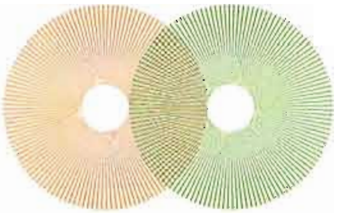
The equation $x^2 + y^2 = 16$ represents a conic section with two sets of intercepts, $(\pm 4, 0)$ and $(0, \pm 4)$. Each intercept is 4 units from the center. The equation models a circle.

Got It? **5.** Unintended moiré patterns cause problems for printers. Describe the unintended pattern. Which equation in Problem 5 is a possible model for each pattern?

a.



b.



Lesson Check

Do you know HOW?

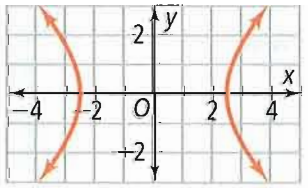
Graph each equation. Find the lines of symmetry, the domain, and the range.

1. $x^2 + 4y^2 = 36$

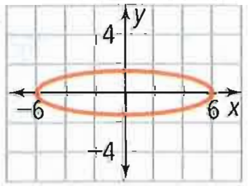
2. $4x^2 - 9y^2 = 36$

Identify the domain and range.

3.



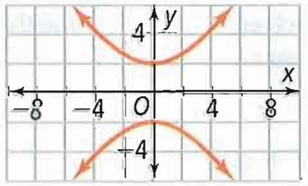
4.



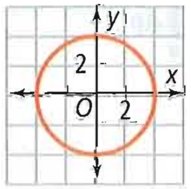
Do you UNDERSTAND?

5. **Vocabulary** Identify the type of conic section graphed.

a.



b.



6. **Compare and Contrast** How is the domain of an ellipse different from the domain of a hyperbola?

Practice and Problem-Solving Exercises

A Practice

Graph each equation. Identify the conic section and describe the graph and its lines of symmetry. Then find the domain and range.

See Problems 1, 2, and 3.

7. $3y^2 - x^2 = 25$

8. $2x^2 + y^2 = 36$

9. $x^2 + y^2 = 16$

10. $3y^2 - x^2 = 9$

11. $4x^2 + 25y^2 = 100$

12. $x^2 + y^2 = 49$

13. $x^2 - y^2 + 1 = 0$

14. $x^2 - 2y^2 = 4$

15. $6x^2 + 6y^2 = 600$

16. $x^2 + y^2 - 4 = 0$

17. $6x^2 + 24y^2 - 96 = 0$

18. $4x^2 + 4y^2 - 20 = 0$

19. $x^2 + 9y^2 = 1$

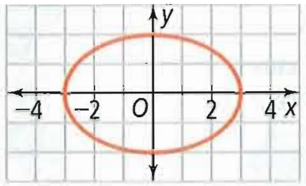
20. $4x^2 - 36y^2 = 144$

21. $4y^2 - 36x^2 = 1$

Identify the conic section. Then give the center, intercepts, domain, and range of each graph.

See Problem 4.

22.



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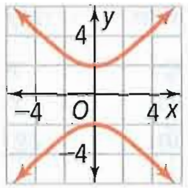

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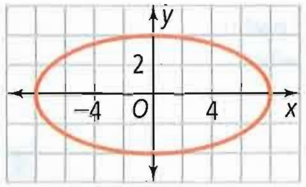

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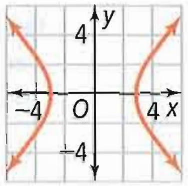

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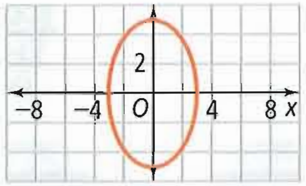

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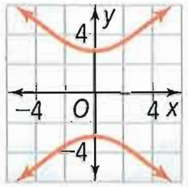


Match each equation with a graph in Exercises 22–27.

See Problem 5.

28. $x^2 - y^2 = 9$

29. $4x^2 + 9y^2 = 36$

30. $y^2 - x^2 = 4$

31. $x^2 + 4y^2 = 64$

32. $25x^2 + 9y^2 = 225$

33. $y^2 - x^2 = 9$

B Apply

Graph each equation. Describe the graph and its lines of symmetry. Then find the domain and range.

34. $9x^2 - y^2 = 144$

35. $11x^2 + 11y^2 = 44$

36. $-8x^2 + 32y^2 - 128 = 0$

37. $25x^2 + 16y^2 - 320 = 0$

38. Think About a Plan The light emitted from a lamp with a shade forms a shadow on the wall. How can you turn the lamp in relation to the wall so that the shadow cast by the shade forms a parabola and a circle?

- How can a drawing or model help you solve this problem?
- Can you form a hyperbola and an ellipse? If so, explain how.

39. a. Writing Describe the relationship between the center of a circle and the axes of symmetry of the circle.

b. Make a Conjecture Where is the center of an ellipse or a hyperbola located in relation to the axes of symmetry? Verify your conjecture with examples.

Graph each circle with the given radius or diameter so that the center is at the origin. Then write the equation for each graph.

40. radius 6

41. radius $\frac{1}{2}$

42. diameter 8

43. diameter 2.5

Mental Math Each given point is on the graph of the given equation. Use symmetry to find at least one more point on the graph.

44. $(2, -4), y^2 = 8x$

45. $(-\sqrt{2}, 1), x^2 + y^2 = 3$

46. $(2, 2\sqrt{2}), x^2 + 4y^2 = 36$

47. $(-2, 0), 9x^2 + 9y^2 - 36 = 0$

48. $(-3, -\sqrt{51}), 6y^2 - 9x^2 - 225 = 0$

49. $(0, \sqrt{7}), x^2 + 2y^2 = 14$

50. Sound An airplane flying faster than the speed of sound creates a cone-shaped pressure disturbance in the air. This is heard by people on the ground as a sonic boom. What is the shape of the path on the ground?

51. Open-Ended Describe any other figures you can see that can be formed by the intersection of a plane and another shape, such as a sphere.

Challenge

52. a. Graph the equation $xy = 16$. Use both positive and negative values for x.

b. Which conic section does the equation appear to model?

c. Identify any intercepts and lines of symmetry.

d. Does your graph represent a function? If so, rewrite the equation using function notation.

53. Graphing Calculator An xy-term has an interesting effect on the graph of a conic section. Sketch the graph of each conic section below using your graphing calculator. (*Hint:* To solve for y, you will need to complete a square.)

a. $4x^2 + 2xy + y^2 = 9$

b. $4x^2 + 2xy - y^2 = 9$

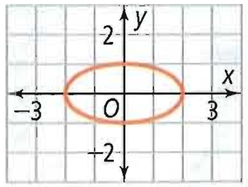
Sunshine State Standards Practice

MA.912.A.5.2 **54.** Which expression can be simplified to $\frac{x-1}{x-3}$?

Ⓐ $\frac{x^2 - x - 6}{x^2 - x - 2}$ Ⓑ $\frac{x^2 - 2x + 1}{x^2 + 2x - 3}$ Ⓒ $\frac{x^2 - 3x - 4}{x^2 - 7x + 12}$ Ⓓ $\frac{x^2 - 4x + 3}{x^2 - 6x + 9}$

MA.912.A.9.2 **55.** Which is the graph of $4x^2 - y^2 = 4$?

Ⓕ



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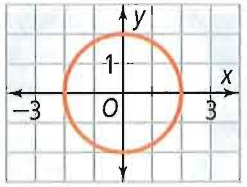

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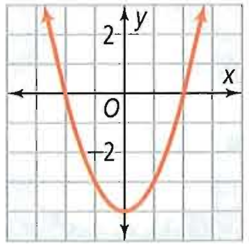

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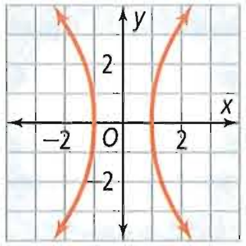


MA.912.A.6.2 **56.** Which product is NOT equal to 13?

Ⓐ $(4 + \sqrt{3})(4 - \sqrt{3})$ Ⓒ $(6 + \sqrt{23})(6 - \sqrt{23})$

Ⓑ $(5 - 2\sqrt{3})(5 + 2\sqrt{3})$ Ⓓ $(7 - \sqrt{6})(7 + \sqrt{6})$

MA.912.A.8.1 **57.** Which function represents exponential growth?

Ⓕ $y = 35x^{1.35}$ Ⓖ $y = 35 \cdot (0.35)^x$ Ⓗ $y = 35 \cdot (1.35)^x$ Ⓘ $y = 35 \div (1.35)^x$

MA.912.D.11.1 **58. Short Response** What is an explicit formula for the sequence 4, 9, 16, 25, 36, . . . ? What is the ninth term in this sequence?

Mixed Review

Determine whether each geometric series *diverges* or *converges*. If the series converges, state the sum. **See Lesson 9-5.**

59. $1 + 3 + 9 + \dots$ **60.** $1 + \frac{4}{3} + \frac{16}{9} + \dots$ **61.** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

Expand each binomial. **See Lesson 5-7.**

62. $(x - y)^3$ **63.** $(p + q)^6$ **64.** $(x - 2)^4$ **65.** $(3 - x)^5$

Get Ready! **To prepare for Lesson 10-2, do Exercises 66–69.**

Make a table of values for each equation. Then graph the equation. **See Lesson 2-7.**

66. $y = |x|$ **67.** $y = |x| + 3$ **68.** $y = |x - 2|$ **69.** $y = |x + 1| - 4$

Concept Byte

For Use With Lesson 10-1

TECHNOLOGY

Graphing Conic Sections

Sunshine State Standard

MA.912.A.9.2 Graph conic sections with and without using graphing technology.

You can use your graphing calculator to graph relations that are not functions.

Example

Graph the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Step 1 Solve the equation for y.

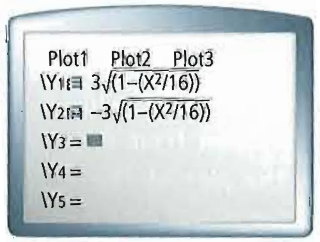
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\frac{y^2}{9} = 1 - \frac{x^2}{16}$$

$$y^2 = 9\left(1 - \frac{x^2}{16}\right)$$

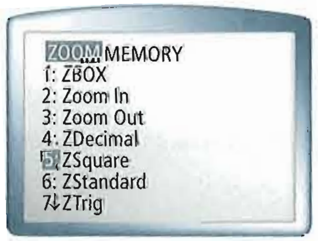
$$y = \pm 3\sqrt{1 - \frac{x^2}{16}}$$

Step 2 Enter the equations as $\mathbf{Y_1}$ and $\mathbf{Y_2}$.



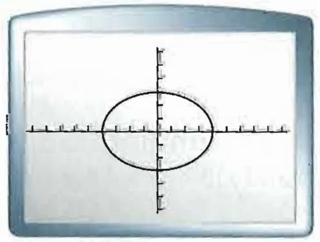
```
Plot1  Plot2  Plot3
\Y1= 3√(1−(X²/16))
\Y2= −3√(1−(X²/16))
\Y3=
\Y4=
\Y5=
```

Step 3 Select a square window.



```
ZOOM MEMORY
1: ZBOX
2: Zoom In
3: Zoom Out
4: ZDecimal
5: ZSquare
6: ZStandard
7↓ZTrig
```

Step 4 Graph.



Exercises

Graph each conic section.

1. $x^2 + y^2 = 25$
2. $4x^2 + y^2 = 16$
3. $9x^2 - 16y^2 = 144$
4. $x^2 - y^2 = 3$
5. $\frac{x^2}{4} - \frac{y^2}{9} = 1$
6. $x^2 + \frac{y^2}{4} = 16$
7. **a.** Graph $y = \sqrt{\frac{81}{4} - x^2}$ and $y = -\sqrt{\frac{81}{4} - x^2}$.
   b. Estimate the x-intercepts and find the y-intercepts.
   c. Adjust the window to $-9.3 \leq x \leq 9.5$. What are the x-intercepts?
   d. What conic section does the graph represent?

Graph each conic section. Find the x- and y-intercepts.

8. $4x^2 + y^2 = 25$
9. $x^2 + y^2 = 30$
10. $9x^2 - 4y^2 = 72$

11. **Writing** Explain how to use a graphing calculator to graph $x = |y - 3|$.

12. **Reasoning** Which conic sections can you graph using only one equation? Explain.

10-2 Parabolas

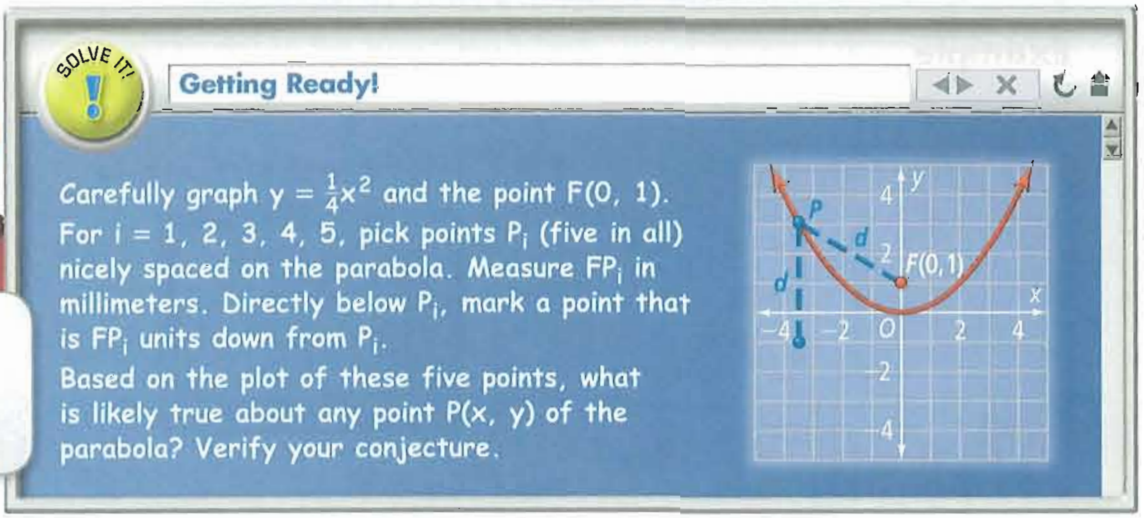
Sunshine State Standards

MA.912.A.9.1 Write the equations of conic sections in standard form and general form, in order to identify the conic section and to find its geometric properties (foci, asymptotes, eccentricity, etc.).

MA.912.A.9.2 Graph conic sections with and without using graphing technology.

Objective To write the equation of a parabola and to graph parabolas

P_i, $i = 1$ to 5, stands for the five points P_1, P_2, P_3, P_4, and P_5.



Getting Ready!

Carefully graph $y = \frac{1}{4}x^2$ and the point F(0, 1). For $i = 1, 2, 3, 4, 5$, pick points P_i (five in all) nicely spaced on the parabola. Measure FP_i in millimeters. Directly below P_i, mark a point that is FP_i units down from P_i.

Based on the plot of these five points, what is likely true about any point P(x, y) of the parabola? Verify your conjecture.

Dynamic Activity
Parabolas

Lesson Vocabulary
- focus of a parabola
- directrix
- focal length

From Chapter 4, you know that a parabola has a vertex and an axis of symmetry. A parabola also has other characteristics.

Essential Understanding Each point of a parabola is equidistant from a point called the *focus* and a line called the *directrix*.

take note

Key Concept Parabola

Definition

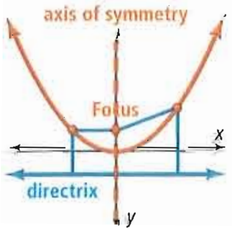
A parabola is the set of all points in a plane that are the same distance from a fixed line and a fixed point not on the line.

The fixed point is called the **focus of a parabola**.

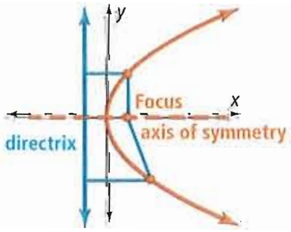
The fixed line is called the **directrix**.

The distance between the vertex and the focus is the **focal length** of the parabola.

Graph



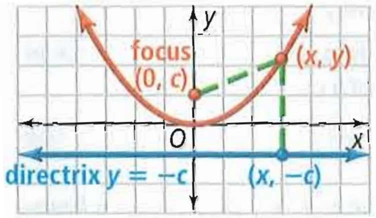
Vertical Parabola



Horizontal Parabola

In this lesson, you will consider vertical parabolas (each of which has a vertical axis of symmetry and a horizontal directrix) and horizontal parabolas (each of which has a horizontal axis of symmetry and a vertical directrix).

You can find the equation of a vertical parabola with vertex at the origin by using the geometric definition. If you denote the focus by $(0, c)$, the directrix is the line with equation $y = -c$.



Here's Why It Works Any point (x, y) on the parabola must be equidistant from the focus and the directrix. Use the Distance Formula.

$\sqrt{(x-0)^2+(y-c)^2} = \sqrt{(x-x)^2+(y-(-c))^2}$	Distance Formula
$x^2+(y-c)^2 = 0^2+(y+c)^2$	Square each side.
$x^2+y^2-2cy+c^2 = y^2+2cy+c^2$	Expand.
$x^2-2cy = 2cy$	Subtract y^2 and c^2 from each side.
$x^2 = 4cy$	Add $2cy$ to each side.
$y = \frac{1}{4c}x^2$	Standard quadratic form

Note that the equation has the expected quadratic form $y = ax^2$ for a vertical parabola with vertex at $(0, 0)$. The coefficient $a = \frac{1}{4c}$ determines both the focus $(0, c)$ and the directrix $y = -c$. This is the key to shifting between the algebraic and geometric representations of a parabola.

Problem 1 Parabolas with Equation $y = ax^2$

Plan

How can you tell if this is a vertical or a horizontal parabola? The focus and the vertex are on the axis of symmetry. They both lie on the y-axis so the parabola is vertical.

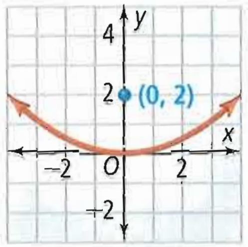
A What is an equation of the parabola with vertex at the origin and focus (0, 2)?

The focus is directly above the vertex.

This is a vertical parabola with vertex at the origin.

The focus is $(0, c)$, so $c = 2$.

$$y = \frac{1}{4c}x^2 = \frac{1}{4(2)}x^2 = \frac{1}{8}x^2$$



Think

What does the sign of a tell you about the graph? Since a is negative, the parabola opens downward.

B What are the focus and directrix of the parabola with equation $y = -\frac{1}{12}x^2$?

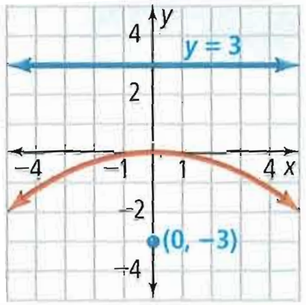
This is a vertical parabola with vertex at the origin and $a = -\frac{1}{12}$.

$$a = \frac{1}{4c} = -\frac{1}{12}$$

$$4c = -12$$

$$c = -3$$

Since the vertex is at the origin, knowing c, you can conclude that the focus is the point $(0, -3)$ and the directrix is the line with equation $y = 3$.

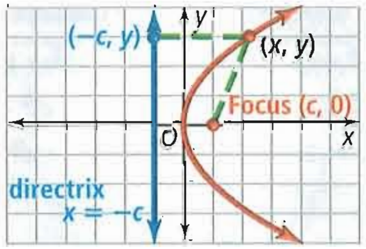


Got It? **1. a.** What is an equation of the parabola with vertex (0, 0) and focus (0, −1.5)?

b. What are the vertex, focus, and directrix of the parabola with equation $y = \frac{x^2}{4}$?

c. **Reasoning** How does the distance of the focus from the vertex affect the shape of a parabola?

The quadratic equation $x = ay^2$ determines a *horizontal parabola* with vertex at (0, 0). The coefficient $a = \frac{1}{4c}$ determines both the focus $(c, 0)$ and the directrix $x = -c$.



Problem 2 Parabolas with Equation $x = ay^2$

Plan

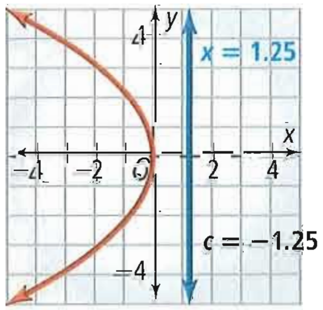
How can you tell if this is a vertical or a horizontal parabola? The directrix is parallel to the y-axis, so this is a horizontal parabola.

A **What is an equation of a parabola with vertex at the origin and directrix $x = 1.25$?**

The directrix lies directly to the right of the vertex.

The parabola is horizontal.

The directrix has equation $x = -c$, so $c = -1.25$. Thus,
$x = \frac{1}{4c}y^2 = \frac{1}{4(-1.25)}y^2 = -\frac{1}{5}y^2$



Check for Reasonableness

The graph is reasonable since it opens in the negative direction and $a < 0$.

Think

What does the sign of *a* tell you about the graph? Since a is positive, the parabola opens to the right.

B **What are the vertex, focus, and directrix of the parabola with equation $x = 0.75y^2$?**

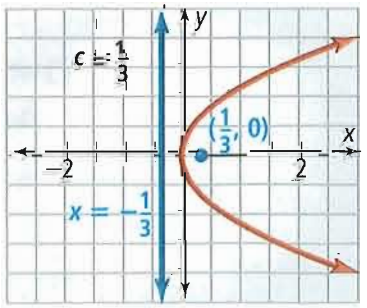
This is a horizontal parabola. The vertex is at the origin and $a = 0.75$. Thus,

$$a = \frac{1}{4c} = 0.75$$

$$4c = \frac{1}{0.75}$$

$$c = \frac{1}{3}$$

Knowing c, you can conclude that the focus is the point $\left(\frac{1}{3}, 0\right)$. The directrix is the line with equation $x = -\frac{1}{3}$.

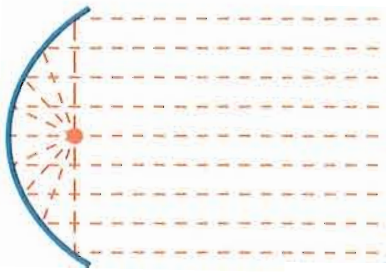


Got It? **2. a.** What is an equation of the parabola with vertex at the origin and directrix $x = -\frac{5}{2}$?

b. What are the vertex, focus, and directrix of the parabola with equation $x = -4y^2$?

The geometry of a parabola implies a very important reflective property that gives real-world meaning to the word "focus."

As the diagram of the *parabolic reflector* shows, lines from the focus reflect off the parabola along lines parallel to the axis of symmetry. This is how a flashlight works. Conversely, lines parallel to the axis of symmetry reflect off the parabola directly into the focus. This is how a satellite dish works.



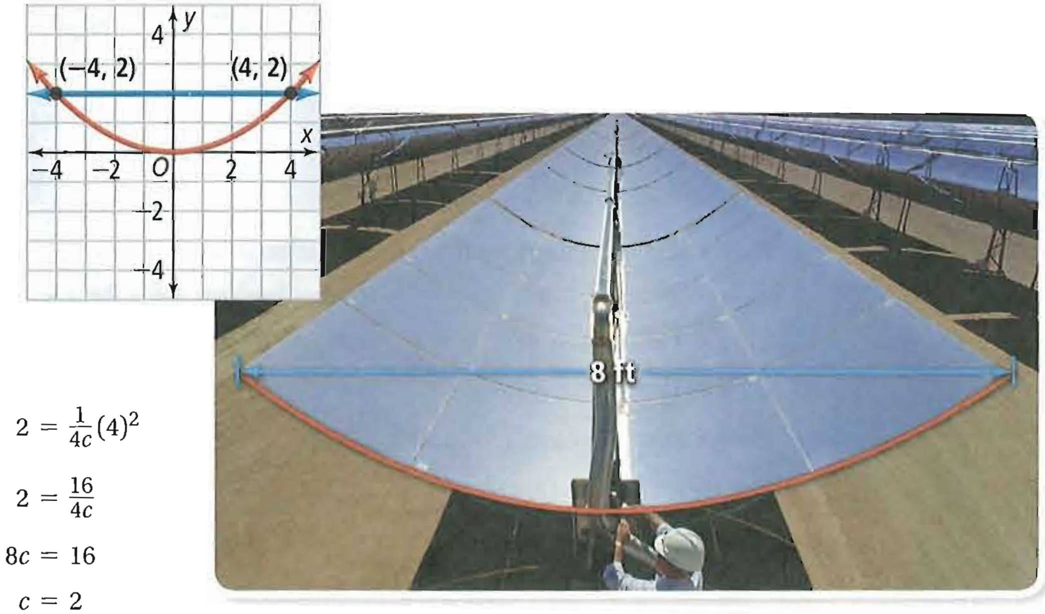
Problem 3 Using Parabolas to Solve Problems

Think

What is the shape of the solar reflector?
A cross section is part of a parabola and is 8 ft across.

Solar Reflector **The parabolic solar reflector pictured has a depth of 2 feet at the center. How far from the vertex is the focus? (What is the focal length?)**

Graph the parabola in a coordinate system with vertex (0, 0). The vertical parabola has the form $y = \frac{1}{4c}x^2$. Substitute either the point (−4, 2) or the point (4, 2).



$$2 = \frac{1}{4c}(4)^2$$

$$2 = \frac{16}{4c}$$

$$8c = 16$$

$$c = 2$$

Therefore the focus is at (0, 2), 2 ft from the vertex.
The focal length is 2 ft.

Got It? **3.** The mirrored reflector of a flashlight is 8 cm across and 4 cm deep. How far from the vertex should the light bulb be positioned?

In Chapter 4, you studied how to translate a parabola from one with vertex (0, 0) to one with vertex (h, k). For such a translation, all of the other features—axis of symmetry, focus, and directrix—translate along with the parabola and its vertex.

take note

Key Concept Transformations of a Parabola

Vertical Parabola	**Vertex (0, 0)**	**Vertex (*h*, *k*)**
Equation	$y = \frac{1}{4c}x^2$	$y = \frac{1}{4c}(x - h)^2 + k$
Focus	$(0, c)$	$(h, k + c)$
Directrix	$y = -c$	$y = k - c$
Horizontal Parabola	**Vertex (0, 0)**	**Vertex (*h*, *k*)**
Equation	$x = \frac{1}{4c}y^2$	$x = \frac{1}{4c}(y - k)^2 + h$
Focus	$(c, 0)$	$(h + c, k)$
Directrix	$x = -c$	$x = h - c$

Problem 4 Analyzing a Parabola

What are the vertex, focus, and directrix of the parabola with equation $y = x^2 - 4x + 8$?

Know	Need	Plan
The equation of the parabola	• vertex • focus • directrix	• Find c, h, and k • Use these values to find the vertex, focus, and directrix.

Think

How can you change the equation to an equivalent form?
Subtract the same value outside the parentheses that you added inside the parentheses.

First, complete the square to get the equation in vertex form.

$y = x^2 - 4x + 8$	Standard form $y = ax^2 + bx + c$
$y = (x^2 - 4x + 4) + 8 - 4$	Add $\left(\frac{1}{2} \cdot -4\right)^2$ inside parentheses; subtract it outside.
$y = (x - 2)^2 + 4$	Vertex form $y = \frac{1}{4c}(x - h)^2 + k$

Note that, in this case, $\frac{1}{4c} = 1$, so $c = 0.25$.

The vertex (h, k) is $(2, 4)$.

The focus $(h, k + c)$ is $(2, 4.25)$.

The directrix $y = k - c$ is $y = 3.75$.

Got It? 4. What are the vertex, focus, and directrix of the parabola with equation $y = x^2 + 8x + 18$?

Problem 5 Writing an Equation of a Parabola

Plan

How do you determine which equation to use? Use the focus and the vertex to determine the orientation of the parabola.

Multiple Choice Which is an equation of the parabola with vertex (3, 7) and focus (5, 7)?

Ⓐ $x = \frac{1}{4}(y - 7)^2 + 3$ Ⓒ $x = \frac{1}{8}(y - 7)^2 + 3$

Ⓑ $y = \frac{1}{8}(x - 7)^2 + 3$ Ⓓ $x = \frac{1}{8}(y + 7)^2 - 3$

The focus is to the right of the vertex, so the parabola is horizontal. Also, (h, k) is (3, 7) and $(h + c, k)$ is (5, 7), so $c = 2$. Substitute all this information into the equation for a horizontal parabola, $x = \frac{1}{4c}(y - k)^2 + h$, to get $x = \frac{1}{8}(y - 7)^2 + 3$. The correct answer is C.

Got It? **5.** What is an equation of the parabola with vertex (1, 4) and focus (1, 6)?

Lesson Check

Do you know HOW?

Write an equation of a parabola with the given information.

1. vertex (0, 0), focus $\left(0, \frac{1}{2}\right)$

2. vertex (3, 2), focus (4, 2)

Find the vertex, focus, and the directrix of each parabola.

3. $x = \frac{1}{16}y^2$

4. $y = x^2 + 6x + 5$

Do you UNDERSTAND?

5. Vocabulary If the vertex of a parabola is 3 units from the focus, how far is the focus from the directrix?

6. Error Analysis The vertex of a parabola is at the origin, one unit away from the focus. A student concludes that the equation is $y = \frac{1}{4}x^2$. Identify at least two ways in which the student's equation might be in error.

Practice and Problem-Solving Exercises

A Practice

Write an equation of a parabola with vertex at the origin and the given focus. **See Problem 1.**

7. focus at (6, 0)
8. focus at (0, −4)
9. focus at (0, 7)
10. focus at (−1, 0)
11. focus at (2, 0)
12. focus at (0,−5)

Identify the vertex, the focus, and the directrix of the parabola with the given equation. Then sketch the graph of the parabola. **See Problems 1 and 2.**

13. $y = 4x^2$
14. $y = -\frac{1}{8}x^2$
15. $x = y^2$
16. $x = \frac{1}{2}y^2$

Write an equation of a parabola with vertex at the origin and the given directrix. **See Problem 2.**

17. directrix $x = -3$
18. directrix $y = 5$
19. directrix $y = -\frac{1}{3}$
20. directrix $x = 9$
21. directrix $y = 2.8$
22. directrix $x = -3.75$

23. **Optics** A cross section of a flashlight reflector is a parabola. The bulb is located at the focus. Suppose the bulb is located $\frac{1}{4}$ in. from the vertex of the reflector. Model a cross section of the reflector by writing an equation of a parabola that opens upward and has its vertex at the origin. What is an advantage of this parabolic design?

See Problem 3.

Identify the vertex, the focus, and the directrix of the parabola with the given equation. Then sketch the graph of the parabola.

See Problem 4.

24. $y = x^2 + 4x + 3$

25. $y = x^2 - 6x + 11$

26. $y = x^2 + 8x + 13$

27. $y = x^2 - 2x - 4$

28. $y = x^2 - 8x + 17$

29. $y = 2x^2 + 4x - 2$

Write an equation of a parabola with the given vertex and focus.

See Problem 5.

30. vertex (4, 1), focus (6, 1)

31. vertex (0, 3), focus (−8, 3)

32. vertex (−5, 4), focus (−5, 0)

33. vertex (7, 2), focus (7, −2)

B Apply

Identify the vertex, the focus, and the directrix of a parabola with each equation. Then sketch a graph of the parabola with the given equation.

34. $y^2 - 25x = 0$

35. $x^2 = -4y$

36. $(x - 2)^2 = 4y$

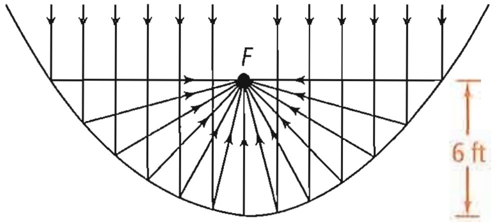
37. $-8x = y^2$

38. $y^2 - 6x = 18$

39. $x^2 + 24y - 8x = -16$

40. **Think About a Plan** In some solar collectors, a mirror with a parabolic cross section is used to concentrate sunlight on a pipe, which is located at the focus of the mirror as shown in the diagram. What is an equation of the parabola that models the cross section of the mirror?

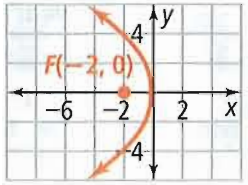
- What information can you get from the diagram?
- What information do you need to be able to write an equation that models the cross section of the mirror?



41. **Earth Science** The equation $d = \frac{1}{10}s^2$ relates the depth d (in meters) of the ocean to the speed s (in m/s) at which tsunamis travel. What is the graph of the equation?

Use the information in each graph to write the equation for the parabola.

42.



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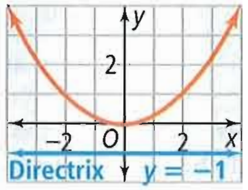
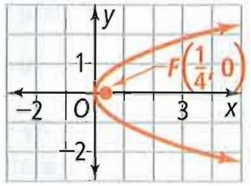


44.



45. **Sound** Broadcasters use a parabolic microphone on football sidelines to pick up field audio for broadcasting purposes. A certain parabolic microphone has a reflector dish with a diameter of 28 inches and a depth of 14 inches. If the receiver of the microphone is located at the focus of the reflector dish, how far from the vertex should the receiver be positioned?

Graph each equation.

46. $y^2 - 8x = 0$

47. $y^2 - 8y + 8x = -16$

48. $2x^2 - y + 20x = -53$

49. $x^2 = 12y$

50. $y = 4(x - 3)^2 - 2$

51. $(y - 2)^2 = 4(x + 3)$

Write an equation of a parabola with vertex at (1, 1) and the given information.

52. directrix $y = -\frac{1}{2}$

53. directrix $x = \frac{3}{2}$

54. focus at (1,0)

55. **Writing** Explain how to find the distance from the focus to the directrix of the parabola $x = 2y^2$.

Challenge

56. **Reasoning** Use the definition of a parabola to show that the parabola with vertex (h, k) and focus $(h, k + c)$ has the equation $(x - h)^2 = 4c(y - k)$.

57. **a.** What part of a parabola is modeled by the function $y = \sqrt{x}$?
    b. State the domain and range for the function in part (a).

58. **Proof** If the radius and depth of a satellite dish are equal, prove that the radius is four times the focal length.

Sunshine State Standards Practice

MA.912.A.9.1

59. What is the equation of a parabola with vertex at the origin and focus at $\left(0, \frac{5}{2}\right)$?

Ⓐ $x = -\frac{1}{10}y^2$ Ⓑ $x = \frac{1}{10}y^2$ Ⓒ $x = -\frac{1}{10}x^2$ Ⓓ $y = \frac{1}{10}x^2$

MA.912.A.9.2

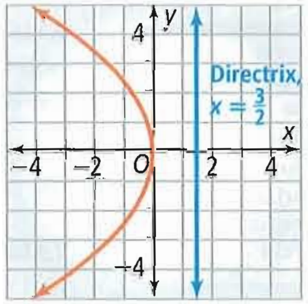
60. Use the information in the graph to find the equation for the graph.

Ⓕ $y^2 + 6x = 0$

Ⓖ $y^2 - 6x = 0$

Ⓗ $x^2 + 6y = 0$

Ⓘ $x^2 - 6y = 0$



MA.912.A.6.4

61. Which expression is NOT equivalent to $(25x^4y)^{\frac{1}{3}}$?

Ⓐ $x\sqrt[3]{25xy}$

Ⓑ $5x\sqrt[3]{xy}$

Ⓒ $\sqrt[3]{25x^4y}$

Ⓓ $\sqrt[6]{625x^8y^2}$

MA.912.A.8.2

62. **Extended Response** Use the properties of logarithms to write log 12 in four different ways. Name each property you use.

Mixed Review

Graph each equation. Identify the conic section and describe the graph and its lines of symmetry. Then find the domain and range.

See Lesson 10-1.

63. $x^2 + y^2 = 64$

64. $x^2 + 9y^2 = 9$

65. $4x^2 - 9y^2 = 36$

Get Ready! **To prepare for Lesson 10-3, do Exercises 66–69.**

Complete the square.

See Lesson 4-6.

66. $x^2 - 2x + ■$

67. $x^2 + 4x + ■$

68. $x^2 + 10x + ■$

69. $x^2 - 6x + ■$

10-3 Circles

Sunshine State Standards

MA.912.G.6.6 Given the center and radius, find the equation of a circle or given the equation of a circle, state the center and the radius of the circle.

MA.912.G.6.7 Given the equation of a circle in center-radius form or given the center and the radius of a circle, sketch the graph of the circle.

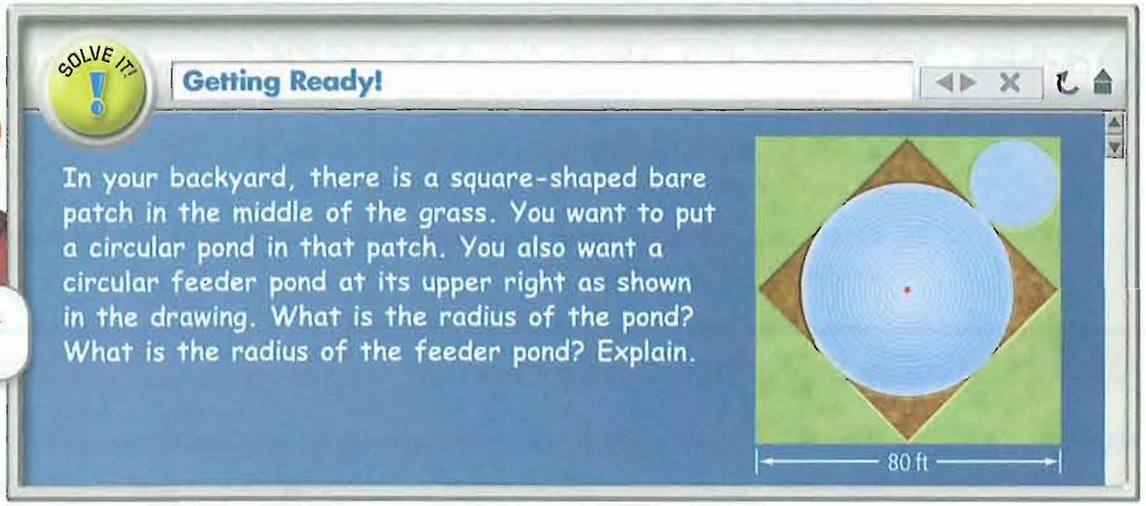
Objectives To write and graph the equation of a circle

To find the center and radius of a circle and use them to graph the circle

Dynamic Activity
Circles in the Coordinate Plane

Lesson Vocabulary

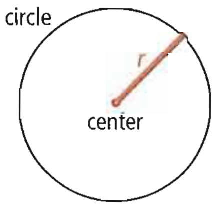
- circle
- center of a circle
- radius
- standard form of an equation of a circle



Getting Ready!

In your backyard, there is a square-shaped bare patch in the middle of the grass. You want to put a circular pond in that patch. You also want a circular feeder pond at its upper right as shown in the drawing. What is the radius of the pond? What is the radius of the feeder pond? Explain.

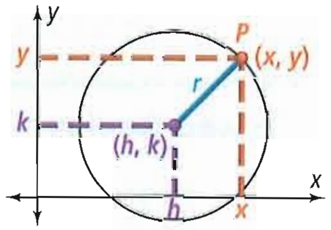
A **circle** is the set of all points in a plane that are a distance r from a given point, the **center of a circle**. The distance r is the **radius** of the circle. The use of *distance* in these definitions makes the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ a useful tool for describing a circle in the coordinate plane.



Essential Understanding An equation of a circle with center (0, 0) and radius r in the coordinate plane is $x^2 + y^2 = r^2$.

Not every circle has its center at the origin. Suppose a circle with radius r has center (h, k). Then r is the distance from (h, k) to any point (x, y) on the circle.

$$r = \sqrt{(x - h)^2 + (y - k)^2} \quad \text{Distance formula}$$
$$r^2 = (x - h)^2 + (y - k)^2 \quad \text{Square each side.}$$



take note

Key Concept Standard Form of an Equation of a Circle

The **standard form of an equation of a circle** with center (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

You can use the center and the radius of a circle to write an equation for the circle.

Plan

How do you know which equation to use?
Since this circle is not centered at the origin, use the standard form equation.

Problem 1 Writing an Equation of a Circle

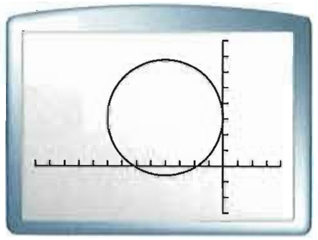
What is an equation of the circle with center $(-4, 3)$ and radius 4?

$$\begin{aligned} (x - h)^2 + (y - k)^2 &= r^2 && \text{Use the standard form.} \\ (x - (-4))^2 + (y - 3)^2 &= 4^2 && \text{Substitute } -4 \text{ for } h\text{, 3 for } k\text{, and 4 for } r. \\ (x + 4)^2 + (y - 3)^2 &= 16 && \text{Simplify.} \end{aligned}$$

An equation of the circle is $(x + 4)^2 + (y - 3)^2 = 16$.

Check Solve the equation for y. Enter both functions into your graphing calculator.

$$\begin{aligned} (x + 4)^2 + (y - 3)^2 &= 16 \\ (y - 3)^2 &= 16 - (x + 4)^2 \\ y - 3 &= \pm\sqrt{16 - (x + 4)^2} \\ y &= 3 \pm \sqrt{16 - (x + 4)^2} \end{aligned}$$



Got It? **1.** What is an equation of the circle with center $(5, -2)$ and radius 8? Check your answer.

Problem 2 Using Translations to Write an Equation

What is an equation for the translation of $x^2 + y^2 = 9$ by 4 units left and 3 units up? Draw the graph.

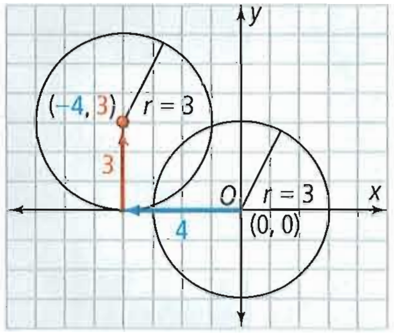
Know	Need	Plan
The original equation that is translated 4 units left and 3 units up	The equation and graph of the translation	• Use the standard form to write the equation of the translation. • Graph the equation.

Think

How does the translation help you draw the graph?
Use the translation to determine the new center coordinates.

$$\begin{aligned} x^2 + y^2 &= 9 && \text{Write the given equation.} \\ (x - (-4))^2 + (y - 3)^2 &= 9 && \text{The graph is translated left 4 units and up 3 units.} \\ (x + 4)^2 + (y - 3)^2 &= 3^2 && \text{Simplify inside the parentheses. The form } (x - h)^2 + (y - k)^2 = r^2 \text{ shows the radius } r. \end{aligned}$$

Graph the given equation and the translation.



Got It? **2.** What is an equation for each translation?

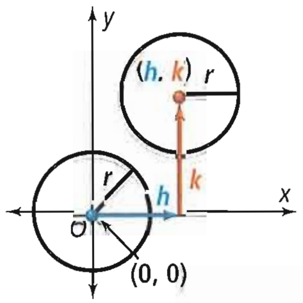
a. $x^2 + y^2 = 1$; left 5 units and down 3 units

b. $x^2 + y^2 = 9$; right 2 units and up 3 units

Key Concept Transforming a Circle

You can use the parameter r to stretch or shrink the unit circle $x^2 + y^2 = 1$ to the circle $x^2 + y^2 = r^2$ with radius r.

You can use the parameters h and k to translate the circle $x^2 + y^2 = r^2$ with center (0, 0) to the circle $(x - h)^2 + (y - k)^2 = r^2$ with center (h, k).



Problem 3 Using a Graph to Write an Equation

Multiple Choice Which equation models the circular irrigation field?

Ⓐ $(x - 450)^2 + (y - 500)^2 = 160{,}000$

Ⓑ $(x + 500)^2 + (y + 450)^2 = 160{,}000$

Ⓒ $(x - 450)^2 + (y - 500)^2 = 400$

Ⓓ $(x + 450)^2 + (y + 500)^2 = 400$

Plan

What information do you need to write an equation for the circle?
You need the center and radius of the circle.

According to the photograph, this circular irrigation field has radius 400 and center at the point (450, 500).

$(x - h)^2 + (y - k)^2 = r^2$ Use the standard form.

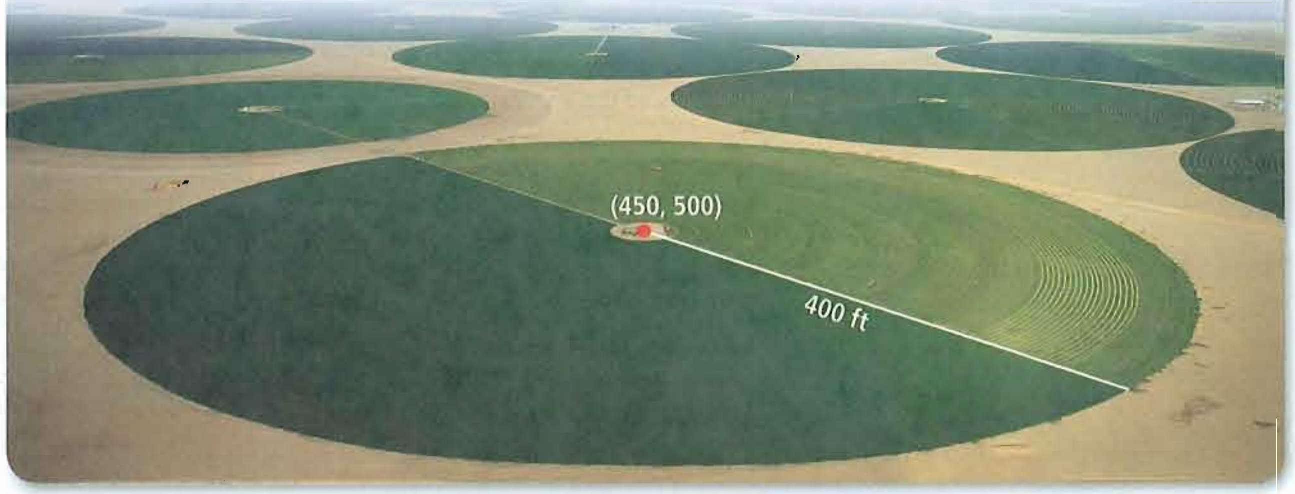
$(x - 450)^2 + (y - 500)^2 = 400^2$ Substitute the values of h, k, and r from the photograph.

$(x - 450)^2 + (y - 500)^2 = 160{,}000$ Simplify.

The correct answer is A.

Got It? **3. a.** What is an equation of the circle for a circular irrigation field that has radius 12 and center (7, −10)?

b. Reasoning Will the graph of every equation of the form $(x - h)^2 + (y - k)^2 = r^2$, where h, k, and r are real numbers, be a circle? Explain your reasoning.



You can find the center and radius of a circle by rewriting the equation in standard form. In some cases, you may need to complete the square.

Problem 4 Finding the Center and Radius

Plan

What do you need to do to the equation to find the center and radius of the circle?
Write the equation in standard form.

What are the center and radius of the circle with the given equation?

A $(x - 16)^2 + (y + 9)^2 = 144$

$(x - 16)^2 + (y - (-9))^2 = 12^2$ Rewrite the equation in standard form.

$h = 16$ $k = -9$ $r = 12$ Find h, k, and r.

The center of the circle is $(16, -9)$. The radius is 12.

B $x^2 + y^2 + 8x - 10y = 8$

$(x^2 + 8x) + (y^2 - 10y) = 8$ Set up to complete the squares.

$(x^2 + 8x + 16) + (y^2 - 10y + 25) = 8 + 16 + 25$ Complete the squares and balance the equation.

$(x + 4)^2 + (y - 5)^2 = 49$ Simplify.

$(x - (-4))^2 + (y - 5)^2 = 7^2$ Rewrite the equation in standard form.

$h = -4$ $k = 5$ $r = 7$ Find h, k, and r.

The center of the circle is $(-4, 5)$. The radius is 7.

Got It? **4.** What are the center and radius of the circle with the given equation?

a. $(x + 8)^2 + (y + 3)^2 = 121$ **b.** $x^2 + y^2 - 6x + 14y = 8$

You can use the center and the radius to graph a circle.

Problem 5 Graphing a Circle Using Center and Radius

Plan

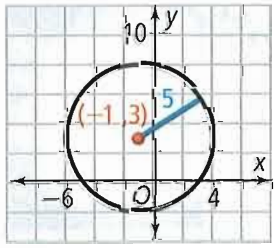
What information do you need to graph a circle?
You need the center and radius of the circle.

What is the graph of $(x + 1)^2 + (y - 3)^2 = 25$?

$(x - (-1))^2 + (y - 3)^2 = 5^2$ Rewrite the equation in standard form.

$h = -1$ $k = 3$ $r = 5$ Identify h, k and r.

center: $(-1, 3)$ radius: $r = 5$ Find the center and the radius of the circle.



Locate the center $(-1, 3)$.

Draw a circle of radius 5.

Got It? **5.** What is the graph of $(x - 4)^2 + (y + 2)^2 = 49$?

Lesson Check

Do you know HOW?

Use the given information to write an equation of a circle.

1. center at $(-1, -5)$, radius 2

2. center at $(0, 0)$, radius 6

Write an equation for each translation.

3. $x^2 + y^2 = 121$; up 3 units

4. $x^2 + y^2 = 16$; left 5 units and down 3 units

Do you UNDERSTAND?

5. **Error Analysis** A student claims that the circle $(x + 7)^2 + (y - 7)^2 = 8$ is a translation of the circle $x^2 + y^2 = 8$, 7 units right and 7 units down. What is the student's mistake?

6. **Reasoning** Let $P(x, y)$ be any point on the circle with center $(0, 0)$ and radius r. Prove that $x^2 + y^2 = r^2$ is an equation for the circle.

Practice and Problem-Solving Exercises

A Practice

Write an equation of a circle with the given center and radius. Check your answers.

See Problem 1.

7. center $(0, 0)$, radius 10

8. center $(-4, -6)$, radius 7

9. center $(2, 3)$, radius 4.5

10. center $(-6, 10)$, radius 1

11. center $(1, -3)$, radius 10

12. center $(-1.5, -3)$, radius 2

Write an equation for each translation.

See Problem 2.

13. $x^2 + y^2 = 9$; down 1 unit

14. $x^2 + y^2 = 1$; left 1 unit

15. $x^2 + y^2 = 25$; right 2 units and down 4 units

16. $x^2 + y^2 = 81$; left 1 unit and up 3 units

17. $x^2 + y^2 = 100$; down 5 units

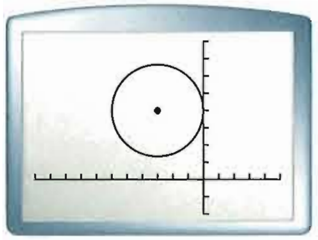
18. $x^2 + y^2 = 49$; right 3 units and up 2 units

19. An archery target has a circular bull's-eye (diameter 24 cm) surrounded by four concentric rings, each with a width of 12 cm. Draw the target in a coordinate plane with center at the origin. Write the equations of the circles that form the boundaries of the different regions of the target.

Write an equation for each circle. Each interval represents one unit.

See Problem 3.

20.



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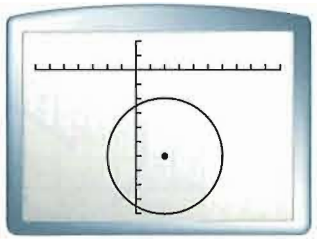


For each equation, find the center and radius of the circle.

See Problem 4.

22. $(x - 1)^2 + (y - 1)^2 = 1$

23. $(x + 2)^2 + (y - 10)^2 = 4$

24. $(x - 3)^2 + (y + 1)^2 = 36$

25. $(x + 3)^2 + (y - 5)^2 = 81$

26. $x^2 + (y + 3)^2 = 25$

27. $(x + 6)^2 + y^2 = 121$

Use the center and the radius to graph each circle.

See Problem 5.

28. $(x + 4)^2 + (y - 4)^2 = 4$

29. $(x - 6)^2 + y^2 = 64$

30. $x^2 + y^2 = 9$

31. $(x + 3)^2 + (y - 9)^2 = 49$

32. $(x - 7)^2 + (y - 1)^2 = 100$

33. $x^2 + (y + 4)^2 = 144$

B Apply

Write the equation of the circle that passes through the given point and has a center at the origin. (*Hint:* You can use the distance formula to find the radius.)

34. $(0, 4)$

35. $(0, -3)$

36. $(-5, 0)$

37. $(\sqrt{3}, 0)$

38. $(4, -3)$

39. $(12, -5)$

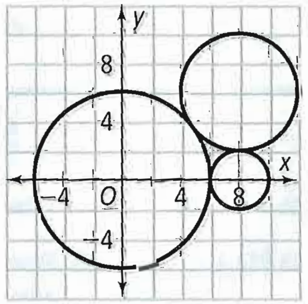
40. $(-2, 3)$

41. $(1, -5)$

42. $(-6, -4)$

43. **Think About a Plan** Three gears of radii 6 in., 4 in., and 2 in. mesh with each other in a motor assembly as shown to the right. What is the equation of each circle in standard form?

- How can the diagram of the gears in the coordinate plane help you solve this problem?
- How can you write an equation for each circle?



44. **Open-Ended** Write two functions that together represent a circle.

Use the given information to write an equation of the circle.

45. radius 7, center $(-6, 13)$

46. area 25π, center $(5, -3)$

47. center $(-2, 7.5)$, circumference 3π

48. center $(1, -2)$, through $(0, 1)$

49. center $(2, 1)$, through $(6, 4)$

50. center $(6, 4)$, through $(2, 1)$

51. translation of $(x - 1)^2 + (y + 3)^2 = 36$, 2 units left and 4 units down

52. **Machinery** Three gears, *A*, *B*, and *C*, mesh with each other in a motor assembly. Gear *A* has a radius of 4 in., *B* has a radius of 3 in., and *C* has a radius of 1 in. If the largest gear is centered at $(-7, 0)$, the smallest gear is centered at $(4, 0)$, and Gear *B* is centered at the origin, what is the equation of each circle in standard form?

Find the center and the radius of each circle.

53. $x^2 + y^2 = 2$

54. $x^2 + (y + 1)^2 = 5$

55. $x^2 + y^2 = 14$

56. $x^2 + (y - 4)^2 = 11$

57. $(x + 5)^2 + y^2 = 18$

58. $(x + 2)^2 + (y + 4)^2 = 50$

59. $(x + 3)^2 + (y - 5)^2 = 38$

60. $x^2 + 2x + 1 + y^2 = 4$

61. $x^2 + y^2 - 6x - 2y + 4 = 0$

62. $x^2 + y^2 - 4y - 16 = 0$

Challenge **Graph each pair of equations. Identify the conic section represented by the graph. Then write a single equation for the conic section.**

63. $y = 3 + \sqrt{16 - (x - 4)^2}$
$y = 3 - \sqrt{16 - (x - 4)^2}$

64. $y = -2 + \sqrt{x - 3}$
$y = -2 - \sqrt{x - 3}$

65. Astronomy The table gives the diameters of three planets.

a. Use a center of (0, 0) to graph a circle that represents the size of each planet.

b. Write an equation representing the circular cross section through the center of each planet.

Planet	Diameter (miles)
Mercury	3031
Mars	4222
Earth	7926

66. a. A circle contains the points (0, 0), (6, 8), and (7, 7). Find its equation by solving a system of three equations.

b. Several parabolas contain the three points of part (a), but only one is described by a quadratic function. What is that function?

Sunshine State Standards Practice

GRIDDED RESPONSE

MA.912.G.6.6 **67.** What is the radius of the circle with equation $(x + 5)^2 + (y - 3)^2 = 144$?

MA.912.A.7.10 **68.** Find the positive zero of the function $y = x^2 + 2x - 5$ by graphing. Enter your answer as a decimal to the nearest hundredth.

MA.912.A.9.1 **69.** What is the distance between $T(9, -5)$ and the center of the circle with equation $(x - 6)^2 + (y + 1)^2 = 10$?

MA.912.D.11.2 **70.** What is the common ratio in a geometric series if $a_2 = \frac{1}{3}$ and $a_5 = \frac{8}{81}$? Enter your answer as a fraction.

MA.912.D.11.4 **71.** Evaluate the sum $\sum_{n=1}^{3}\left(\frac{1}{n + 1}\right)^2$. Enter your answer as a decimal to the nearest hundredth.

Mixed Review

72. What is an equation of a parabola opening left with vertex (0, 0) and focus (−3, 0)? **See Lesson 10-2.**

For each rational function, find any points of discontinuity. **See Lesson 8-3.**

73. $y = \frac{2}{x + 1}$ **74.** $y = \frac{1}{x^2 - 5x + 6}$ **75.** $y = \frac{2x - 1}{x^2 + 4}$

Evaluate each logarithm. **See Lesson 7-3.**

76. $\log_2 16$ **77.** $\log_5 25$ **78.** $\log_3 \frac{1}{27}$

79. $\log 10{,}000$ **80.** $\log_{36} 6$ **81.** $\log_{100} 100$

Get Ready! **To prepare for Lesson 10-4, do Exercises 82–84.**

Solve each equation. **See Lesson 5-3.**

82. $x^2 + 11x = -18$ **83.** $m^5 - 256m = 0$ **84.** $p^4 + 32 = 12p^2$

10 Mid-Chapter Quiz

MathXL® for School
Go to PowerAlgebra.com

Do you know HOW?

Describe the graph and identify the domain and range for each equation.

1. $y^2 - 2x^2 = 16$

2. $3x^2 + 3y^2 - 12 = 0$

3. $9x^2 - 25y^2 = 225$

4. $36 - 4x^2 - 9y^2 = 0$

Identify the vertex, focus, and directrix of each parabola. Then graph the parabola.

5. $y = 3x^2$

6. $x = 4(y + 2)^2$

7. $y + 1 = (x - 3)^2$

Write an equation for the parabola with the given vertex and focus.

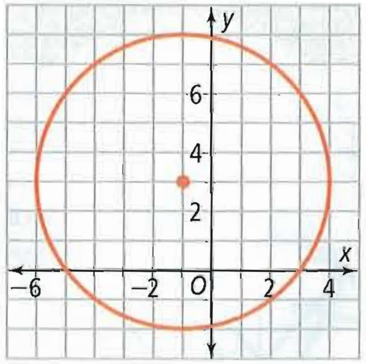
8. vertex $(-5, 4)$; focus $(-5, 0)$

9. vertex $(7, 2)$; focus $(7, -2)$

10. vertex $(0, 0)$; focus $(-7, 0)$

11. vertex $(2, 4)$; focus $(1, 4)$

12. Write an equation that models the graph below.



Write an equation in standard form of the circle with the given center and radius.

13. center $(-6, 3)$, radius 8

14. center $(1, 1)$, radius 1.5

Do you UNDERSTAND?

Determine whether each point lies on the graph of the conic section with the given equation.

15. $x^2 + y^2 = 36$
a. $(-6, 0)$ **b.** $(-2, -\sqrt{3})$ **c.** $(0, \sqrt{2})$

16. $4x^2 - y^2 - 4 = 0$
a. $(-1, 0)$ **b.** $(2, 2)$ **c.** $(1, 0)$

17. The table below represents points on the graph of a conic section. Identify the conic section.

x	−12	−8	0	12
y	0	±12	±16	0

18. Writing Suppose that $x^2 = 4py$ and $y = ax^2$ represent the same parabola. Explain how a and p are related.

Reasoning **Without graphing, describe how each graph differs from the graph of $y = x^2$.**

19. $y = 2x^2$

20. $y = -x^2$

21. $y = x^2 + 2$

22. $y = \frac{1}{3}x^2$

23. A circle has center $(0, 0)$ and radius 1. Write an equation that represents the translation of the circle 7 units left and 8 units up. Then graph the equation.

Write the standard form of the equation of the circle that passes through the given point and whose center is at the origin.

24. $(-6, 0)$

25. $(0, 5)$

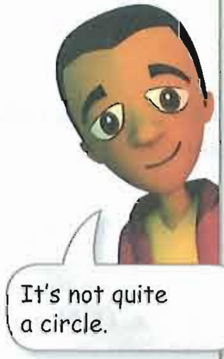
26. $(-11, -11)$

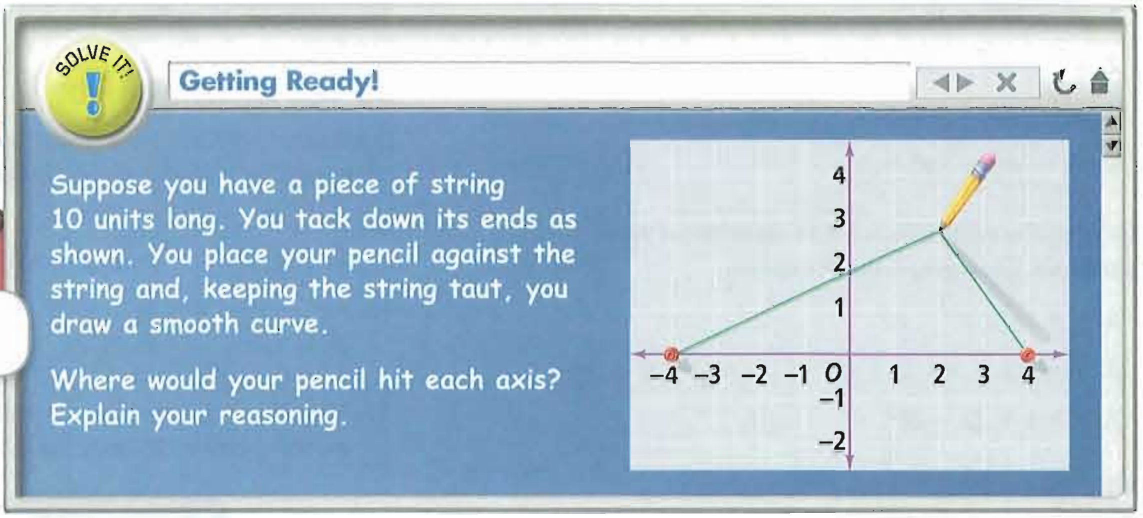
27. $(-8, 14)$

10-4 Ellipses

Sunshine State Standard
MA.912.A.9.2 Graph conic sections with and without using graphing technology.

Objectives To write the equation of an ellipse
To find the foci of an ellipse
To graph an ellipse





Suppose you have a piece of string 10 units long. You tack down its ends as shown. You place your pencil against the string and, keeping the string taut, you draw a smooth curve.

Where would your pencil hit each axis? Explain your reasoning.

Lesson Vocabulary
- ellipse
- focus of an ellipse
- major axis
- center of an ellipse
- minor axis
- vertices of an ellipse
- co-vertices of an ellipse

Points on the smooth curve in the Solve It have a total distance of 10 units to the points $(-4, 0)$ and $(4, 0)$. In fact, all of the points on the smooth curve have a total distance of 10 units to the two fixed points. You can describe this smooth curve with an equation.

Essential Understanding A circle is the set of points a fixed distance from one point. An *ellipse* "stretches" a circle in one direction and is the set of points that have a total fixed distance from two points.

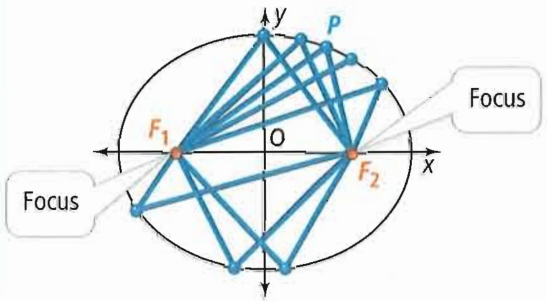
take note

Key Concept Ellipse

Definition
An **ellipse** is a set of all points P in a plane such that the sum of the distances from P to two fixed points F_1 and F_2 is a constant k. A **focus of an ellipse** (plural: foci) is one of the two fixed points.

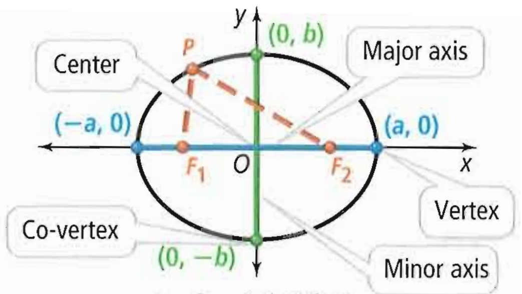
Symbols
$PF_1 + PF_2 = k$, where $k > F_1F_2$.

Graph

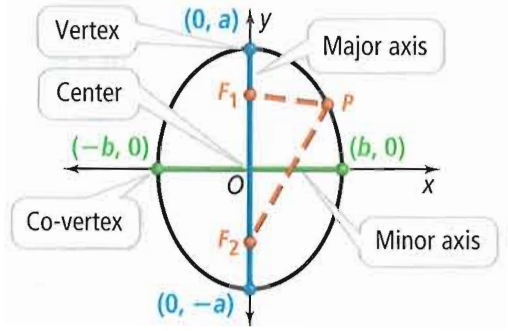


Dynamic Activity
Ellipses

The **major axis** is the segment that contains the foci and has its endpoints on the ellipse. Its midpoint is the **center of the ellipse**. The **minor axis** is perpendicular to the major axis at the center. The **vertices of an ellipse** (singular: *vertex*) are the endpoints of the major axis. The **co-vertices of an ellipse** are the endpoints of the minor axis.



Horizontal Ellipse



Vertical Ellipse

take note

Key Concept Properties of Ellipses with Center (0, 0)

	Horizontal Ellipses	Vertical Ellipses
Standard Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b > 0$
Major Axis	horizontal	vertical
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Co-vertices	$(0, \pm b)$	$(\pm b, 0)$
Foci	$(\pm c, 0)$ on x-axis	$(0, \pm c)$ on y-axis

The length of the major axis is $2a$ and the length of the minor axis is $2b$.
For any point P on an ellipse, $PF_1 + PF_2 = 2a$.

Problem 1 Writing an Equation of an Ellipse

What is an equation in standard form of an ellipse centered at the origin with vertex $(-6, 0)$ and co-vertex $(0, 3)$?

Think

What is the orientation of the ellipse?
Since the vertices $(-6, 0)$ and $(6, 0)$ are aligned horizontally, the ellipse is horizontal.

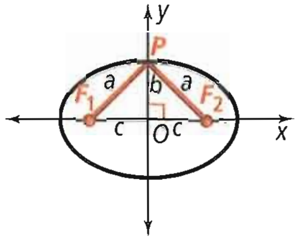
Since one vertex is $(-6, 0)$, the other vertex is $(6, 0)$. The major axis is horizontal. Since one co-vertex is $(0, 3)$, the other co-vertex is $(0, -3)$. The minor axis is vertical. So $a = 6$, $b = 3$, $a^2 = 36$, and $b^2 = 9$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Standard form of a horizontal ellipse

$\frac{x^2}{36} + \frac{y^2}{9} = 1$ Substitute for a^2 and b^2.

Got It? **1.** What is the equation in standard form of an ellipse centered at the origin with vertex $(0, 5)$ and co-vertex $(2, 0)$?

Since the co-vertex $P(0, b)$ is on the ellipse, $PF_1 + PF_2 = 2a$. If you denote the distance from each focus to the center of the ellipse by c, then a, b, and c are the lengths of the sides of a right triangle, as shown in the ellipse at the right. Thus, the distances from the center to each vertex, to each co-vertex, and to each focus are related by the Pythagorean Theorem: $a^2 = b^2 + c^2$.



If $(\pm a, 0)$, $(0, \pm b)$, and $(\pm c, 0)$ are the vertices, the co-vertices, and the foci of an ellipse, respectively,

$$c^2 = a^2 - b^2$$

Problem 2 Finding the Foci of an Ellipse

What are the foci of the ellipse with the equation $25x^2 + 9y^2 = 225$? Graph the foci and the ellipse.

Know	Need	Plan
The equation of an ellipse.	The coordinates of the vertices, co-vertices, and foci.	• Write the equation in standard form to find a^2 and b^2. Use $c^2 = a^2 - b^2$ to find c. • Use a, b, and c to graph the ellipse.

$$25x^2 + 9y^2 = 225$$

$$\frac{25x^2}{225} + \frac{9y^2}{225} = 1 \quad \text{Divide each side by 225.}$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \quad \text{Simplify to standard form.}$$

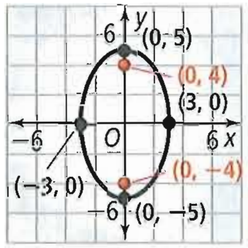
Since $25 > 9$ and 25 is with y^2, the major axis is vertical.

$$a^2 = 25 \text{ and } b^2 = 9$$

$$c^2 = a^2 - b^2 = 25 - 9 = 16$$

$$c = \pm 4$$

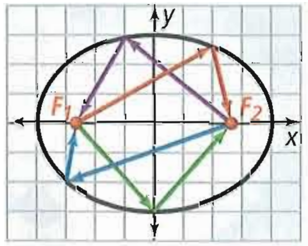
The foci are $(0, 4)$ and $(0, -4)$. Plot points for the vertices, co-vertices, and foci, then graph the ellipse.



Got It? **2. a.** What are the coordinates of the foci of the ellipse with the equation $36x^2 + 100y^2 = 3600$? Graph the ellipse.

b. Reasoning What happens to the foci as c gets closer to 0? What would the graph of an ellipse be if $c = 0$?

Like parabolas, ellipses have an important reflective property related to their foci: Any line emanating from one focus of an ellipse will reflect off the ellipse directly into the other focus. This property is related to the two-focus definition of an ellipse and can give a new interpretation to the same picture.

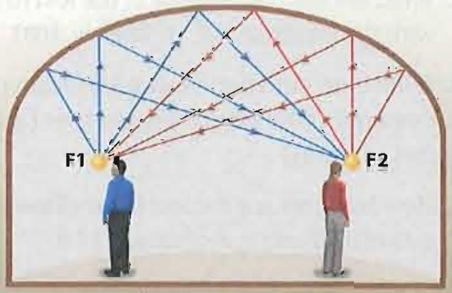


Problem 3 Using the Foci of an Ellipse

Plan

How can you find the distance between foci, given the major and minor axes?
Find the values of *a* and *b*. Then, use a^2 and b^2 to solve for *c*. The distance between the foci is $2c$.

Whispering Gallery **A room with an elliptical ceiling (called an *ellipsoid*, since it is 3-dimensional) forms a "whispering gallery." Thanks to the reflective property of the ellipse, a whispered message at one focus can be heard clearly by someone standing across the room at the other focus. If the elliptical ceiling has a major axis of 120 feet and a minor axis of 72 feet, how far apart are the foci?**



The major axis has length $2a = 120$, so $a = 60$.

The minor axis has length $2b = 72$, so $b = 36$.

Thus $c = \sqrt{a^2 - b^2} = \sqrt{60^2 - 36^2} = 48$.

The foci are $2c = 96$ feet apart.

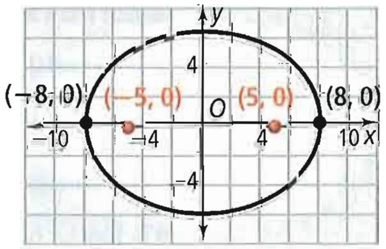
Got It? **3.** How far apart are the foci of an ellipse with a major axis of 26 ft and a minor axis of 10 ft?

Problem 4 Using the Foci of an Ellipse

Think

How can you write the equation of an ellipse given a focus and a vertex?
Find the values of *a* and *c*. Use a^2 and c^2 to find b^2.

What is the standard form equation of the ellipse shown?



The foci are on the *x*-axis, so the major axis is horizontal.

Since $c = 5$ and $a = 8$, $c^2 = 25$ and $a^2 = 64$.

$c^2 = a^2 - b^2$

$25 = 64 - b^2$ Substitute.

$b^2 = 39$ Solve.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Standard form of a horizontal ellipse

$\frac{x^2}{64} + \frac{y^2}{39} = 1$ Substitute.

Got It? **4.** What is the standard form equation of an ellipse with foci at $(0, \pm\sqrt{17})$ and co-vertices at $(\pm 6, 0)$?

Lesson Check

Do you know HOW?

1. What is an equation in standard form of an ellipse with co-vertices $(0, \pm 6)$ and major axis with length 16?

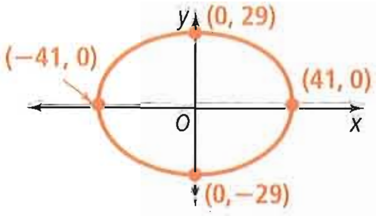
2. What are the coordinates of the foci of an ellipse with the equation $4x^2 + 25y^2 = 100$?

3. What is an equation in standard form of an ellipse centered at the origin with vertices $(\pm 13, 0)$ and foci $(\pm 12, 0)$?

4. How far apart are the foci of an ellipse with a major axis of 32 ft and minor axis of 14 ft?

Do you UNDERSTAND?

5. Error Analysis A student claims that an equation of the ellipse shown is $\frac{x^2}{41} + \frac{y^2}{29} = 1$. Describe the student's error. What is the correct equation in standard form of the ellipse?



6. Reasoning Explain why a circle is a special case of an ellipse.

Practice and Problem-Solving Exercises

A Practice

Write an equation of an ellipse in standard form with center at the origin and with the given vertex and co-vertex listed respectively.

See Problem 1.

7. (4, 0), (0, 3)
8. (0, 1), (2, 0)
9. (3, 0), (0, −1)
10. (0, 6), (1, 0)
11. (0, −7), (4, 0)
12. (−6, 0), (0, 5)
13. (−9, 0), (0, −2)
14. (0, 5), (−3, 0)

Find the foci for each equation of an ellipse. Then graph the ellipse.

See Problem 2.

15. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
16. $\frac{x^2}{9} + \frac{y^2}{25} = 1$
17. $\frac{x^2}{81} + \frac{y^2}{49} = 1$
18. $\frac{x^2}{25} + \frac{y^2}{16} = 1$
19. $\frac{x^2}{64} + \frac{y^2}{100} = 1$
20. $3x^2 + y^2 = 9$
21. $x^2 + 4y^2 = 16$
22. $\frac{x^2}{225} + \frac{y^2}{144} = 1$

Find the distance between the foci of an ellipse. The lengths of the major and minor axes are listed respectively.

See Problem 3.

23. 40 and 24
24. 30 and 18
25. 10 and 8
26. 16 and 10
27. 20 and 16
28. 18 and 14
29. 36 and 12
30. 8 and 6

Write an equation of an ellipse for the given foci and co-vertices.

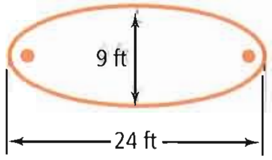
See Problem 4.

31. foci $(\pm 6, 0)$, co-vertices $(0, \pm 8)$
32. foci $(0, \pm 8)$, co-vertices $(\pm 8, 0)$
33. foci $(\pm 5, 0)$, co-vertices $(0, \pm 8)$
34. foci $(0, \pm 4)$, co-vertices $(\pm 2, 0)$

35. Miniature Golf The figure at the right represents a miniature golf green. The green is elliptical with the tee at one focus and the hole at the other.

a. How far is the hole from the tee ?

b. Knowing that the border is elliptical, how should you aim your putt from the tee?



B Apply

Find the foci for each equation of an ellipse.

36. $4x^2 + 9y^2 = 36$

37. $16x^2 + 4y^2 = 64$

38. $4x^2 + 36y^2 = 144$

39. $25x^2 + 4y^2 = 100$

40. $36x^2 + 8y^2 = 288$

41. $25x^2 + 24y^2 = 600$

42. **Think About a Plan** The open area south of the White House is known as the Ellipse, or President's Park South. It is 902 ft wide and 1058 ft long. Assume the origin is at the center of the President's Park South. What is the equation of the ellipse in standard form?

- How does the length and width of the ellipse relate to the equation?
- What does the center at the origin tell you?
- How can you write the equation of the ellipse in standard form?

43. The eccentricity of an ellipse is a measure of how nearly circular it is. Eccentricity is defined as $\frac{c}{a}$, where c is the distance from the center to a focus and a is the distance from the center to a vertex.

a. Find the eccentricity of an ellipse with foci $(\pm 9, 0)$ and vertices $(\pm 10, 0)$.

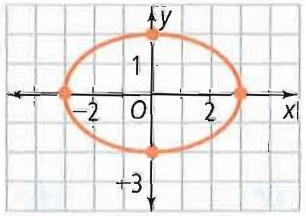
b. Find the eccentricity of an ellipse with foci $(\pm 1, 0)$ and vertices $(\pm 10, 0)$.

c. Describe the shape of an ellipse that has an eccentricity close to 0.

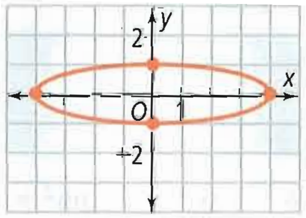
d. Describe the shape of an ellipse that has an eccentricity close to 1.

Write an equation for each ellipse.

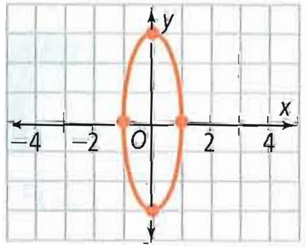
44.



45.



46.



47. **Open-Ended** Find a real-world design that uses ellipses. Place a coordinate grid over the design and write an equation of the ellipse.

Write an equation of an ellipse in standard form with center at the origin and with the given characteristics.

48. focus $(1, 0)$, width 4

49. $a = 5$, $b = 2$, width 10

50. vertex $(-11, 0)$, co-vertex $(0, 9)$

51. height 29, width 53

52. focus $(-5, 0)$, co-vertex $(0, -12)$

53. $c^2 = 68$, vertex $(0, -18)$

54. focus $(0, 3\sqrt{2})$, height 19

55. focus $(2, 0)$, x-intercept 4

56. focus $(0, -5)$, y-intercept 8

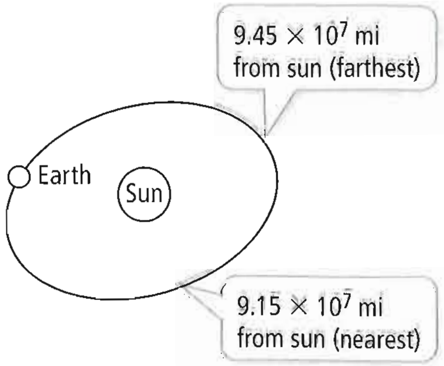
57. focus $(3, 0)$, x-intercept -6

58. $a = 3$, $b = 2$, width 4

59. $a = 2\sqrt{5}$, $b = 3\sqrt{2}$, width $6\sqrt{2}$

60. **Aerodynamics** Scientists used the Transonic Tunnel at NASA Langley Research Center, Virginia, to study the dynamics of air flow. The elliptical opening of the Transonic Tunnel is 82 ft wide and 58 ft high. What is an equation of the ellipse?

Challenge



61. **Astronomy** The sun is at a focus of Earth's elliptical orbit.

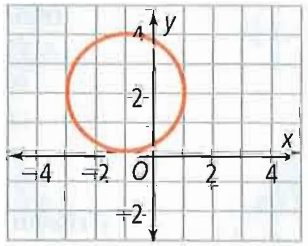
a. Find the distance from the sun to the other focus.

b. Refer to Exercise 43 for the definition of eccentricity. What is the eccentricity of the orbit?

c. Write an equation of Earth's orbit. Assume that the major axis is horizontal.

62. **Writing** The area of a circle is πr^2. The area of an ellipse is πab. Explain the connection.

Sunshine State Standards Practice



MA.912.G.6.7 **63.** Which equation is represented by the circle shown?

Ⓐ $(x - 1)^2 + (y + 2)^2 = 4$

Ⓑ $(x + 1)^2 + (y - 2)^2 = 4$

Ⓒ $(x + 2)^2 + (y - 1)^2 = 4$

Ⓓ $(x - 2)^2 + (y + 1)^2 = 4$

MA.912.A.6.5 **64.** Solve $\sqrt{x + \sqrt{2x}} = 2$. Check for extraneous solutions.

Ⓕ 2, −8 Ⓖ 0, 2 Ⓗ 2 Ⓘ −8, 1

MA.912.A.9.1 **65.** The graph of which equation contains all the points in the table below?

x	−4	−2	0	2	4
y	0	$\pm\sqrt{3}$	±2	$\pm\sqrt{3}$	0

Ⓐ $x^2 + 4y^2 = 16$ Ⓑ $4x^2 + 16y^2 = 144$ Ⓒ $4x^2 + 25y^2 = 64$ Ⓓ $9x^2 + 4y^2 = 81$

MA.912.A.5.6 **66.** **Short Response** Find the horizontal asymptote of $y = \frac{5x + 7}{x + 3}$ by dividing the numerator by the denominator. Explain your steps.

Mixed Review

Write an equation of a circle with the given center and radius. See Lesson 10-3.

67. center (1, −5), radius 3

68. center (−2, 4), radius 9

Simplify each expression. State any restrictions on the variable. See Lesson 8-4.

69. $\frac{3x}{6x^2 - 9x^5}$

70. $\frac{x^2 - 36}{x^2 + 5x - 6}$

71. $\frac{x^2 - 3x - 10}{x^3 + 8}$

Write each expression as a single logarithm. See Lesson 7-4.

72. $\log 3 + \log 5$

73. $\log_3 12 - \log_3 2$

74. $3 \log 2 - \log 4$

Get Ready! **To prepare for Lesson 10-5, do Exercises 75–76.**

Write an equation of a line in slope-intercept form using the given information. See Lesson 2-3.

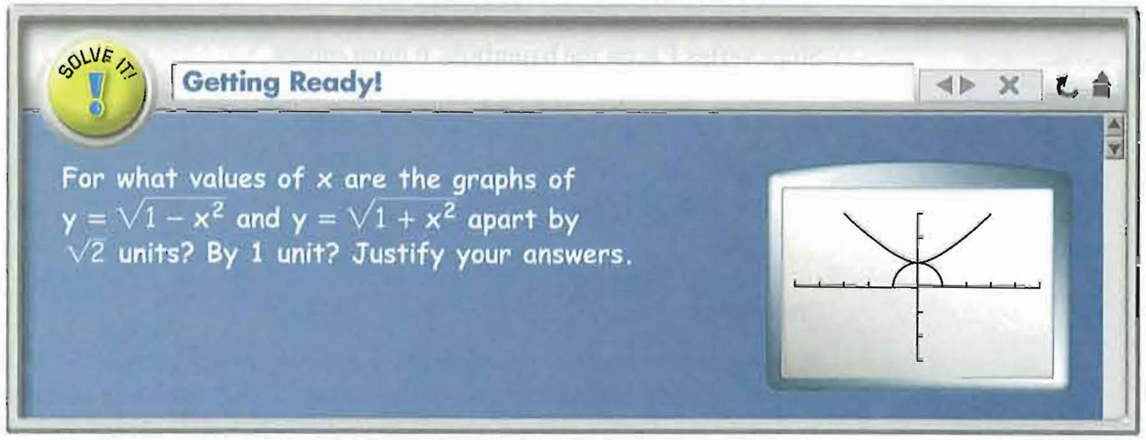
75. $m = 2$ and the y-intercept is 4

76. passes through (3, 1) and (9, 3)

10-5 Hyperbolas

Sunshine State Standard
MA.912.A.9.2 Graph conic sections with and without using graphing technology.

Objectives To graph hyperbolas
To find and use the foci of a hyperbola

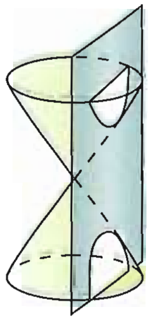


For what values of x are the graphs of $y = \sqrt{1 - x^2}$ and $y = \sqrt{1 + x^2}$ apart by $\sqrt{2}$ units? By 1 unit? Justify your answers.

Dynamic Activity
Hyperbolas

Lesson Vocabulary
- hyperbola
- focus of the hyperbola
- vertex of a hyperbola
- transverse axis
- axis of symmetry
- center of a hyperbola
- conjugate axis

In the Solve It, you saw the top halves of two different conic sections. You can complete each conic section by graphing $y = -\sqrt{1 - x^2}$ and $y = -\sqrt{1 + x^2}$ respectively.



Recall from Lesson 10-1, that you can get a variety of conic sections by slicing the double cone with a plane. Changing the angle at which the plane slices the double cone determines the shape of the curve and whether or not the plane will slice both cones. If the plane is parallel to the axis of the double cone, it slices both cones and the result is a *hyperbola*.

Essential Understanding Like the ellipse, the hyperbola's shape is determined by its distance from two foci.

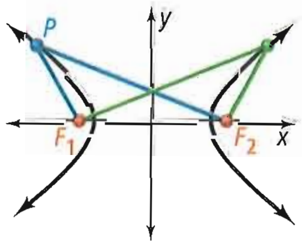
take note Key Concept Hyperbola

A **hyperbola** is the set of points P in a plane such that the absolute value of the difference between the distances from P to two fixed points F_1 and F_2 is a constant k.

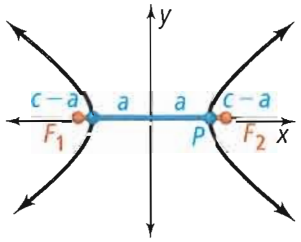
$$|PF_1 - PF_2| = k, \text{ where } k < F_1F_2$$

Each fixed point F is a **focus of the hyperbola**.

Since F_1 and F_2 are the foci of the hyperbola, the long and short segments in each of the 2 colored paths differ in length by k.



A hyperbola consists of two smooth branches. The turning point of each branch is a **vertex** of the hyperbola. The segment connecting the two vertices is the **transverse axis**, which lies on the **axis of symmetry**. The two foci also lie on the axis of symmetry. The **center of the hyperbola** is the midpoint between the two vertices, which also is the midpoint between the two foci.



Just as for an ellipse, if the foci are $(\pm c, 0)$, the distance between the two foci is $2c$. If the vertices are $(\pm a, 0)$, the distance between the vertices is $2a$.

Since vertex P is on the hyperbola, it must satisfy the equation $|PF_1 - PF_2| = k$, but you can also see that

$$\begin{aligned} |PF_1 - PF_2| &= |[2a + (c - a)] - (c - a)| \\ &= |2a + c - a - c + a| \\ &= |2a| = 2a \end{aligned}$$

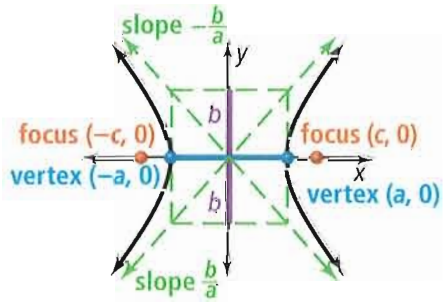
Therefore, $k = 2a$.

In a standard hyperbola, c is related to a and b by the equation $c^2 = a^2 + b^2$. The length of the **conjugate axis** is $2b$. The transverse and conjugate axes determine a rectangle that lies between the vertices, and the diagonals of that central rectangle determine the asymptotes of the hyperbola. Recall that an asymptote is a line that a graph approaches. The branches of the hyperbola will approach the asymptotes.

take note

Key Concept Properties of Hyperbolas with Center (0, 0)

Horizontal Hyperbola



Equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

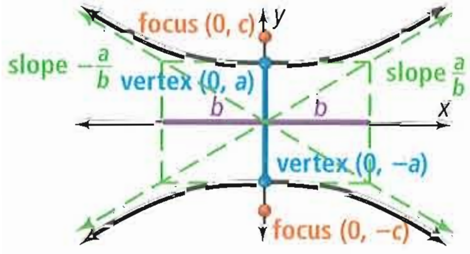
Transverse axis: Horizontal

Vertices: $(\pm a, 0)$

Foci: $(\pm c, 0)$, where $c^2 = a^2 + b^2$

Asymptotes: $y = \pm \frac{b}{a}x$

Vertical Hyperbola



Equation: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Transverse axis: Vertical

Vertices: $(0, \pm a)$

Foci: $(0, \pm c)$, where $c^2 = a^2 + b^2$

Asymptotes: $y = \pm \frac{a}{b}x$

Because of symmetry, for both the ellipse and a hyperbola, the value of c is half the distance between the two foci.

Problem 1 Writing and Graphing the Equation of a Hyperbola

A hyperbola centered at (0, 0) has vertices (±4, 0) and one focus (5, 0).

Think

Is the transverse axis horizontal or vertical?

The vertices and focus are on a horizontal line. The transverse axis is horizontal.

A What is the standard-form equation of the hyperbola?

The vertices are $(\pm 4, 0)$, so $a = 4$. One focus is $(5, 0)$, so $c = 5$.

The transverse axis is horizontal.

Use $c^2 = a^2 + b^2$ to find b: $5^2 = 4^2 + b^2$, so $b = 3$.

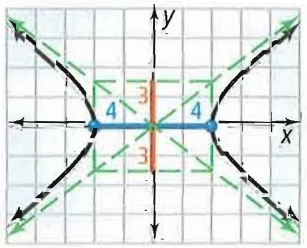
Write the equation of a horizontal hyperbola in standard form, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$ Substitute values for a and b.

$\frac{x^2}{16} - \frac{y^2}{9} = 1$ Simplify.

B Sketch the hyperbola. Use a graphing calculator to check.

Step 1 Draw the horizontal transverse axis, vertices, and central rectangle. The central rectangle guides the drawing of the graph. It shares a center with the hyperbola and in this case has a height of $2b$ and a width of $2a$. If the hyperbola were vertical the dimensions would be reversed.



Step 2 Extend the diagonals of the rectangle to show the asymptotes.

Step 3 Sketch the branches from the vertices.

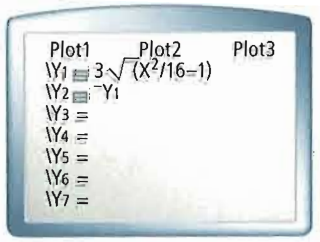
Check Solve for y,

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

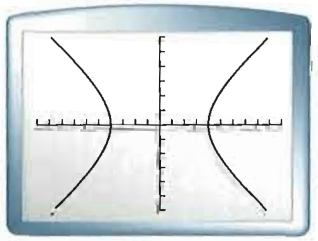
$$\frac{y^2}{9} = \frac{x^2}{16} - 1$$

$$y^2 = 9\left(\frac{x^2}{16} - 1\right)$$

$$y = \pm 3\sqrt{\frac{x^2}{16} - 1}.$$



```
Plot1  Plot2  Plot3
\Y1 = 3√(X²/16−1)
\Y2 = −Y1
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```



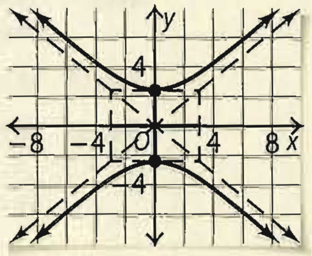
Got It? **1. a.** What is the standard-form equation of the hyperbola with vertices $(0, \pm 4)$ and foci $(0, \pm 5)$?

b. Sketch the hyperbola. Use a graphing calculator to check.

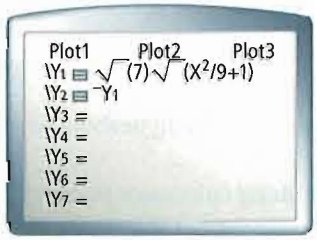
c. Reasoning Under what circumstances are the asymptotes of the graph of a hyperbola perpendicular?

Problem 2 Analyzing a Hyperbola from Its Equation

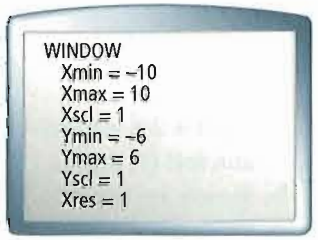
What are the vertices, foci, and asymptotes of the hyperbola with equation $9y^2 - 7x^2 = 63$? Sketch the graph. Use a graphing calculator to check your sketch.

Think	Write
Write the equation.	$9y^2 - 7x^2 = 63$
In standard form, the right side must be 1. Divide each side by 63.	$\frac{9y^2}{63} - \frac{7x^2}{63} = 1$
Simplify. Since y^2 has the positive coefficient, the hyperbola is vertical. The vertices and foci are on the y-axis.	$\frac{y^2}{7} - \frac{x^2}{9} = 1$
Compare to $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ to find a^2 and b^2.	$a^2 = 7$ and $b^2 = 9$ $a = \pm\sqrt{7}$ $b = \pm 3$
Find c^2. Use $c^2 = a^2 + b^2$.	$c^2 = 7 + 9$, so $c = \pm\sqrt{7+9} = \pm 4$
You know a, b, and c. You can answer the questions and draw the graph. $\sqrt{7} \approx 2.65$	Vertices: $(0, \pm\sqrt{7})$, Foci: $(0, \pm 4)$. Slopes of asymptotes: $m = \pm\frac{\sqrt{7}}{3}$ Asymptotes: $y = \pm\frac{\sqrt{7}}{3}x$ 

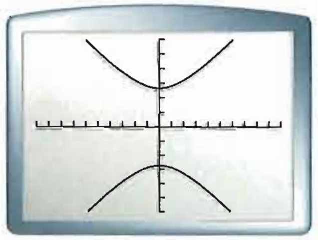
Check



```
Plot1   Plot2   Plot3
\Y1 ≡ √(7)√(X²/9+1)
\Y2 ≡ ⁻Y1
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

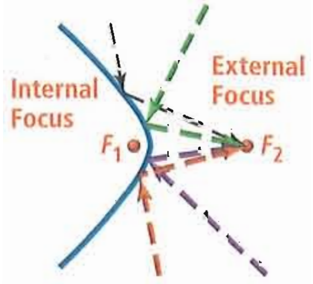


```
WINDOW
  Xmin = −10
  Xmax = 10
  Xscl = 1
  Ymin = −6
  Ymax = 6
  Yscl = 1
  Xres = 1
```



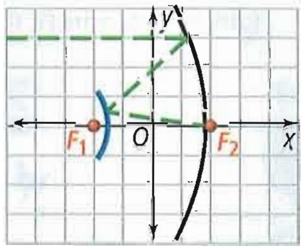
Got It? **2.** What are the vertices, foci, and asymptotes of the hyperbola with equation $9x^2 - 4y^2 = 36$? Sketch a graph. Use a graphing calculator to check your sketch.

The *reflection property of a hyperbola* is important in optics. As with an ellipse, the reflection property of a hyperbola involves both foci, but only one branch reflects. Any ray on the *external side* of a branch directed at its internal focus will reflect off the branch toward the *external focus*.



Problem 3 Modeling with a Hyperbola

Communications The graph shows a 2-dimensional view of a satellite dish. The focus is located at F_1 but the receiving device is located on the bottom of the dish at the point F_2. The rays are reflected by the first reflector (the black curve), toward F_1 and then reflected by the second reflector (the blue curve) toward F_2.



Think

What information does the diagram give up?
It helps you see the relative positions of the reflectors and foci.

A What kind of curve is the second reflector? How can you tell?

The second reflector is a hyperbola because it reflects rays aimed at its internal focus toward its external focus.

B The vertex of the second reflector is 3 in. from F_1 and 21 in. from F_2. What is an equation for the second reflector? Assume the conic is horizontal and centered at the origin.

Step 1 Determine the standard-form equation of the conic.
The conic is a horizontal hyperbola centered at the origin.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Step 2 Find c.

The distance between the foci is $3 + 21 = 24$ in.
Since c is half this distance, $c = 12$.

Step 3 Find a.

The distance from the internal focus to the vertex of the reflector is 3 in.
So, $c - a = 12 - a = 3$ and $a = 9$.

Step 4 Use c and a to find b^2.

$$b^2 = c^2 - a^2$$
$$= 12^2 - 9^2 \quad \text{Substitute and simplify.}$$
$$= 63 \quad \text{Simplify.}$$

Step 5 Use a and b^2 to write the equation.

An equation is $\frac{x^2}{81} - \frac{y^2}{63} = 1$.

Got It? **3.** Suppose the vertex of the second reflector in Problem 3 were 4 in. from F_1 and 18 in. from F_2. What is the equation for the second reflector? Assume the conic is horizontal and centered at the origin.

Lesson Check

Do you know HOW?

Find the vertices and foci of each hyperbola. Write the slopes of the asymptotes. Then sketch the graph.

1. $\frac{x^2}{36} - \frac{y^2}{25} = 1$

2. $\frac{y^2}{16} - \frac{x^2}{25} = 1$

3. $4y^2 - x^2 = 16$

4. $16x^2 - 25y^2 = 400$

5. What is an equation of a hyperbola with vertices $(\pm 5, 0)$ and focus $(7, 0)$?

Do you UNDERSTAND?

6. **Compare and Contrast** How is graphing a hyperbola like graphing an ellipse? How is it different?

7. **Error Analysis** Your friend says that a graph must be a vertical hyperbola because the larger denominator is under the y^2 term. What error did your friend make?

Practice and Problem-Solving Exercises

A Practice

Write an equation of a hyperbola with the given values, foci, or vertices. Assume that the transverse axis is horizontal.

See Problem 1.

8. $a = 3, b = 4$

9. $a = 12, c = 13$

10. $b = 9, c = 10$

11. $a = 7, b = 11$

12. foci $(\pm 13, 0)$, vertices $(\pm 12, 0)$

13. foci $(\pm 3, 0)$, vertices $(\pm 2, 0)$

Find the vertices, foci, and asymptotes of each hyperbola. Then sketch the graph.

See Problem 2.

14. $\frac{y^2}{81} - \frac{x^2}{16} = 1$

15. $\frac{y^2}{49} - \frac{x^2}{64} = 1$

16. $\frac{x^2}{121} - \frac{y^2}{144} = 1$

17. $\frac{x^2}{64} - \frac{y^2}{36} = 1$

18. $\frac{y^2}{25} - \frac{x^2}{100} = 1$

19. $81y^2 - 9x^2 = 729$

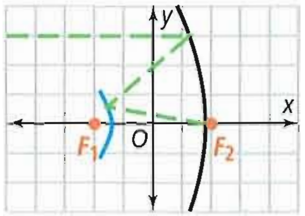
20. $4y^2 - 25x^2 = 100$

21. $36x^2 - 8y^2 = 288$

22. $14y^2 - 28x^2 = 448$

See Problem 3.

23. **Satellite Dish** The diagram at the right models a satellite dish and the small reflector inside it. Suppose F_1 and F_2 are 7 meters apart, and F_1 is 1 meter from the vertex of the small reflector. What equation best models the small reflector?



B Apply

24. **Think About a Plan** The path that Voyager 2 made around Jupiter followed one branch of a hyperbola. Find an equation that models the path of Voyager 2 around Jupiter, given that $a = 2{,}184{,}140$ km and $c = 2{,}904{,}906.2$ km. Use the horizontal model.

- What information do you need to write the equation?
- How can you use the given information to find the information you need?

Write an equation of a hyperbola with the given foci and vertices.

25. foci $(\pm 5, 0)$, vertices $(\pm 3, 0)$

26. foci $(0, \pm 13)$, vertices $(0, \pm 5)$

27. foci $(0, \pm 2)$, vertices $(0, \pm 1)$

28. foci $(\pm\sqrt{5}, 0)$, vertices $(\pm 2, 0)$

Write an equation of a hyperbola from the given information. Assume the center of each hyperbola is (0, 0).

29. Transverse axis is vertical and is 9 units; central rectangle is 9 units by 4 units

30. Perimeter of central rectangle is 16 units; vertices are (0, 3) and (0, −3)

31. (Distance from the center of a hyperbola to a focus)$^2 = 96$; endpoints of the transverse axis are at $(-\sqrt{32}, 0)$ and $(\sqrt{32}, 0)$.

Graphing Calculator **Solve each equation for y. Graph each relation on your graphing calculator. Use the TRACE feature to locate the vertices.**

32. $x^2 - 2y^2 = 4$

33. $x^2 - y^2 = 1$

34. $3x^2 - y^2 = 2$

Graph each equation.

35. $5x^2 - 12y^2 = 120$

36. $16x^2 - 20y^2 = 560$

37. $\frac{y^2}{20} - \frac{x^2}{5} = 1$

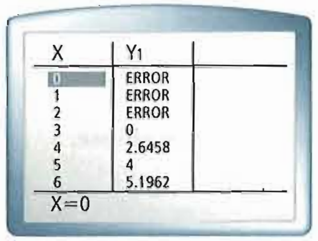
38. **Comets** The path of a comet around the sun followed one branch of a hyperbola. Find an equation that models its path around the sun, given that $a = 40$ million miles and $c = 250$ million miles. Use the horizontal model.

39. **Open-Ended** Choose two points on an axis to be the vertices of a hyperbola. Choose two other points on the same axis to be the foci. Write the equation of your hyperbola and draw its graph.

40. **Error Analysis** On a test, a student found that the foci of the hyperbola with equation $\frac{y^2}{100} - \frac{x^2}{21} = 1$ were $(0, \pm\sqrt{79})$. The teacher credited the student three points out of a possible five. What did the student do right? What did the student do wrong?

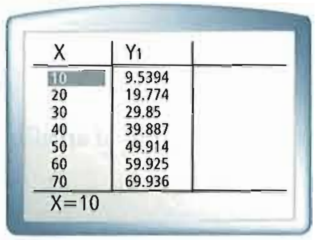
Challenge

41. The function $y = \sqrt{x^2 - 9}$ represents part of a hyperbola. The tables show the coordinates of several points on the graph.



X	Y1
0	ERROR
1	ERROR
2	ERROR
3	0
4	2.6458
5	4
6	5.1962

X=0



X	Y1
10	9.5394
20	19.774
30	29.85
40	39.887
50	49.914
60	59.925
70	69.936

X=10

a. Explain why ERROR appears for some entries.

b. Describe the relationship between the x- and y-coordinates as x increases.

c. **Reasoning** Do you think that the x- and y-coordinates will ever be equal? Explain.

d. **Make a Conjecture** What are the equations of the asymptotes of this hyperbola? Verify your answer by drawing the complete graph.

42. Air Traffic Control Suppose you are an air traffic controller directing the pilot of a plane on a hyperbolic flight path. You and another air traffic controller from a different airport send radio signals to the pilot simultaneously. The two airports are 48 km apart. The pilot's instrument panel tells him that the signal from your airport always arrives 100 μs (microseconds) before the signal from the other airport.

- **a.** Which airport is the plane closer to?
- **b.** If the signals travel at a rate of 300 m/μs, what is the difference in distances from the plane to the two airports?
- **c.** Write the equation of the flight path. (*Hint:* $k = 2a$)
- **d.** Draw the hyperbola. Which branch represents the flight path?

Sunshine State Standards Practice

MA.912.A.9.2

43. The graph of $\frac{x^2}{16} - \frac{y^2}{4} = 1$ is a hyperbola. Which set of equations represents the asymptotes of the hyperbola's graph?

Ⓐ $y = \frac{1}{2}x, y = -\frac{1}{2}x$

Ⓒ $x = \frac{1}{2}y, x = -\frac{1}{2}y$

Ⓑ $y = 2x, y = -2x$

Ⓓ $y = \frac{1}{4}x, y = -\frac{1}{4}x$

MA.912.A.5.3

44. Simplify $\dfrac{\frac{1}{y} - \frac{1}{x}}{\frac{1}{xy} - 1}$.

Ⓕ $-\dfrac{y - x}{xy - 1}$

Ⓖ $\dfrac{x - y}{1 - xy}$

Ⓗ $\dfrac{x + y}{1 + xy}$

Ⓘ $x + y$

MA.912.A.8.3

45. How is the graph of $y = 4 \cdot \left(\frac{1}{2}\right)^{x-3}$ translated from the graph of $y = 4 \cdot \left(\frac{1}{2}\right)^{x}$?

Ⓐ 3 units right

Ⓑ 3 units left

Ⓒ 3 units down

Ⓓ 3 units up

MA.912.D.11.2

46. Short Response Using sigma notation, what is an expression for the sum of a 6-term arithmetic sequence with first term of 3 and a common difference of 4? What is the sum?

Mixed Review

Find the foci for each equation of an ellipse. Then graph the ellipse. **See Lesson 10-4.**

47. $\frac{x^2}{34} + \frac{y^2}{25} = 1$

48. $3x^2 + 2y^2 = 6$

49. $25x^2 + 16y^2 = 1600$

Solve each equation. Check your answers. **See Lesson 7-5.**

50. $8^{2x} = 4$

51. $\log 8x = 3$

52. $2 \log_3 x - \log_3 4 = 2$

Get Ready! **To prepare for Lesson 10-6, do Exercises 53–55.**

Rewrite each function in vertex form. **See Lesson 4-2.**

53. $y = x^2 - 6x + 1$

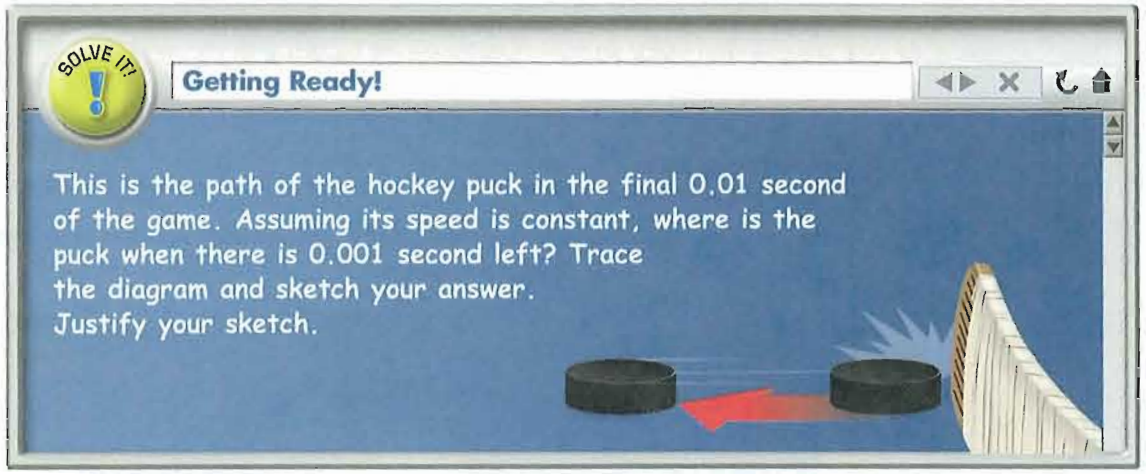
54. $y = 2x^2 + 12x$

55. $y = 3x^2 + 24x - 2$

10-6 Translating Conic Sections

nshine State Standards
MA.912.A.9.1 Write the equations of conic sections in standard form and general form, in order to identify the conic section and to find its geometric properties.
MA.912.A.9.2 Graph conic sections.

Objectives To write the equation of a translated conic section
To identify a translated conic section from an equation



This is the path of the hockey puck in the final 0.01 second of the game. Assuming its speed is constant, where is the puck when there is 0.001 second left? Trace the diagram and sketch your answer. Justify your sketch.

In this lesson you will practice translation skills to locate ellipses and hyperbolas when their centers move from (0, 0) to (h, k). You will not need to relearn the geometry of these curves. Each will still depend on the same distances, a, b, and c that relate to their vertices and foci.

Essential Understanding In a relation with an x-y relationship, replacing x by $x - h$ and y by $y - k$ (with $h > 0$ and $k > 0$) translates the graph of the relation h units to the right and k units up.

The summary tables in this lesson are like those from the lesson on parabolas. Notice how each entry in the "Center (0, 0)" column relates to the corresponding entry in the "Center (h, k)" column.

take note **Summary** **Translating Horizontal Ellipses**

Horizontal Ellipse	**Center (0, 0)**	**Center (h, k)**
Standard-Form Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
Vertices	$(\pm a, 0)$	$(h \pm a, k)$
Co-vertices	$(0, \pm b)$	$(h, k \pm b)$
Foci	$(\pm c, 0)$	$(h \pm c, k)$
a, b, c relationship, $a > b > 0$	$c^2 = a^2 - b^2$	$c^2 = a^2 - b^2$

Summary Translating Vertical Ellipses

Vertical Ellipse	Center (0, 0)	Center (*h*, *k*)
Standard-Form Equation	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Vertices	$(0, \pm a)$	$(h, k \pm a)$
Co-vertices	$(\pm b, 0)$	$(h \pm b, k)$
Foci	$(0, \pm c)$	$(h, k \pm c)$
a, b, c relationship, $a > b > 0$	$c^2 = a^2 - b^2$	$c^2 = a^2 - b^2$

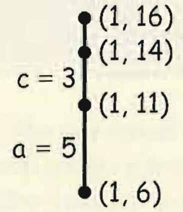
Problem 1 Writing an Equation of a Translated Ellipse

What is the standard-form equation of an ellipse with vertices (1, 6) and (1, 16), and one focus at (1, 14)? Sketch the ellipse.

Think

Graph the points to find the orientation. The ellipse is vertical with center at (1, 11), halfway between the two vertices.

Write



The vertices are 5 units from the center. The focus is 3 units from the center. Substitute values into $b^2 = a^2 - c^2$.

$a = 5$, so $a^2 = 25$

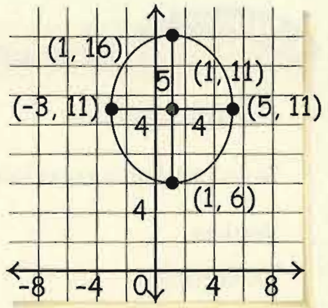
$c = 3$, so $c^2 = 9$

$b^2 = 25 - 9 = 16$

Write an equation for the vertical ellipse using $h = 1$, $k = 11$, $a^2 = 25$, and $b^2 = 16$.

$$\frac{(x-1)^2}{16} + \frac{(y-11)^2}{25} = 1$$

Sketch the ellipse. Draw a smooth curve through the vertices and co-vertices.



Got It? **1.** What is the standard-form equation of an ellipse with vertices (2, 3) and (22, 3), and one focus at (6, 3)? Sketch the ellipse.

take note Summary Translating Horizontal and Vertical Hyperbolas

Horizontal Hyperbola	**Center (0, 0)**	**Center (*h*, *k*)**
Standard-Form Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
Vertices	$(\pm a, 0)$	$(h \pm a, k)$
Foci	$(\pm c, 0)$	$(h \pm c, k)$
Asymptotes	$y = \pm \frac{b}{a}x$	$y - k = \pm \frac{b}{a}(x - h)$
a, b, c relationship	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$

Vertical Hyperbola	**Center (0, 0)**	**Center (*h*, *k*)**
Standard-Form Equation	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Vertices	$(0, \pm a)$	$(h, k \pm a)$
Foci	$(0, \pm c)$	$(h, k \pm c)$
Asymptotes	$y = \pm \frac{a}{b}x$	$y - k = \pm \frac{a}{b}(x - h)$
a, b, c relationship	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$

As with the ellipse, you can identify the characteristics of a hyperbola just by analyzing its equation.

Problem 2 Analyzing a Hyperbola from Its Equation

What are the center, vertices, foci, and asymptotes of the hyperbola with equation $\frac{(y-1)^2}{25} - \frac{(x-3)^2}{144} = 1$? Sketch the graph.

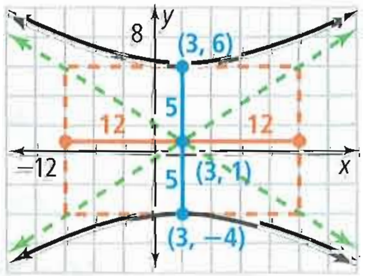
Think
How can you find the asymptotes?
Use the point-slope form of a line to find the equations for the asymptotes.

The equation is of the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, so the hyperbola is vertical with $h = 3$, $k = 1$, $a^2 = 25$, $b^2 = 144$, and $c^2 = a^2 + b^2 = 25 + 144 = 169$.

Center: $(h, k) = (3, 1)$
Vertices: $(h, k \pm a) = (3, 1 \pm 5)$; $(3, 6)$ and $(3, -4)$
Foci: $(h, k \pm c) = (3, 1 \pm 13)$; $(3, 14)$ and $(3, -12)$

The equations of the asymptotes are $y - 1 = \frac{5}{12}(x - 3)$ and $y - 1 = -\frac{5}{12}(x - 3)$.

To graph the hyperbola, first graph the vertices and central rectangle. Draw the asymptotes. Then sketch the branches through the vertices and along the asymptotes.



Got It? **2.** What are the center, vertices, foci, and asymptotes of the hyperbola with equation $\frac{(x-2)^2}{36} - \frac{(y+2)^2}{64} = 1$? Sketch the graph.

All equations for conic sections expand to the general equation form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A and C are not both equal to zero. If the conic section is horizontal or vertical, $B = 0$ and the general form becomes

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

To locate the center or the two foci of a conic section, you must first convert the general form equation into the standard form for that conic section. This usually involves completing the square at least once and possibly twice.

Problem 3 Identifying a Translated Conic Section

Multiple Choice **Which conic section has the equation $4x^2 + y^2 - 24x + 6y + 9 = 0$?**

Ⓐ circle; center $(3, -3)$

Ⓑ ellipse; foci $(3, -3 - 3\sqrt{3})$ and $(3, -3 + 3\sqrt{3})$

Ⓒ circle; center $(-3\sqrt{3}, 3\sqrt{3})$

Ⓓ ellipse; foci $(0, 3)$ and $(0, -3)$

Think

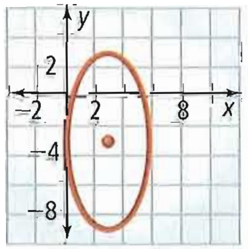
How do you complete the square?
Divide the x (or y) coefficient by 2. Square the result.

Complete the square for the x- and y-terms to write the equation in standard form.

$$\begin{aligned}
4x^2 + y^2 - 24x + 6y + 9 &= 0 \\
4x^2 - 24x + y^2 + 6y &= -9 && \text{Group the } x\text{- and } y\text{-terms.} \\
4(x^2 - 6x) + (y^2 + 6y) &= -9 && \text{Factor.} \\
4(x^2 - 6x + (-3)^2) + (y^2 + 6y + 3^2) &= -9 + 4(-3)^2 + 3^2 && \text{Complete the square.} \\
4(x^2 - 6x + 9) + (y^2 + 6y + 9) &= -9 + 36 + 9 && \text{Simplify.} \\
4(x - 3)^2 + (y + 3)^2 &= 36 && \text{Factor.} \\
\frac{4(x - 3)^2}{36} + \frac{(y + 3)^2}{36} &= 1 && \text{Divide by 36 so the right side is 1.} \\
\frac{(x - 3)^2}{9} + \frac{(y + 3)^2}{36} &= 1 && \text{Simplify.}
\end{aligned}$$

The equation represents a vertical ellipse. The center is $(3, -3)$. Since $a^2 = 36$, $a = 6$. Since $b^2 = 9$, $b = 3$. Use these values to locate the foci.

$$\begin{aligned}
c^2 &= a^2 - b^2 \\
&= 36 - 9 \\
&= 27 \\
c &= 3\sqrt{3}
\end{aligned}$$



The distance from the center $(3, -3)$ of the vertical ellipse to the foci is $3\sqrt{3}$. The foci are at $(3, -3 - 3\sqrt{3})$ and $(3, -3 + 3\sqrt{3})$.

The correct choice is B.

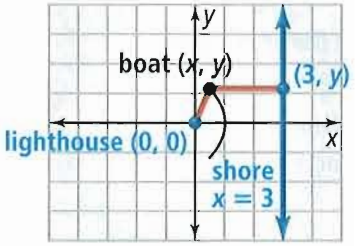
Got It? **3. a.** Which type of conic section has equation $x^2 + y^2 - 12x + 4y = 8$? What is its center? Sketch the graph.

b. **Reasoning** Using as few changes as possible, modify the equation $4x^2 + y^2 - 24x + 6y + 9 = 0$ to make it an equation of a hyperbola.

Problem 4 Modeling With a Conic Section

Navigation **A lighthouse is on an island 3 miles from a long, straight shoreline. A boat sails around the island; deliberately following a path that always keeps it twice as far from the shoreline as it is from the lighthouse. What is an equation of the conic section describing the boat's path?**

Draw a diagram of the situation with the lighthouse at (0, 0), the shore at the line $x = 3$, and the boat at an arbitrary point (x, y) satisfying the given distance condition.



The distance from the boat to the lighthouse is $\sqrt{x^2 + y^2}$ and the distance from the boat to the shore is the horizontal difference $3 - x$. The distance condition translates to:

$$2\sqrt{x^2 + y^2} = 3 - x$$

$$4(x^2 + y^2) = (3 - x)^2 \qquad \text{Square each side.}$$

$$4x^2 + 4y^2 = 9 - 6x + x^2 \qquad \text{Expand.}$$

$$3x^2 + 6x + 4y^2 = 9 \qquad \text{Combine like terms.}$$

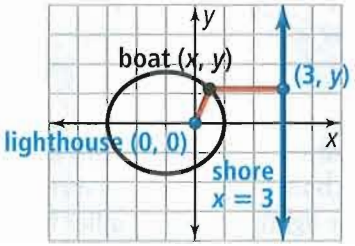
$$3(x^2 + 2x + 1) + 4y^2 = 9 + 3(1) \qquad \text{Complete the square.}$$

$$3(x + 1)^2 + 4y^2 = 12 \qquad \text{Simplify.}$$

$$\frac{3(x + 1)^2}{12} + \frac{4y^2}{12} = 1 \qquad \text{Divide each side by 12 so the right side equals 1.}$$

$$\frac{(x + 1)^2}{4} + \frac{y^2}{3} = 1 \qquad \text{Simplify.}$$

This is the standard-form equation for a horizontal ellipse centered at $(-1, 0)$ with major axis of length $2a = 4$ and minor axis of length $2b = 2\sqrt{3}$. Note that $c = \sqrt{a^2 - b^2} = \sqrt{4 - 3} = 1$, so the lighthouse is at one focus of the ellipse.



The boat follows an elliptical path modeled by the equation $\frac{(x + 1)^2}{4} + \frac{y^2}{3} = 1$.

Got It? **4.** If the lighthouse were 8 miles from the shore and the boat were to stay 3 times as far from the shore as from the lighthouse, what would be the equation of the conic section describing the boat's path?

Lesson Check

Do you know HOW?

Identify the center, vertices, and foci of the ellipse or hyperbola.

1. ellipse: $\frac{(x+7)^2}{225} + \frac{(y+1)^2}{144} = 1$

2. hyperbola: $\frac{(x-1)^2}{9} - \frac{(y-3)^2}{4} = 1$

3. Write the standard-form equation of the ellipse with vertices $(0, -4)$ and $(0, 12)$ and with a focus $(0, 0)$.

4. Write the standard-form equation of the hyperbola with vertices $(-3, 6)$ and $(-3, -8)$ and with foci $(-3, -11)$ and $(-3, 9)$.

Do you UNDERSTAND?

5. **Vocabulary** Which of the conic sections have more than one focus: circle, parabola, ellipse, hyperbola?

6. **Error Analysis** Your friend said that the points $(-14, 1)$ and $(10, 1)$ are the vertices of the graph of the equation $\frac{(x-2)^2}{144} - \frac{(y+1)^2}{25} = 1$. What error did your friend make?

7. **Reasoning** A student claims that the graph of the equation $x^2 + y^2 + 8x - 10y + 41 = 0$ is a circle with center at $(-4, 5)$ and radius $\sqrt{41}$. Explain the student's error and describe the correct graph.

Practice and Problem-Solving Exercises

A Practice

Write the standard-form equation of an ellipse with the given characteristics. Sketch the ellipse.

See Problem 1.

8. vertices $(-5, 1)$ and $(1, 1)$, focus $(-3, 1)$

9. vertices $(3, -1)$ and $(3, -11)$, focus $(3, -4)$

10. vertices $(9, 9)$ and $(9, -5)$, focus $(9, 6)$

11. vertices $(-5, 4)$ and $(8, 4)$, focus $(-4, 4)$

Identify the center, vertices, and foci of each hyperbola.

See Problem 2.

12. $\frac{(x+11)^2}{16} - \frac{y^2}{9} = 1$

13. $\frac{(y-4)^2}{9} - \frac{(x-3)^2}{4} = 1$

14. $\frac{(y+8)^2}{4} - \frac{(x+3)^2}{49} = 1$

Identify each conic section by writing the equation in standard form and sketching the graph. For a parabola, give the vertex. For a circle, give the center and the radius. For an ellipse or a hyperbola, give the center and the foci.

See Problem 3.

15. $x^2 - 8x - y + 19 = 0$

16. $3x^2 + 6x + y^2 - 6y = -3$

17. $y^2 - x^2 + 6x - 4y = 6$

18. $x^2 - 4y^2 - 2x - 8y = 7$

19. $y^2 - 2x - 4y = -10$

20. $x^2 + y^2 - 4x - 6y - 3 = 0$

See Problem 4.

21. **Navigation** A lighthouse is on an island 4 miles from a long, straight shoreline. When a boat is directly between the lighthouse and the shoreline, it is 1 mile from the lighthouse and 3 miles from the shore. As it sails away from the shore and lighthouse, it continues so that the difference in distances between boat and lighthouse and between boat and shore is always 2 miles.

a. What conic section models this problem?

b. What part of the graph does the lighthouse represent? The shoreline?

c. What equation represents the path of the boat?

B Apply

22. Think About a Plan An ellipse has center (3, 2), one vertex (9, 2), and one co-vertex (3, −1). Sketch its graph. Then write its equation.

- How can the sketch help you write the equation?
- What information do you need to write the equation?

23. Reasoning Use the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ to identify the shape of the graph that results in each case.

a. $A = C = D = E = 0, B \neq 0, F \neq 0$

b. $A = B = C = D = 0, E \neq 0, F \neq 0$

Sketch each conic section. Then write its equation.

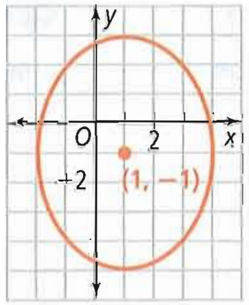
24. A parabola has vertex (2, −3) and focus (2, 5).

25. A hyperbola has center (6, −3), one focus (6, 0), and one vertex (6, −1).

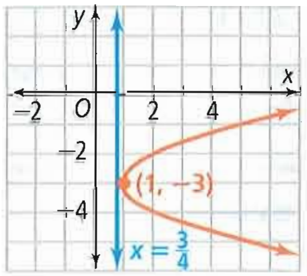
26. Theater Arts The director of a stage show asks you to design an elliptical platform. Her sketch shows the platform centered at (9, 7) from the front left corner of the stage. The platform has a 12-ft major axis parallel to the front edge of the stage and extends to within 3 ft of the edge. Write an equation that models the platform.

Write an equation for each graph.

27.



28.



The graph of each equation is to be translated 2 units left and 4 units up. Write each new equation.

29. $(x - 2)^2 + (y + 4)^2 = 16$

30. $\frac{(x - 3)^2}{64} + \frac{(y - 3)^2}{36} = 1$

31. $y = 2x^2$

32. $9x^2 + 3x + 10 = 16y^2 + 154 + 3x$

Graph each pair of functions. Identify the conic section represented by the graph and write the functions as a single equation in standard form.

33. $y = \sqrt{36 - 4x^2}$
$y = -\sqrt{36 - 4x^2}$

34. $y = \sqrt{4x^2 - 36}$
$y = -\sqrt{4x^2 - 36}$

35. $y = 0.5\sqrt{36 - x^2}$
$y = -0.5\sqrt{36 - x^2}$

C Challenge

36. Open-Ended On a graphing calculator, create a design using three translated quadratic relations.

37. **Astronomy** The dimensions of the elliptical orbits of three planets are given in millions of kilometers in the table. The sun is at one focus. The other focus is on the positive x-axis.

a. Write an equation for each orbit and draw the curves on your graphing calculator. (Remember to adjust the viewing window.)

b. **Reasoning** Which orbit is most circular? Justify your reasoning.

Planet	a	b
Earth	149.60	149.58
Mars	227.9	226.9
Mercury	57.9	56.6

Sunshine State Standards Practice

MA.912.A.9.1 **38.** What is the standard form of the equation of the conic given by $2x^2 + 2y^2 + 4x - 12y - 22 = 0$?

Ⓐ $\frac{(x+1)^2}{21} - \frac{(y-3)^2}{21} = 1$

Ⓑ $\frac{(x+1)^2}{21} + \frac{(y-3)^2}{21} = 1$

Ⓒ $\frac{(x-3)^2}{21} + \frac{(y+1)^2}{21} = 1$

Ⓓ $\frac{(x-1)^2}{7} + \frac{(y+3)^2}{3} = 1$

MA.912.A.7.10 **39.** Using a calculator, what are the approximate solutions of $x^2 - 7x + 5 = 0$?

Ⓕ $-0.65, 7.65$ Ⓖ $-7.65, 0.65$ Ⓗ $-1.14, 6.14$ Ⓘ $0.81, 6.19$

MA.912.G.6.6 **40.** What is the center of the circle with equation $(x+3)^2 + (y-2)^2 = 49$?

Ⓐ $(3, -2)$ Ⓑ $(-3, 2)$ Ⓒ $(3, 2)$ Ⓓ $(-3, -2)$

MA.912.D.11.3 **41.** **Short Response** How can you use the arithmetic mean to find the missing terms in the arithmetic sequence 15, ■, ■, ■, 47, . . . ?

Mixed Review

Find the foci of each hyperbola. Draw the graph. **See Lesson 10-5.**

42. $\frac{x^2}{49} - \frac{y^2}{36} = 1$ **43.** $8y^2 - 6x^2 = 72$ **44.** $4y^2 - 100x^2 = 400$

Solve each equation. Check your answers. **See Lesson 8-6.**

45. $\frac{1}{3x+1} = \frac{1}{x^2-3}$ **46.** $\frac{2}{x+2} = \frac{6}{x^2-4}$ **47.** $\frac{5}{x^2-x} + \frac{3}{x-1} = 6$

Simplify each expression. **See Lesson 7-6.**

48. $\ln e$ **49.** $2 \ln e$ **50.** $\ln e^3$ **51.** $4 \ln e^2$

Evaluate each expression for the given value of the variable. **See Lesson 1-3.**

52. $x + 5x - x - 9; x = -2$ **53.** $(n-4)^2 + n; n = 5$

Concept Byte

For Use With Lesson 10-6

Solving Quadratic Systems

Sunshine State Standard

MA.912.A.7.7 Solve non-linear systems of equations with and without using technology.

In Chapter 3, you solved systems of linear equations algebraically and graphically. You can use the same methods to solve systems of quadratic equations.

Example 1

Solve the system algebraically. $\begin{cases} x^2 - y^2 = 9 \\ x^2 + 9y^2 = 169 \end{cases}$

$$\begin{array}{r} x^2 - y^2 = 9 \\ \underline{x^2 + 9y^2 = 169} \\ -10y^2 = -160 \end{array}$$

Subtract like terms to eliminate the x^2 term.

$y = 4$ or $y = -4$ Solve for y.

Substitute the values of y into the original equations.

$$x^2 - (4)^2 = 9 \qquad\qquad x^2 - (-4)^2 = 9$$

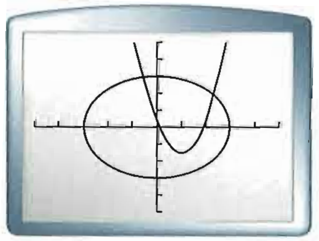
$$x^2 = 25 \qquad\qquad x^2 = 25$$

$x = 5$ or $x = -5$ Solve for x. $x = 5$ or $x = -5$

The ordered pairs (5, 4), (−5, 4), (5, −4), and (−5, −4) are solutions to the system.

Example 2

Solve the system by graphing. $\begin{cases} x^2 + y^2 = 36 \\ y = (x - 2)^2 - 3 \end{cases}$



$$x^2 + y^2 = 36$$

$$y = \pm\sqrt{36 - x^2}$$

Solve the first equation for y.

Graph the equations and find the point(s) of intersection. The solutions are approximately (−1, 5.9) and (4.6, 3.8).

Exercises

Solve each quadratic system.

1. $\begin{cases} x^2 + 64y^2 = 64 \\ x^2 + y^2 = 64 \end{cases}$

2. $\begin{cases} 2x^2 - y^2 = 2 \\ x^2 + y^2 = 25 \end{cases}$

3. $\begin{cases} 9x^2 + 25y^2 = 225 \\ y = -x^2 + 5 \end{cases}$

4. a. Writing The system that consists of $y = -3x + 6$ and $y = x^2 - 4x$ is a linear-quadratic system. How would you solve the system algebraically? Graphically?

b. Solve the system in part (a).

Identify each system as linear-quadratic or quadratic-quadratic. Then solve.

5. $\begin{cases} y = x - 1 \\ x^2 + y^2 = 25 \end{cases}$

6. $\begin{cases} 9x^2 + 4y^2 = 36 \\ x^2 - y^2 = 4 \end{cases}$

7. $\begin{cases} -x + y = 4 \\ y = x^2 - 4x + 2 \end{cases}$

8. $\begin{cases} 4x^2 + 25y^2 = 100 \\ y = x + 2 \end{cases}$

10

Pull It All Together

BIG idea Modeling

You can represent many real-world mathematical problems algebraically.

TASK 1

Imagine a plane and a cone intersecting to form a parabola.

a. Explain why the plane has to intersect the axis of the cone.

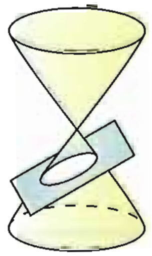
b. Imagine the plane moving so that it keeps the same angle, but its point of intersection with the axis moves in the direction of the apex of the cone and eventually passes through the apex. Describe what happens to the parabola and write equations that describe how the parabola changes.

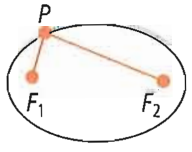
BIG idea Equivalence

You can represent any relationship in an infinite number of ways, where each representation has the same domain and the same pairing of inputs with outputs.

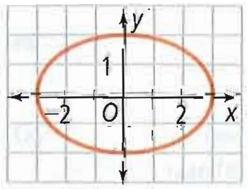
TASK 2

You can define an ellipse using a cone, a set of points, or algebra. For each kind of definition, explain how to describe a circle as a special case of an ellipse.





$PF_1 + PF_2 = k$



$\frac{x^2}{9} + \frac{y^2}{4} = 1$

BIG idea Coordinate Geometry

You can use a coordinate system to represent and analyze geometric relationships.

TASK 3

The focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is c_h units from (0, 0). Imagine the ellipse inscribed in the central rectangle of the hyperbola. Its focus is c_e units from (0, 0). How far apart are the foci $(c_e, 0)$ and $(c_h, 0)$? Give the distance in terms of a and b.

10 Chapter Review

Connecting BIG ideas and Answering the Essential Questions

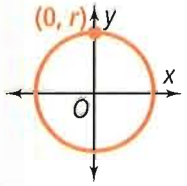
1 Modeling
The intersection of a cone and a plane parallel to the side of a cone is a parabola.

2 Equivalence
$\frac{x^2}{9} + \frac{y^2}{9} = 1$ is an equation of a circle centered at the origin with radius 3. Multiply each side by 9 to get $x^2 + y^2 = 9$.

3 Coordinate Geometry
The x^2 and y^2 terms of the algebraic form of an ellipse are both positive. For a hyperbola, one term is negative.

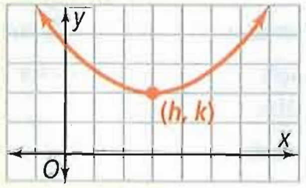
Parabolas (Lesson 10-2)
Centered at the origin,
- a parabola has equation $y = ax^2$, or $x = ay^2$.

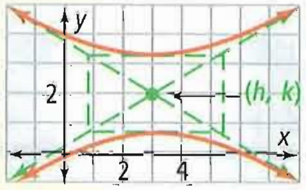
Circles (Lesson 10-3)
Centered at the origin,
- a circle with radius r has equation $x^2 + y^2 = r^2$.



Ellipses and Hyperbolas (Lessons 10-4 and 10-5)
Centered at the origin,
- an ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- a hyperbola has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Translating Conic Sections (Lesson 10-6)
Centered at (h, k),
- substitute $x - h$ for x and $y - k$ for y in the original equation for a conic section.





Chapter Vocabulary

- axis of symmetry (p. 646)
- center of a circle (p. 630)
- center of a hyperbola (p. 646)
- center of an ellipse (p. 639)
- circle (p. 630)
- conic section (p. 614)
- conjugate axis (p. 646)
- co-vertices of an ellipse (p. 639)
- directrix (p. 622)
- ellipse (p. 638)
- focal length (p. 622)
- focus of a parabola (p. 622)
- foci of an ellipse (p. 638)
- foci of a hyperbola (p. 645)
- hyperbola (p. 645)
- major axis (p. 639)
- minor axis (p. 639)
- radius (p. 630)
- standard form of an equation of a circle (p. 630)
- transverse axis (p. 646)
- vertices of an ellipse (p. 639)
- vertices of a hyperbola (p. 646)

Fill in the blanks.

1. In the definition of a parabola, a point on the curve is equidistant from the focus and the _?_.

2. The vertices of an ellipse are on its _?_.

3. $(x - h)^2 + (y - k)^2 = r^2$ is the _?_.

4. The distance from a point on a circle to its center is the _?_ of the circle.

5. The vertices of a hyperbola are on its _?_.

10-1 Exploring Conic Sections

Quick Review

A **conic section** is formed by the intersection of a plane and a double cone. Circles, ellipses, parabolas, and hyperbolas are all conic sections.

Example

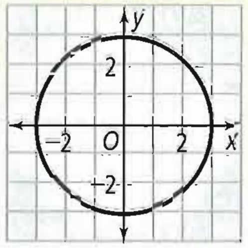
Graph the equation $x^2 + y^2 = 9$. Identify the conic section, the domain and range.

Plot points that satisfy the equation. Connect them with a smooth curve.

The graph is a circle with center (0, 0) and radius 3.

The domain is $-3 \leq x \leq 3$.

The range is $-3 \leq y \leq 3$.



Exercises

Graph each equation. Identify the conic section, any lines of symmetry, and the domain and range.

6. $\frac{x^2}{49} + \frac{y^2}{121} = 1$

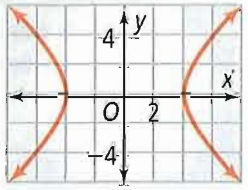
7. $x^2 + y^2 = 4$

8. $\frac{x^2}{25} - \frac{y^2}{4} = 1$

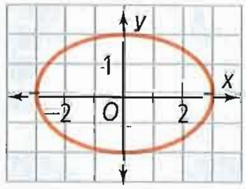
9. $x = 2y^2 + 5$

Identify the center and domain and range of each graph.

10.



11.



10-2 Parabolas

Quick Review

In a plane, a parabola is the set of all points that are the same distance, c, from a fixed point, the **focus** and a fixed line, the **directrix**.

For $y = ax^2$, if $a > 0$, the parabola opens up, and has focus $(0, c)$ and directrix $y = -c$; if $a < 0$, the parabola opens down, and has focus $(0, -c)$ and directrix $y = c$.

For $x = ay^2$, if $a > 0$, the parabola opens right, and has focus $(c, 0)$ and directrix $x = -c$; if $a < 0$, the parabola opens left, and has focus $(-c, 0)$ and directrix $x = c$. In all cases, $a = \frac{1}{4c}$.

Example

Write an equation of a parabola that opens up, with vertex at the origin and focus 1 unit from the vertex.

Since the parabola opens up, use $y = ax^2$. Since the focus is 1 unit from the vertex, $c = 1$.

$$a = \frac{1}{4c} = \frac{1}{4(1)} = \frac{1}{4}$$

An equation for the parabola is $y = \frac{1}{4}x^2$.

Exercises

Write an equation of a parabola with vertex at the origin and the given focus.

12. (5, 0)

13. (0, −5)

14. (0, 6)

Write an equation of a parabola that opens up, with vertex at the origin and a focus as described.

15. focus is 2.5 units from the vertex

16. focus is $\frac{1}{12}$ of a unit from the vertex

Write an equation of a parabola with the given focus and directrix.

17. focus: (0, 3); directrix: $y = -1$

18. focus: (−2, 0); directrix: $x = 4$

Find the focus and the directrix of the graph of each equation. Sketch the graph.

19. $y = 5x^2$

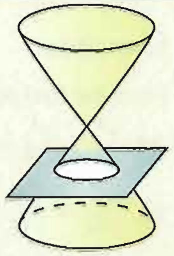
20. $x = 2y^2$

21. $x = -\frac{1}{8}y^2$

10-3 Circles

Quick Review

In a plane, a **circle** is the set of all points that are a given distance, the **radius**, r, from a given point, the **center**, (h, k).



Example

Write an equation in standard form of a circle with center $(-3, 4)$ and radius 2.

Use the standard form of the equation of a circle. Substitute -3 for h, 4 for k, and 2 for r.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - (-3))^2 + (y - 4)^2 = 2^2$$

$$(x + 3)^2 + (y - 4)^2 = 4$$

An equation for the circle is $(x + 3)^2 + (y - 4)^2 = 4$.

Exercises

Write an equation in standard form of a circle with the given center and radius.

22. center $(0, 0)$; radius 4

23. center $(8, 1)$; radius 5

Write an equation for each translation of $x^2 + y^2 = r^2$ with the given radius.

24. left 3 units, up 2 units; radius 10

25. right 5 units, down 3 units; radius 8

Find the center and the radius of each circle. Graph each circle. Describe the translation from center $(0, 0)$.

26. $(x - 1)^2 + y^2 = 64$

27. $(x + 7)^2 + (y + 3)^2 = 49$

10-4 Ellipses

Quick Review

An **ellipse** is the set of all points P, where the sum of the distances between P and two fixed points, the **foci**, is constant. The **major axis** contains the foci, and its endpoints are the **vertices of the ellipse.** For $a > b$, there are two standard forms of ellipses centered at the origin. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the major axis is horizontal with vertices $(\pm a, 0)$, foci $(\pm c, 0)$, and co-vertices $(0, \pm b)$. If $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, the major axis is vertical with vertices $(0, \pm a)$, foci $(0, \pm c)$, and co-vertices $(\pm b, 0)$. In either case, $c^2 = a^2 - b^2$.

Example

Write an equation of an ellipse with foci $(\pm 5, 0)$ and co-vertices $(0, \pm 3)$.

Since the foci are $(\pm 5, 0)$, the major axis is horizontal. Since $c = 5$ and $b = 3$, $c^2 = 25$ and $b^2 = 9$. Using the equation $c^2 = a^2 - b^2$, $a^2 = 34$.

An equation of the ellipse is $\frac{x^2}{34} + \frac{y^2}{9} = 1$.

Exercises

Write an equation of an ellipse centered at the origin, satisfying the given conditions.

28. foci $(\pm 1, 0)$; co-vertices $(0, \pm 4)$

29. vertex $(0, \sqrt{29})$; co-vertex $(-5, 0)$

30. focus $(0, 1)$; vertex $(0, \sqrt{10})$

31. foci $(\pm 2, 0)$; co-vertices $(0, \pm 6)$

32. Write an equation of an ellipse centered at the origin with height 8 units and width 16 units.

33. Find the foci of the graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Graph the ellipse.

10-5 Hyperbolas

Quick Review

A **hyperbola** is the set of all points P such that the absolute value of the difference of the distances from P to two fixed points, the **foci**, is constant. There are two standard forms of hyperbolas centered at the origin. If $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the asymptotes are $y = \pm\frac{b}{a}x$, the **transverse axis** is horizontal with vertices $(\pm a, 0)$, and the foci are $(\pm c, 0)$. If $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, the asymptotes are $y = \pm\frac{a}{b}x$, the transverse axis is vertical with vertices $(0, \pm a)$, and the foci are $(0, \pm c)$. In either case, $c^2 = a^2 + b^2$.

Example

Find the foci of the graph of $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

The equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, so the transverse axis is horizontal; $a^2 = 25$ and $b^2 = 9$.

Using the Pythagorean Theorem to find c,
$c = \sqrt{25 + 9} = \sqrt{34} \approx 5.8$.

The foci, $(\pm c, 0)$, are approximately $(5.8, 0)$ and $(-5.8, 0)$.

Exercises

Find the foci of each hyperbola. Graph the hyperbola.

34. $\frac{x^2}{36} - \frac{y^2}{225} = 1$

35. $\frac{y^2}{400} - \frac{x^2}{169} = 1$

36. $\frac{x^2}{121} - \frac{y^2}{81} = 1$

Write an equation of a hyperbola with the given foci and vertices.

37. foci $(\pm 17, 0)$, vertices $(\pm 8, 0)$

38. foci $(0, \pm 25)$, vertices $(0, \pm 7)$

39. Find an equation that models the hyperbolic path of a spacecraft around a planet if $a = 107{,}124$ km and $c = 213{,}125.9$ km.

10-6 Translating Conic Sections

Quick Review

Substitute $(x - h)$ for x and $(y - k)$ for y to translate graphs of the conic sections.

Example

Identify and describe the conic section represented by the equation $2x^2 + 3y^2 + 4x + 12y - 22 = 0$.

By completing the square, the equation becomes
$\frac{(x + 1)^2}{18} + \frac{(y + 2)^2}{12} = 1$, which is an ellipse.

The center is $(-1, -2)$ and the major axis is horizontal.

Using the equation $c^2 = a^2 - b^2$, $c = \sqrt{6}$, so the distance from the center of the ellipse to the foci is $\sqrt{6}$.

Since the ellipse is centered at $(-1, -2)$ and the major axis is horizontal, the foci are located $\sqrt{6}$ to the left and right of this center.

The foci are at $(-1 + \sqrt{6}, -2)$ and $(-1 - \sqrt{6}, -2)$.

Exercises

Write an equation of a conic section with the given characteristics.

40. a circle with center $(1, 1)$; radius 5

41. an ellipse with center $(3, -2)$; vertical major axis of length 6; minor axis of length 4

42. a hyperbola with vertices $(3, 3)$ and $(9, 3)$; foci $(1, 3)$ and $(11, 3)$

Identify the conic section and sketch the graph. If it is a parabola, give the vertex. If it is a circle, give the center and radius. If it is an ellipse or a hyperbola, give the center and foci.

43. $-x^2 + y^2 + 4y - 16 = 0$

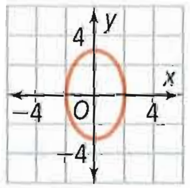
44. $x^2 + y^2 + 3x - 4y - 9 = 0$

45. $x^2 + x - y - 42 = 0$

10 Chapter Test

Do you know HOW?

Identify the type of each conic section. Give the center, domain, and range of each graph.

1.


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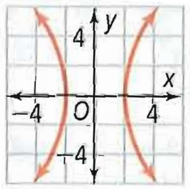

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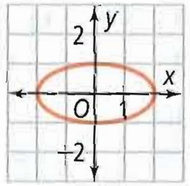

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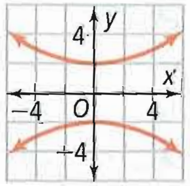


Identify the focus and the directrix of the graph of each equation.

5. $y = 3x^2$

6. $x = -2y^2$

7. $x + 5y^2 = 0$

8. $9x^2 - 2y = 0$

Write an equation of a parabola with its vertex at the origin and the given characteristics.

9. focus at $(0, -2)$

10. focus at $(3, 0)$

11. directrix $x = 7$

12. directrix $y = -1$

For each equation, find the center and radius of the circle. Graph the circle.

13. $(x - 2)^2 + (y - 3)^2 = 36$

14. $(x + 5)^2 + (y + 8)^2 = 100$

15. $(x - 1)^2 + (y + 7)^2 = 81$

16. $(x + 4)^2 + (y - 10)^2 = 121$

Write an equation of an ellipse for each given height and width. Assume that the center of the ellipse is (0, 0).

17. height 10 units; width 16 units

18. height 2 units; width 12 units

19. height 9 units; width 5 units

Find the foci of each ellipse. Then graph the ellipse.

20. $x^2 + \frac{y^2}{49} = 1$

21. $4x^2 + y^2 = 4$

Find the foci of each hyperbola. Then graph the hyperbola.

22. $\frac{x^2}{64} - \frac{y^2}{4} = 1$

23. $y^2 - \frac{x^2}{225} = 1$

Write an equation of an ellipse with the given characteristics.

24. center $(-2, 7)$; horizontal major axis of length 8; minor axis of length 6

25. center $(3, -2)$; vertical major axis of length 12; minor axis of length 10

Write an equation of a hyperbola with the given characteristics.

26. vertices $(\pm 3, 7)$; foci $(\pm 5, 7)$

27. vertices $(2, \pm 5)$; foci $(2, \pm 8)$

Identify the conic section represented by each equation. If it is a parabola, give the vertex. If it is a circle, give the center and radius. If it is an ellipse or a hyperbola, give the center and foci. Sketch the graph.

28. $3y^2 - x - 6y + 5 = 0$

29. $4x^2 + y^2 - 16x - 6y + 9 = 0$

Do you UNDERSTAND?

30. **Writing** Explain how you can tell what kind of conic section a quadratic equation describes without graphing the equation.

31. **Reasoning** What shape is an ellipse whose height and width are equal?

32. **Open-Ended** Write an equation of a hyperbola whose transverse axis is on the x-axis.

Sunshine State Standards End-of-Course Test

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

1. What is the solution of $|x + 2| \geq 5$?

(A) $x \geq 3$

(B) $-7 \leq x \leq 3$

(C) $x \geq 3$ or $x \geq 7$

(D) $x \leq -7$ or $x \geq 3$

2. A boat took 4 hours to make a trip downstream with a current of 6 km/h. The return trip against the same current took 10 hours. How far did the boat travel?

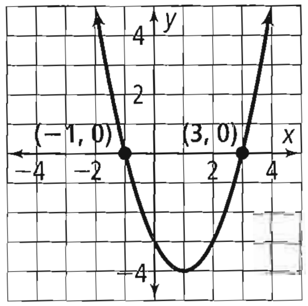
(F) 48 km

(G) 84 km

(H) 160 km

(I) 196 km

3. The graph of a quadratic function $f(x)$ is shown below.



What is the solution of $f(x) < 0$?

(A) $-1 < x < 3$

(B) $-1 \leq x \leq 3$

(C) $x < -1$ or $x > 3$

(D) $x \leq -1$ or $x \geq 3$

4. What are the complex solutions of $x^2 - 4x = -5$?

(F) $x = -1, x = 5$

(G) $x = 1, x = 3$

(H) $x = 2 + i, x = 2 - i$

(I) $x = 2 + 3i, x = 2 - 3i$

5. Newton's Law of Universal Gravitation is $F = \frac{Gm_1m_2}{r^2}$. Solve this equation for r.

(A) $r = \sqrt{\frac{F}{Gm_1m_2}}$

(B) $r = \sqrt{\frac{Gm_1m_2}{F}}$

(C) $r = \frac{F}{2Gm_1m_2}$

(D) $r = \frac{Gm_1m_2}{2F}$

6. Let $f(x) = x^3 - 4x^2 + 9x$ and let $g(x) = 6x^3 + x^2 - 5x - 12$. What is $f(x) - g(x)$?

(F) $-5x^3 - 5x^2 + 14x + 12$

(G) $-5x^3 - 3x^2 + 4x - 12$

(H) $7x^3 - 3x^2 + 4x - 12$

(I) $-5x^4 - 5x^3 + 14x^2 + 12x$

7. Let $f(x) = x - 3$ and let $g(x) = 2x^2 - 6$. What is $g(f(x))$?

(A) $2x^2 - 9$

(B) $2x^2 - 12$

(C) $2x^2 - 12x + 12$

(D) $2x^3 - 6x^2 - 6x + 18$

8. Let $f^{-1}(x) = 2x + 3$. What is the solution of $f(x) = f^{-1}(x)$?

(F) $x = -1$ or $x = -2$

(G) $x = -3$

(H) $(x, y) = (-1, -2)$

(I) $(x, y) = (-3, -3)$

9. The function below can be used to find the total amount $C(x)$ an electric company charges a customer who uses x kilowatt-hours (kWh) in a month.

$$C(x) = \begin{cases} 0.07275x + 6.00, \text{ if } 0 \leq x \leq 400 \\ 0.05535x + 35.10, \text{ if } x > 400 \end{cases}$$

If a customer uses 546 kWhr in a month, what is the total amount charged?

(A) \$45.72

(B) \$65.32

(C) \$78.28

(D) \$111.04

10. What is the simpler form of $\frac{\sqrt{5}}{3 - \sqrt{2}}$?

(F) $\frac{\sqrt{10}}{3\sqrt{2} - 2}$

(G) $\frac{5}{3\sqrt{5} - 10}$

(H) $\frac{3\sqrt{5} - 10}{7}$

(I) $\frac{3\sqrt{5} + \sqrt{10}}{7}$

11. Suppose that $\sqrt[4]{n} = 2$. What is $n^{-\frac{1}{2}}$?

Ⓐ -8 Ⓒ $\frac{1}{8}$

Ⓑ -4 Ⓓ $\frac{1}{4}$

12. What is the quotient $\frac{2 + 5i}{4 + 3i}$, written in standard form?

Ⓕ $\frac{1}{2} + \frac{5}{3}i$ Ⓗ $-\frac{7}{7} + \frac{26}{7}i$

Ⓖ $\frac{8}{25} - \frac{12}{25}i$ Ⓘ $\frac{23}{25} + \frac{14}{25}i$

13. Which is the simpler form of $\frac{r^{\frac{1}{2}}}{r^{-\frac{1}{4}}}$?

Ⓐ $-r^{\frac{1}{4}}$ Ⓒ $r^{\frac{1}{8}}$

Ⓑ $-r^{2}$ Ⓓ $r^{\frac{3}{4}}$

14. Multiply $\frac{x^3}{x^2 - 4} \cdot \frac{5x + 10}{10x}$.

Ⓕ $\frac{5x}{4}$ Ⓗ $\frac{x^2(x + 2)}{2(x^2 - 4)}$

Ⓖ $\frac{x^2}{2(x - 2)}$ Ⓘ $\frac{5x^3}{10x(x - 2)}$

15. Which is the simpler form of the complex fraction $\frac{\frac{1}{b} + c}{b + \frac{1}{c}}$?

Ⓐ 1 Ⓒ $\left(\frac{1}{b} + c\right)^2$

Ⓑ $\frac{c}{b}$ Ⓓ $(1 + c)(b + 1)$

16. The graph of a quadratic function, $y = ax^2 + bx + c$ passes through the points $(2, 1)$, $(3, -2)$, and $(4, -7)$. What is the axis of symmetry of the parabola?

Ⓕ $x = -2$ Ⓗ $x = 1$

Ⓖ $x = -1$ Ⓘ $x = 2$

17. What is the equation of a parabola with the following characteristics?

Axis of symmetry: $x = -3$

Range: all real numbers less than or equal to 4

Ⓐ $y = -(x - 4)^2 - 3$

Ⓑ $y = (x - 4)^2 - 3$

Ⓒ $y = -(x + 3)^2 + 4$

Ⓓ $y = (x + 3)^2 + 4$

18. What is the sum of the x-intercepts of the graph of the quadratic equation $y = x^2 - 4x - 12$?

Ⓕ 6 Ⓗ -1

Ⓖ 4 Ⓘ -4

19. What is the end behavior of the graph of the polynomial function $f(x) = -2x^5 + x^4 + 3x^3 - x + 1$?

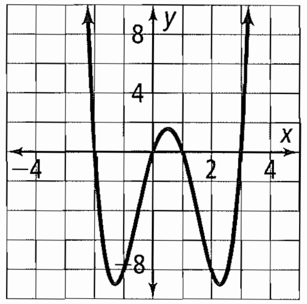
Ⓐ down and down

Ⓑ down and up

Ⓒ up and down

Ⓓ up and up

20. The graph of a degree 4 polynomial function with integer zeros is shown below.



What is the equation of the polynomial function?

Ⓕ $y = x^4 - 6x^3 + 11x^2 - 6x$

Ⓖ $y = x^4 - 2x^3 - 5x^2 + 6x$

Ⓗ $y = x^4 - 2x^3 + x^2 + 3x$

Ⓘ $y = x^4 + 2x^3 - 5x^2 - 6x$

21. The graph of the exponential equation $y = 2^x$ is reflected across the y-axis and shifted down 1 unit. What is the equation of the resulting graph?

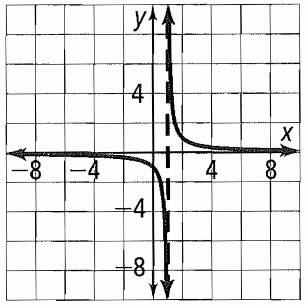
Ⓐ $y = 2^{-x-1}$

Ⓑ $y = -2^{x-1}$

Ⓒ $y = 2^{-x} - 1$

Ⓓ $y = -2^{x} - 1$

22. Which function is best represented by the graph below?



(F) $y = \frac{1}{x - 1}$ (H) $y = \frac{x}{x - 1}$

(G) $y = \frac{1}{x + 1}$ (I) $y = \frac{x}{x + 1}$

23. How many distinct real roots does the equation $x^4 + 3x^3 - 4x = 0$ have?

(A) 1 (C) 3

(B) 2 (D) 4

24. The principal amount invested in an account with 1.5% interest compounded continuously is \$500. The equation $A(x) = 500e^{0.015x}$ can be used to find the balance in the account after x years. To the nearest year, in how many years will the account have a balance of \$820?

(F) 2 years (H) 72 years

(G) 33 years (I) 109 years

25. Which ellipse has the same foci as the hyperbola described by $\frac{(x-2)^2}{9} - \frac{(y+6)^2}{16} = 1$?

(A) $\frac{(x-2)^2}{9} + \frac{(y+6)^2}{16} = 1$

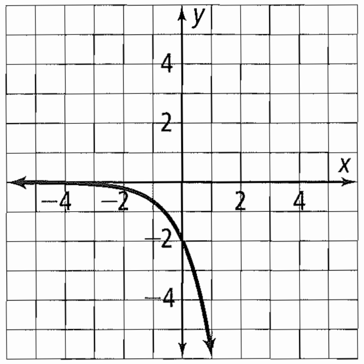
(B) $\frac{(x-2)^2}{25} + \frac{(y+6)^2}{16} = 1$

(C) $\frac{(x-2)^2}{34} + \frac{(y+6)^2}{16} = 1$

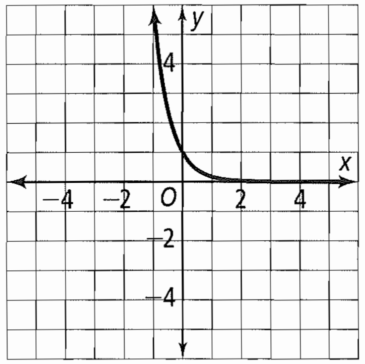
(D) $\frac{(x-2)^2}{41} + \frac{(y+6)^2}{16} = 1$

26. Which graph represents $f(x) = 2 \cdot 3^{-x}$?

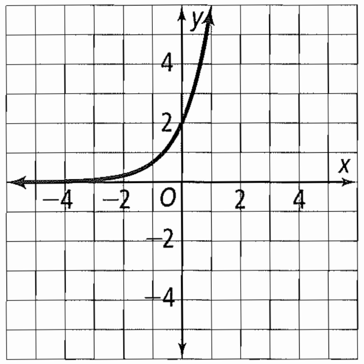
(F)



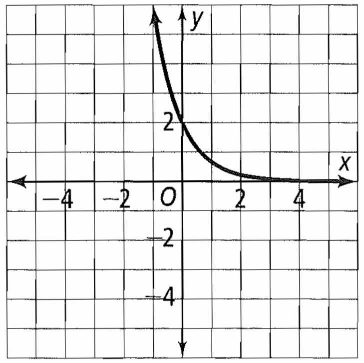
(G)



(H)



(I)



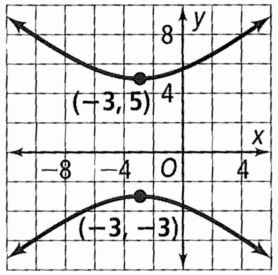
27. The function $C(x) = \frac{10}{2x^2 + 1}$ can be used to find the concentration $C(x)$ in mg/L of a certain drug in the bloodstream of a patient x hours after the injection is given. In approximately how many hours after the injection will the concentration of the drug be 1.3 mg/L?

Ⓐ 0.5 hours
Ⓑ 0.7 hours
Ⓒ 1.8 hours
Ⓓ 2.3 hours

28. The equation of a circle is $(x + 5)^2 + (y - 8)^2 = 81$. What are the center and radius of the circle?

Ⓕ Center: $(-5, 8)$ Radius: 81
Ⓖ Center: $(5, -8)$ Radius: 81
Ⓗ Center: $(-5, 8)$ Radius: 9
Ⓘ Center: $(5, -8)$ Radius: 9

29. The graph of a hyperbola is shown below.



Which equation is best represented by the graph?

Ⓐ $\frac{(x + 3)^2}{16} - \frac{(y - 1)^2}{25} = 1$
Ⓑ $\frac{(y - 1)^2}{16} - \frac{(x + 3)^2}{25} = 1$
Ⓒ $\frac{(x + 3)^2}{16} - \frac{(y - 5)^2}{25} = 1$
Ⓓ $\frac{(y + 3)^2}{16} - \frac{(x + 3)^2}{25} = 1$

30. An employee's initial salary is \$30,000. The person receives a 5% raise each year. What is the formula for the term s_n which represents the salary at the beginning of the nth year?

Ⓕ $s_n = 30{,}000 + 1.05n$
Ⓖ $s_n = 30{,}000 + 5(n - 1)$
Ⓗ $s_n = 30{,}000(1.05)^{n-1}$
Ⓘ $s_n = 30{,}000(1.05)^n$

31. Which expression represents $4\log_3 x + \log_3 y - 2\log_3 z$ as a single logarithm?

Ⓐ $\log_3 \frac{x^4 y}{z^2}$
Ⓑ $\frac{\log_3 x^4 y}{\log_3 z^2}$
Ⓒ $\log_3 (4x + y - 2z)$
Ⓓ $\log_3 (x^4 + y - z^2)$

32. Which expression gives the following sum using summation notation?

$1 + 3 + 5 + 7 + \cdots + 19$

Ⓕ $\sum_{k=1}^{17} (k + 2)$
Ⓖ $\sum_{k=3}^{19} (k + 2)$
Ⓗ $\sum_{k=1}^{10} (2k - 1)$
Ⓘ $\sum_{k=3}^{19} (2k - 1)$

33. Which equation is $\log_5 m = k$, rewritten in exponential form?

Ⓐ $5^m = k$
Ⓑ $5^k = m$
Ⓒ $m^5 = k$
Ⓓ $k^5 = m$

34. Solve $\ln x + \ln 3x = \ln 12$ for x.

Ⓕ $x = 0.6$
Ⓖ $x = 2$
Ⓗ $x = -2, x = 2$
Ⓘ $x = 3$

35. Which is always a solution to $\log_m m^2 = x$?

Ⓐ $x = 0$
Ⓑ $x = 1$
Ⓒ $x = 2$
Ⓓ $x = m$

36. The wavelength of a radio wave varies inversely as its frequency. A wave with a wavelength of 300 meters has a frequency of 1200 kilohertz. What is the length in meters of a wave with a frequency of 1500 kilohertz?

Ⓕ 240 m
Ⓖ 375 m
Ⓗ 600 m
Ⓘ 6,000 m

37. The polynomial $2x^3 + 3x^2 - 8x + 3$ has a zero at $x = 1$. What are the remaining zeros of the polynomial?

Ⓐ −3, 2, 5
Ⓑ −3, 0.5
Ⓒ −1.5, 0
Ⓓ −1.5, 1

38. If $(4x^3 - 12x^2 - 19x + 12) \div (2x + 3) = ax^2 + bx + c$, what is the value of b?

(F) -3 (H) -9

(G) -6 (I) -18

39. The first five rows of Pascal's Triangle are shown below.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Use Pascal's Triangle to expand $(2x - 1)^4$.

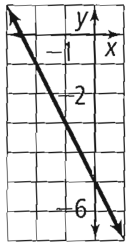
(A) $x^4 - 4x^3 + 6x^2 - 4x + 1$

(B) $2x^4 + 8x^3 + 12x^2 + 8x + 1$

(C) $16x^4 - 32x^3 + 24x^2 - 8x + 1$

(D) $16x^4 + 32x^3 + 24x^2 + 8x + 1$

40. Which is an equation of the line that passes through the point $(-6, 4)$ and is perpendicular to the line shown?



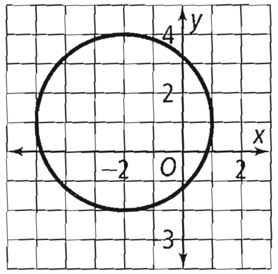
(F) $y = -2x - 8$

(G) $y = -\frac{1}{2}x + 1$

(H) $y = \frac{1}{2}x - 8$

(I) $y = \frac{1}{2}x + 7$

41. Which equation describes the circle?



(A) $(x - 1)^2 + (y + 2)^2 = 3$

(B) $(x - 2)^2 + (y + 1)^2 = 9$

(C) $(x + 2)^2 + (y - 1)^2 = 9$

(D) $(x + 1)^2 + (y - 2)^2 = 3$

42. Which quadrant contains *no* solutions of the system?

$$\begin{cases} 2x + y \geq 0 \\ -3x + y < 2 \end{cases}$$

(F) I (G) II (H) III (I) IV

43. The polynomial equation $x^3 + x^2 - 7x + 65 = 0$, has $2 - 3i$ as an imaginary root. Which of the following must also be a root of the equation?

(A) $-2 + 3i$

(B) -65

(C) $2 + 3i$

(D) $3 + 2i$

44. Which of the following is not true of the graph of $f(x) = \log_2 (x - 3)$?

(F) It has an x-intercept at 4.

(G) It has a vertical asymptote at $x = 2$.

(H) The point $(7, 2)$ is on the graph.

(I) It is a horizontal translation of $y = \log_2 x$.

GRIDDED RESPONSE

45. Find the x-value of the solution to the following system of equations.

$$\begin{cases} 3x + y = -3 \\ 2y - z = 6 \\ x + y - 2z = 1 \end{cases}$$

46. A high school sold 800 tickets for a soccer game. Three types of tickets were sold, adult, student, and child. There were four times as many adult tickets sold as child tickets. And there were 62 more student tickets sold than adult tickets. How many adult tickets were sold?

47. Solve $\frac{3}{2x + 10} + \frac{5}{4} = \frac{7}{x + 5}$ for x.

48. Solve $\sqrt{x - 2} - 7 = -4$.

49. The equation $x^2 - 0.8x + c = 0$ has one real solution. Use the discriminant to find the value of c.

50. Let $f(x) = 3x + 5$ and let $g(x) = x^2 + 2x$. What is $f(-3) \cdot g(-3)$?

51. What is the rational value of x if $\sqrt[5]{b^3} = b^x$?

52. What is the x-coordinate of the vertex of the graph of $f(x) = 2x^2 + 4x - 6$?

53. The horizontal asymptote of the graph of $y = \frac{4x - 4}{2x - 6}$ is $y = t$ for a real number t. What is the value of t?

54. The volume of a square pyramid with a height equal to four less than the length of a side of the base is given by $V(x) = \frac{1}{3}(x^3 + 8x^2 + 16x)$ where x is the height in cm. If the length of a side of the base is 9 cm, what is the volume of the pyramid in cm^3?

55. Solve for $4(3^x) = 26$. Round your answer to the nearest tenth.

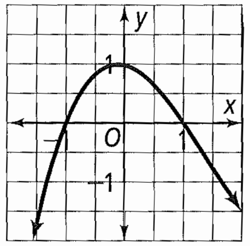
56. The amount of cesium-137 remaining after x years in an initial sample of 200 milligrams can be found using the equation $C(x) = 200e^{-0.02295x}$. In approximately how many years will the sample contain less than 120 milligrams of cesium-137? Round to the nearest year.

57. The magnitude M of an earthquake can be found using the equation $M(x) = \log\left(\frac{x}{0.001}\right)$ where x represents the seismograph reading of the earthquake in mm. An earthquake has a magnitude of 6.2. What is the seismograph reading of the earthquake in mm? Round your answer to the nearest whole number.

58. Use the Change of Base Formula to approximate the value of $\log_2 3.2$ to the nearest tenth.

59. What is the sum of the finite geometric series? $1 - 2 + 4 - 8 + \cdots - 512$?

60. The graph of the function $y = \frac{x^3}{4} - x^2 - \frac{x}{4} + 1$ is shown below.



What is the value of the third zero, not shown in the graph?

61. A sequence is defined below:

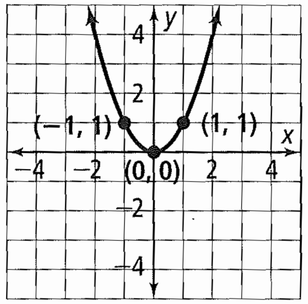
$a_1 = 2$

$a_n = 4a_{n-1} - 5$

What is the 4th term in the sequence?

62. An equation of an ellipse is $9(x + 9)^2 + 4(y + 4)^2 = 36$. What is the y-coordinate of the center of the ellipse?

63. The graph of $f(x) = x^2$ is shown below.

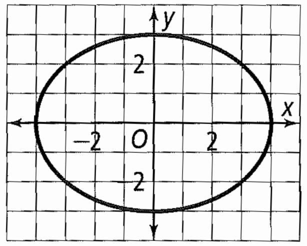


If $g(x)$ represents the graph of $f(x)$ translated down 3 units and right 1 unit, what is $g(2)$?

64. Suppose $g(x)$ is the reflection of $f(x) = |2x + 6| - 1$ in the y-axis. What is the x-value of the vertex of $g(x)$?

65. Solve $2 \ln 4x + 5 = 8$ to the nearest hundredth.

66. What is the distance between the foci? Round to the nearest hundredth.



67. Consider the function $f(x) = \frac{2x^4 - 7x^2 + 3}{5x^3 + 6x}$.

What is the slope of the oblique asymptote?

68. The equation of an ellipse is $4x^2 + 9y^2 + 8x - 54y + 49 = 0$.

What is the length of the major axis?

Skills Handbook

Percents and Percent Applications

Percent means "per hundred." Find fraction, decimal, and percent equivalents by replacing one symbol for *hundredths* with another.

Example 1

Write each number as a percent.

a. $0.082 = 8.2\%$

Move the decimal point two places to the right and write a percent sign.

b. $\frac{3}{5} = \frac{60}{100} = 60\%$

Write the fraction as hundredths. Then replace the hundredths with a percent sign.

c. $1\frac{1}{6} = \frac{7}{6} = 1.166\bar{6} = 116.\bar{6}\%$

First, use $7 \div 6$ to write $1\frac{1}{6}$ as a decimal.

Example 2

Write each percent as a decimal.

a. $50\% = 0.50 = 0.5$

b. $\frac{1}{2}\% = 0.5\% = 00.5\% = 0.005$

Move the decimal point two places to the left and drop the percent sign.

Example 3

Use an equation to solve each percent problem.

a. What is 30% of 12?

$n = 0.3 \times 12$

$n = 3.6$

b. 18 is 0.3% of what?

$18 = 0.003 \times n$

$\frac{18}{0.003} = \frac{0.003n}{0.003}$

$6000 = n$

c. What percent of 60 is 9?

$n \times 60 = 9$

$60n = 9$

$n = \frac{9}{60} = 0.15 = 15\%$

Exercises

Write each decimal as a percent and each percent as a decimal.

1. 0.46
2. 1.506
3. 0.007
4. 8%
5. 103.5%
6. 3.3%

Write each fraction or mixed number as a percent.

7. $\frac{1}{4}$
8. $\frac{3}{8}$
9. $\frac{2}{3}$
10. $\frac{4}{9}$
11. $1\frac{3}{20}$
12. $\frac{1}{200}$

Use an equation to solve each percent problem. Round your answer to the nearest tenth, if necessary.

13. What is 25% of 50?
14. What percent of 58 is 37?
15. 120% of what is 90?
16. 8 is what percent of 40?
17. 15 is 75% of what?
18. 80% of 58 is what?

Operations With Fractions

To add or subtract fractions, use a common denominator. The common denominator is the least common multiple of the denominators.

Example 1

Simplify $\frac{2}{3} + \frac{3}{5}$.

$\frac{2}{3} + \frac{3}{5} = \frac{2}{3} \cdot \frac{5}{5} + \frac{3}{5} \cdot \frac{3}{3}$ For 3 and 5, the least common multiple is 15.

$= \frac{10}{15} + \frac{9}{15}$ Write $\frac{2}{3}$ and $\frac{3}{5}$ as equivalent fractions with denominators of 15.

$= \frac{19}{15}$ or $1\frac{4}{15}$ Add the numerators.

Example 2

Simplify $5\frac{1}{4} - 3\frac{2}{3}$.

$5\frac{1}{4} - 3\frac{2}{3} = 5\frac{3}{12} - 3\frac{8}{12}$ Write equivalent fractions.

$= 4\frac{15}{12} - 3\frac{8}{12}$ Write $5\frac{3}{12}$ as $4\frac{15}{12}$ so you can subtract the fractions.

$= 1\frac{7}{12}$ Subtract the fractions. Then subtract the whole numbers.

To multiply fractions, multiply the numerators and multiply the denominators. You can simplify by using a greatest common factor.

Example 3

Simplify $\frac{3}{4} \cdot \frac{8}{11}$

Method 1 $\frac{3}{4} \cdot \frac{8}{11} = \frac{24}{44} = \frac{24 \div 4}{44 \div 4} = \frac{6}{11}$

Divide 24 and 44 by 4, their greatest common factor.

Method 2 $\frac{3}{\cancel{4}} \cdot \frac{\cancel{8}^{2}}{11} = \frac{6}{11}$

Divide 4 and 8 by 4, their greatest common factor.

To divide fractions, use a reciprocal to change the problem to multiplication.

Example 4

Simplify $3\frac{1}{5} \div 1\frac{1}{2}$

$3\frac{1}{5} \div 1\frac{1}{2} = \frac{16}{5} \div \frac{3}{2}$ Write mixed numbers as improper fractions.

$= \frac{16}{5} \cdot \frac{2}{3}$ Multiply by the reciprocal of the divisor.

$= \frac{32}{15}$ or $2\frac{2}{15}$ Simplify.

Exercises

Perform the indicated operation.

1. $\frac{3}{5} + \frac{4}{5}$

2. $\frac{1}{2} + \frac{2}{3}$

3. $4\frac{1}{2} + 2\frac{1}{3}$

4. $5\frac{3}{4} + 4\frac{2}{5}$

5. $\frac{2}{3} - \frac{3}{7}$

6. $5\frac{1}{2} - 3\frac{2}{5}$

7. $7\frac{3}{4} - 4\frac{4}{5}$

8. $3\frac{4}{5} \cdot 10$

9. $2\frac{1}{2} \cdot 3\frac{1}{5}$

10. $6\frac{3}{4} \cdot 5\frac{2}{3}$

11. $\frac{1}{2} \div \frac{1}{3}$

12. $\frac{6}{5} \div \frac{3}{5}$

13. $8\frac{1}{2} \div 4\frac{1}{4}$

14. $\frac{8}{9} - \frac{2}{3}$

15. $5\frac{1}{4} \cdot 8$

Ratios and Proportions

A *ratio* is a comparison of two quantities by division. You can write *equal ratios* by multiplying or dividing each quantity by the same nonzero number.

Ways to Write a Ratio
$a:b$ $\quad$ a to b $\quad$ $\frac{a}{b}$ $(b \neq 0)$

Example 1

Write $3\frac{1}{3}:\frac{1}{2}$ as a ratio in simplest form.

$$3\tfrac{1}{3}:\tfrac{1}{2} \rightarrow \frac{3\frac{1}{3}}{\frac{1}{2}} \underset{\times 6}{\overset{\times 6}{=}} \frac{20}{3} \text{ or } 20:3$$

In simplest form, both terms should be integers.
Multiply by the common denominator, 6.

A rate is a ratio that compares different types of quantities. In simplest form for a rate, the second quantity is one unit.

Example 2

Write 247 mi in 5.2 h as a rate in simplest form.

$$\frac{247 \text{ mi}}{5.2 \text{ h}} \underset{\div 5.2}{\overset{\div 5.2}{=}} \frac{47.5 \text{ mi}}{1 \text{ h}} \text{ or } 47.5 \text{ mi/h}$$

Divide by 5.2 to make the second quantity one unit.

A proportion is a statement that two ratios are equal. You can find a missing term in a proportion by using the cross products.

Cross Products of a Proportion
$\frac{a}{b} = \frac{c}{d} \rightarrow ad = bc$

Example 3

The Copy Center charges $2.52 for 63 copies. At that rate, how much will the Copy Center charge for 140 copies?

cost → copies → $\frac{2.52}{63} = \frac{c}{140}$	Set up a proportion.
$2.52 \cdot 140 = 63c$	Use cross products.
$c = \frac{2.52 \cdot 140}{63}$	Solve for *c*.
$= 5.6$ or \$5.60	

Exercises

Write each ratio or rate in simplest form.

1. 15 to 20 **2.** 85 : 34 **3.** 38 g in 4 oz **4.** 375 mi in 4.3 h **5.** $\frac{84}{30}$

Solve each proportion. Round your answer to the nearest tenth, if necessary.

6. $\frac{a}{5} = \frac{12}{15}$ **7.** $\frac{21}{12} = \frac{14}{x}$ **8.** $8:15 = n:25$ **9.** $2.4:c = 4:3$ **10.** $\frac{17}{8} = \frac{n}{20}$

11. $\frac{13}{n} = \frac{20}{3}$ **12.** $5:7 = y:5$ **13.** $\frac{0.4}{3.5} = \frac{5.2}{x}$ **14.** $\frac{4}{x} = \frac{7}{6}$ **15.** $4:n = n:9$

16. A canary's heart beats 130 times in 12 s. Use a proportion to find about how many times its heart beats in 50 s.

Simplifying Expressions With Integers

To add two numbers with the same sign, *add* their absolute values. The sum has the same sign as the numbers. To add two numbers with different signs, find the *difference* between their absolute values. The sum has the same sign as the number with the greater absolute value.

Example 1

Add.

a. $-8 + (-5) = -13$ **b.** $-8 + 5 = -3$ **c.** $8 + (-5) = 3$

To subtract a number, add its opposite.

Example 2

Subtract.

a. $4 - 7 = 4 + (-7) = -3$

b. $-4 - (-7) = -4 + 7 = 3$

c. $-4 - 7 = -4 + (-7) = -11$

The product or quotient of two numbers with the same sign is positive. The product or quotient of two numbers with different signs is negative.

Example 3

Multiply or divide.

a. $(-3)(-5) = 15$ **b.** $-35 \div 7 = -5$ **c.** $24 \div (-6) = -4$

Example 4

Simplify $2^2 - 3(4 - 6) - 12$.

$$\begin{aligned} 2^2 - 3(4 - 6) - 12 &= 2^2 - 3(-2) - 12 \\ &= 4 - 3(-2) - 12 \\ &= 4 - (-6) - 12 \\ &= 4 + 6 - 12 = -2 \end{aligned}$$

Order of Operations

1. Perform any operation(s) inside grouping symbols.
2. Simplify any terms with exponents.
3. Multiply and divide in order from left to right.
4. Add and subtract in order from left to right.

Exercises

Simplify each expression.

1. $-4 + 5$ **2.** $12 - 12$ **3.** $-15 + (-23)$ **4.** $4 - 17$ **5.** $-5 - 12$

6. $3 - (-5)$ **7.** $-8 - (-12)$ **8.** $-19 + 5$ **9.** $(-7)(-4)$ **10.** $-120 \div 30$

11. $(-3)(4)$ **12.** $75 \div (-3)$ **13.** $(-6)(15)$ **14.** $(18)(-4)$ **15.** $-84 \div (-7)$

16. $-2(1 + 5) + (-3)(2)$ **17.** $-4(-2 - 5) + 3(1 - 4)$ **18.** $20 - (3)(12) + 4^2$

19. $\frac{-15}{-5} - \frac{36}{-12} + \frac{-12}{-4}$ **20.** $5^2 - 6(5 - 9)$ **21.** $(-3 + 2^3)(4 + \frac{-42}{7})$

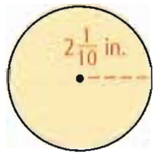
Area and Volume

The *area* of a plane figure is the number of square units contained in the figure.
The *volume* of a space figure is the number of cubic units contained in the figure.
Formulas for area and volume are listed on page 693.

Example 1

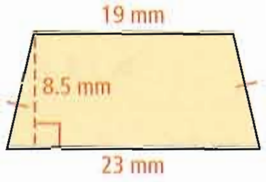
Find the area of each figure.

a.



$$A = \pi r^2$$
$$\approx \frac{22}{7} \cdot \left(\frac{21}{10}\right)^2$$
$$= \frac{693}{50} = 13\frac{43}{50}\text{ in.}^2$$

b.

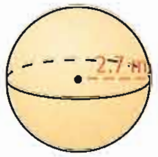


$$A = \frac{1}{2}(b_1 + b_2)h$$
$$= \frac{1}{2}(19 + 23) \cdot 8.5$$
$$= 178.5\text{ mm}^2$$

Example 2

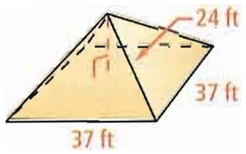
Find the volume of each figure.

a.



$$V = \frac{4}{3}\pi r^3$$
$$\approx \frac{4}{3} \cdot 3.14 \cdot 2.7^3$$
$$= 82.40616 \approx 82.4m^3$$

b.

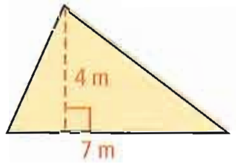


$$V = \frac{1}{3}Bh$$
$$= \frac{1}{3}(37^2) \cdot 24$$
$$= 10{,}952\text{ ft}^3$$

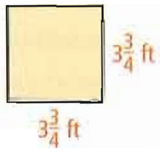
Exercises

Find the exact area of each figure.

1.



2.



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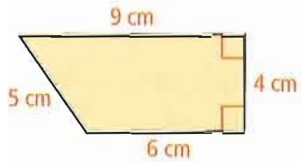

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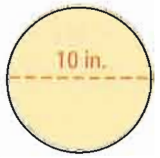
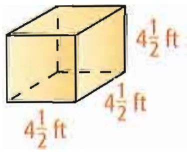


Find the exact volume of each figure.

5.



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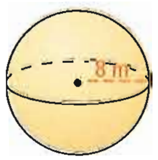

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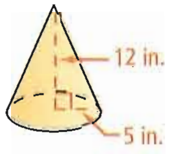

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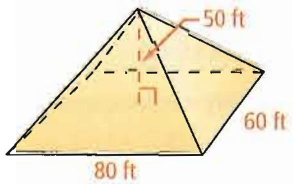


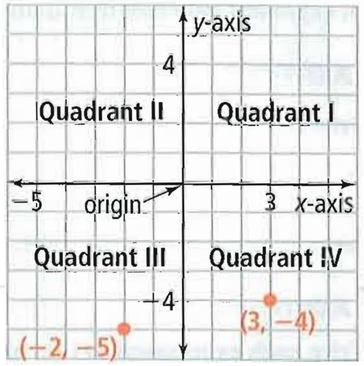
9. Find the area of a triangle with a base of 17 in. and a height of 13 in.

10. Find the volume of a rectangular box 64 cm long, 48 cm wide, and 58 cm high.

11. Find the surface area of the cube in Exercise 5.

The Coordinate Plane, Slope, and Midpoint

The *coordinate plane* is formed when two perpendicular number lines intersect at a point called the origin, forming four quadrants.



Example 1

In which quadrant would you find each point?

a. (3, −4) Move 3 units right and 4 units down. The point is in Quadrant IV.

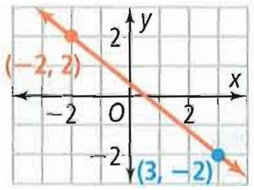
b. (−2, −5) Move 2 units left and 5 units down. The point is in Quadrant III.

To find the slope of a line on the coordinate plane, choose two points on the line and use the slope formula.

Example 2

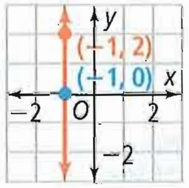
Find the slope of each line.

a.



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - (-2)}{-2 - 3}$$
$$= \frac{4}{-5} \text{ or } -\frac{4}{5}$$

b.



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - 0}{-1 - (-1)} = \frac{2}{0}$$

Since you cannot divide by zero, this line has an undefined slope.

If (x_m, y_m) is the midpoint of the segment joining (x_1, y_1) and (x_2, y_2), then $x_m = \frac{x_1 + x_2}{2}$ and $y_m = \frac{y_1 + y_2}{2}$.

Example 3

Find the coordinates of the midpoint of the segment with endpoints (−2, 5) and (6, −3).

$\frac{-2 + 6}{2} = 2$ and $\frac{5 + (-3)}{2} = 1$ so the midpoint is (2, 1).

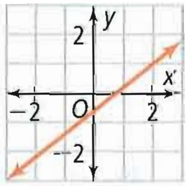
Exercises

In which quadrant would you find each point? Graph each point on a coordinate plane.

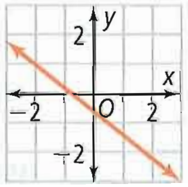
1. (3, 2) **2.** (−4, 3) **3.** (2, −3) **4.** (4, −2) **5.** (−4, −5) **6.** (−1, −3)

Find the slope of each line.

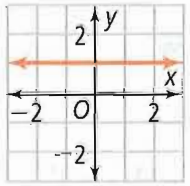
7.



8.



9.



10. the line containing (−3, 4) and (2, −6)

11. the line containing (25, 40) and (100, 55)

Find the midpoint of the segment with the given endpoints.

12. (−4, 4), (2, −5) **13.** (3, 3), (7, −6) **14.** (−1, −8), (0, −3) **15.** (3, 4), (2, −6)

Operations With Exponents

An exponent indicates how many times a number is used as a factor.

$2^n =$ ■	$10^n =$ ■
$2^2 = 4$	$10^2 = 100$
$2^1 = 2$	$10^1 = 10$
$2^0 = 1$	$10^0 = 1$
$2^{-1} = \frac{1}{2}$	$10^{-1} = \frac{1}{10}$
$2^{-2} = \frac{1}{4}$	$10^{-2} = \frac{1}{100}$

Example 1

Write using exponents.

a. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5$

b. $a \cdot a \cdot b \cdot b \cdot b \cdot b = a^2b^4$

The patterns shown at the right indicate that $a^0 = 1$ and that $a^{-n} = \frac{1}{a^n}$.

Example 2

Write each expression so that all exponents are positive.

a. $a^{-2}b^3 = \frac{1}{a^2} \cdot b^3 = \frac{b^3}{a^2}$

b. $x^3y^0z^{-1} = x^3 \cdot 1 \cdot \frac{1}{z} = \frac{x^3}{z}$

You can simplify expressions that contain powers with the same base.

Example 3

Simplify each expression.

a. $b^5 \cdot b^3 = b^{5+3}$
$= b^8$

Add exponents to multiply powers with the same base.

b. $\frac{x^5}{x^7} = x^{5-7}$
$= x^{-2} = \frac{1}{x^2}$

Subtract exponents to divide powers with the same base.

You can simplify expressions that contain parentheses and exponents.

Example 4

Simplify each expression.

a. $\left(\frac{ab}{n}\right)^3 = \frac{a^3b^3}{n^3}$

Raise each factor in the parentheses to the third power.

b. $(c^2)^4 = c^{2 \cdot 4} = c^8$

Multiply exponents to raise a power to a power.

Exercises

Write each expression using exponents.

1. $x \cdot x \cdot x$

2. $x \cdot x \cdot x \cdot y \cdot y$

3. $a \cdot a \cdot a \cdot a \cdot b$

4. $\frac{a \cdot a \cdot a \cdot a}{b \cdot b}$

Write each expression so that all exponents are positive.

5. c^{-4}

6. $m^{-2}n^0$

7. $x^5y^{-7}z^{-3}$

8. $ab^{-1}c^2$

Simplify each expression. Use positive exponents.

9. d^2d^6

10. $\frac{a^5}{a^2}$

11. $\frac{c^7}{c}$

12. $\frac{n^3}{n^6}$

13. $\frac{a^5b^3}{ab^8}$

14. $(3x)^2$

15. $\left(\frac{a}{b}\right)^4$

16. $\left(\frac{xz}{y}\right)^6$

17. $(c^3)^4$

18. $\left(\frac{x^2}{y^5}\right)^3$

19. $(u^4v^2)^3$

20. $(p^5)^{-2}$

21. $\frac{(2a^4)(3a^2)}{6a^3}$

22. $(x^{-2})^3$

23. $(mg^3)^{-1}$

24. $g^{-3}g^{-1}$

25. $\frac{(3a^3)^2}{18a}$

26. $\frac{c^3d^7}{c^{-3}d^{-1}}$

Factoring and Operations With Polynomials

Example 1

Perform each operation.

a. $(3y^2 - 4y + 5) + (y^2 + 9y)$

$= (3y^2 + y^2) + (-4y + 9y) + 5$ To add, group like terms.

$= 4y^2 + 5y + 5$

b. $(n + 4)(n - 3)$

$= n(n) + n(-3) + 4(n) + 4(-3)$ Distribute *n* and 4.

$= n^2 - 3n + 4n - 12$ Combine like terms.

$= n^2 + n - 12$

To factor a polynomial, first find the greatest common factor (GCF) of the terms. Then use the distributive property to factor out the GCF.

Example 2

Factor $6x^3 - 12x^2 + 18x$.

$6x^3 = 6 \cdot x \cdot x \cdot x$; $-12x^2 = 6 \cdot (-2) \cdot x \cdot x$; $18x = 6 \cdot 3 \cdot x$ List the factors of each term. The GCF is $6x$.

$6x^3 - 12x^2 + 18x = 6x(x^2) + 6x(-2x) + 6x(3)$ Use the distributive property to factor out $6x$.

$= 6x(x^2 - 2x + 3)$

When a polynomial is the product of two binomials, you can work backward to find the factors.

$x^2 + bx + c = (x + ■)(x + ■)$

The *sum* of these numbers must equal *b*.

The *product* of these numbers must equal *c*.

Example 3

Factor $x^2 - 13x + 36$.

Choose numbers that are factors of 36. Look for a pair with the sum -13.

The numbers -4 and -9 have a product of 36 and a sum of -13. The factors are $(x - 4)$ and $(x - 9)$. So, $x^2 - 13x + 36 = (x - 4)(x - 9)$.

Factors	Sum
$-6 \cdot (-6)$	-12
$-4 \cdot (-9)$	-13

Exercises

Perform the indicated operations.

1. $(x^2 + 3x - 1) + (7x - 4)$

2. $(5y^2 + 7y) - (3y^2 + 9y - 8)$

3. $4x^2(3x^2 - 5x + 9)$

4. $-5d(13d^2 + 7d + 8)$

5. $(x - 5)(x + 3)$

6. $(n - 7)(n - 2)$

Factor each polynomial.

7. $a^2 - 8a + 12$

8. $n^2 - 2n - 8$

9. $x^2 + 5x + 4$

10. $3m^2 - 9$

11. $y^2 + 5y - 24$

12. $s^3 + 6s^2 + 11s$

13. $2x^3 + 4x^2 - 8x$

14. $y^2 - 10y + 25$

Scientific Notation and Significant Digits

In *scientific notation*, a number has the form $a \times 10^n$, where n is an integer and $1 \le a < 10$.

Example 1

Write 5.59×10^6 in standard form.

$5.59 \times 10^6 = 5\overset{\curvearrowright}{590\,000} = 5,590,000$

A positive exponent indicates a value greater than 1. Move the decimal point six places to the right.

Example 2

Write 0.0000318 in scientific notation.

$0.\underset{\curvearrowright}{00003}18 = 3.18 \times 10^{-5}$

Move the decimal point to create a number between 1 and 10. Since the original number is less than 1, use a negative exponent.

When a measurement is in scientific notation, all the digits of the number between 1 and 10 are *significant digits*. When you multiply or divide measurements, your answer should have as many significant digits as the least number of significant digits in any of the numbers involved.

Example 3

Multiply $(6.71 \times 10^8$ mi/h) and $(3.8 \times 10^4$ h).

$$\begin{aligned}
(\underset{\substack{\uparrow\\ \text{three}\\ \text{significant}\\ \text{digits}}}{6.71} \times 10^8\,\text{mi/h})(\underset{\substack{\uparrow\\ \text{two}\\ \text{significant}\\ \text{digits}}}{3.8} \times 10^4\,\text{h}) &= (6.71 \cdot 3.8)(10^8 \cdot 10^4) && \text{Rearrange factors.}\\
&= 25.498 \times 10^{12} && \text{Add exponents when multiplying powers of 10.}\\
&= 2.5498 \times 10^{13} && \text{Write in scientific notation.}\\
&\approx 2.5 \times 10^{13}\text{mi} && \text{Round to two significant digits.}
\end{aligned}$$

Exercises

Change each number to scientific notation or to standard form.

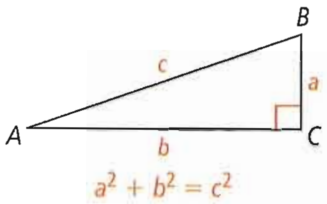
1. 1,340,000
2. 6.88×10^{-2}
3. 0.000775
4. 0.0072
5. 1.113×10^5
6. 8.0×10^{-4}
7. 1895
8. 2.3×10^3
9. 123,400
10. 7.985×10^4

Write each product or quotient in scientific notation. Round to the appropriate number of significant digits.

11. $(1.6 \times 10^2)(4.0 \times 10^3)$
12. $(2.5 \times 10^{-3})(1.2 \times 10^4)$
13. $(4.237 \times 10^4)(2.01 \times 10^{-2})$
14. $\frac{7.0 \times 10^5}{2.89 \times 10^3}$
15. $\frac{1.4 \times 10^4}{8.0 \times 10^2}$
16. $\frac{6.48 \times 10^6}{3.2 \times 10^5}$
17. $(1.78 \times 10^{-7})(5.03 \times 10^{-5})$
18. $(7.2 \times 10^{11})(5 \times 10^6)$
19. $(8.90 \times 10^8) \div (2.36 \times 10^{-2})$
20. $(3.95 \times 10^4) \div (6.8 \times 10^8)$
21. $(4.9 \times 10^{-8}) \div (2.7 \times 10^{-2})$
22. $(3.972 \times 10^{-5})(4.7 \times 10^{-4})$

The Pythagorean Theorem and the Distance Formula

The Pythagorean Theorem

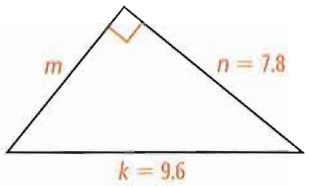


$a^2 + b^2 = c^2$

In a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. Use this relationship, known as the Pythagorean Theorem, to find the length of a side of a right triangle.

Example 1

Find *m* in the triangle below, to the nearest tenth.



$$m^2 + n^2 = k^2$$
$$m^2 + 7.8^2 = 9.6^2$$
$$m^2 = 9.6^2 - 7.8^2 = 31.32$$
$$m = \sqrt{31.32} \approx 5.6$$

To find the distance between two points on the coordinate plane, use the distance formula.

The distance d between any two points (x_1, y_1) and (x_2, y_2) is

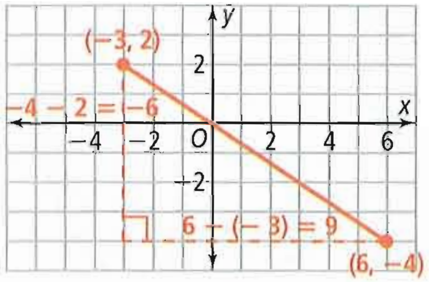
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2

Find the distance between (−3, 2) and (6, −4).

$$d = \sqrt{(6 - (-3))^2 + (-4 - 2)^2}$$
$$= \sqrt{9^2 + (-6)^2}$$
$$= \sqrt{81 + 36}$$
$$= \sqrt{117}$$
$$\approx 10.8$$

Thus, d is about 10.8 units.



Exercises

In each problem, *a* and *b* are the lengths of the legs of a right triangle and *c* is the length of the hypotenuse. Find each missing length. Round your answer to the nearest tenth.

1. c if $a = 6$ and $b = 8$

2. a if $b = 12$ and $c = 13$

3. b if $a = 8$ and $c = 17$

4. c if $a = 10$ and $b = 3$

5. a if $b = 100$ and $c = 114$

6. b if $a = 12.0$ and $c = 30.1$

Find the distance between each pair of points, to the nearest tenth.

7. (0, 0), (4, −3)

8. (−5, −5), (1, 3)

9. (−1, 0), (4, 12)

10. (−4, 2), (4, −2)

11. (0, 15), (17, 0)

12. (−8, 8), (8, 8)

13. (−1, 1), (1, −1)

14. (−2, 9), (0, 0)

15. (−5, 3), (4, 3)

16. (2, 1), (3, 4)

17. (3, −2), (3, 5)

18. (5, 4), (−3, 1)

Bar and Circle Graphs

Sometimes you can draw different graphs to represent the same data, depending on the information you want to share. A *bar graph* is useful for comparing amounts; a *circle graph* is useful for comparing percents.

Example

Display the 2007 data on immigration to the United States in a bar graph and a circle graph.

Immigration to the United States, 2007

Place of Origin	Immigrants (1000's)
Africa	89.2
Asia	359.4
Europe	120.8
North America	331.7
South America	102.6

Source: Department of Homeland Security

To make a circle graph, first find the *percent* of the data in each category. Then express each percent as a decimal and multiply by 360° to find the size of each *central angle.*

$$\begin{array}{r} \text{Africa} \rightarrow \\ \text{Total} \rightarrow \end{array} \frac{89.3}{1003.7} \approx 0.09 \text{ or } 9\%$$

$$0.09 \times 360^\circ \approx 32^\circ$$

Draw a circle and use a protractor to draw each central angle.

To make a bar graph, place the categories along the bottom axis. Decide on a scale for the side axis. An appropriate scale would be 0–300, marked in intervals of 50. For each data item, draw a bar whose height is equal to the data value.

Exercises

Display the data from each table in a bar graph and a circle graph.

1. **NASA Space Shuttle Expenses, 2000**

Operation	Millions of Dollars
Orbiter, integration	698.8
Propulsion	1,053.1
Mission, launch operations	738.8
Flight operations	244.6
Ground operations	510.3

Source: U.S. National Aeronautics and Space Administration

2. **Cable TV Revenue, 2006**

	Millions of Dollars
Airtime	4,566
Basic service	42,918
Pay-per-view, premium services	13,322
Installation	729
Other	27,188

Source: U.S. Census Bureau

Descriptive Statistics and Histograms

For numerical data, you can find the *mean,* the *median,* and the *mode.*

Mean	The sum of the data values in a data set divided by the number of data values
Median	The middle value of a data set that has been arranged in increasing or decreasing order. If the data set has an even number of values, the median is the mean of the middle two values.
Mode	The most frequently occurring value in a data set

Example 1

Find the mean, median, and mode for the following data set. **5 7 6 3 1 7 9 5 10 7**

Mean $$\frac{5+7+6+3+1+7+9+5+10+7}{10} = 6$$

Median 5, 7, 6, 3, 1, 7, 9, 5, 10, 7 Rearrange the numbers from least to greatest.

1, 3, 5, 5, 6, 7, 7, 7, 9, 10 The median is the mean of the two middle numbers, 6 and 7.

The median is $\frac{6+7}{2} = 6.5$.

Mode The most frequently occurring data value is 7.

The frequency of a data value is the number of times it occurs in a data set.
A *histogram* is a bar graph that shows the frequency of each data value.

Example 2

Use the survey results to make a histogram for the cost of a movie ticket at various theaters.

Survey of Movie Ticket Prices

$7	$8	$7	$9	$8	$9	$8	$10	$8

Exercises

Find the mean, the median, and the mode of each data set.

1. −3 4 5 5 −2 7 1 8 9

2. 0 0 1 1 2 3 3 5 3 8 7

3. 2.4 2.4 2.3 2.3 2.4 12.0

4. 1 1 1 1 2 2 2 3 3 4

5. 1.2 1.3 1.4 1.5 1.6 1.7 1.8

6. −4 −3 −2 −1 0 1 2 3 4

Make a histogram for each data set.

7. 7 4 8 6 6 8 7 7 5 7

8. 73 75 76 75 74 75 76 74 76 75

Operations With Rational Expressions

A *rational expression* is an expression that can be written in the form $\frac{\text{polynominal}}{\text{polynominal}}$, where the denominator is not zero. A rational expression is in simplest form if the numerator and denominator have no common factors except 1.

Example 1

Write the expression $\frac{4x + 8}{x + 2}$ in simplest form.

$\frac{4x + 8}{x + 2} = \frac{4(x + 2)}{x + 2}$ Factor the numerator.

$= 4$ Divide out the common factor $x + 2$.

To add or subtract two rational expressions, use a common denominator.

Example 2

Simplify $\frac{x}{2y} + \frac{x}{3y}$.

$\frac{x}{2y} + \frac{x}{3y} = \frac{x}{2y} \cdot \frac{3}{3} + \frac{x}{3y} \cdot \frac{2}{2}$ The common denominator of 3y and 2y is 6y.

$= \frac{3x}{6y} + \frac{2x}{6y}$

$= \frac{5x}{6y}$ Add the numerators.

To multiply rational expressions, first find and divide out any common factors in the numerators and the denominators. Then multiply the remaining numerators and denominators. To divide rational expressions, first use a reciprocal to change the problem to multiplication.

Example 3

Simplify $\frac{40x^2}{21} \div \frac{5x}{14}$.

$\frac{40x^2}{21} \div \frac{5x}{14} = \frac{40x^2}{21} \cdot \frac{14}{5x}$ Change dividing by $\frac{5x}{14}$ to multiplying by the reciprocal, $\frac{14}{5x}$.

$= \frac{{}^{8}\cancel{40}\cancel{x2}^{1}}{{}_{3}\cancel{21}} \times \frac{\cancel{14}^{2}}{\cancel{5x}_{1}}$ Divide out the common factors 5, x, and 7.

$= \frac{16x}{3}$ Multiply the numerators ($8x \cdot 2$). Multiply the denominators ($3 \cdot 1$).

Exercises

Write each expression in simplest form.

1. $\frac{4a^2b}{12ab^3}$ **2.** $\frac{5n + 15}{n + 3}$ **3.** $\frac{x - 7}{2x - 14}$ **4.** $\frac{28c^2(d - 3)}{35c(d - 3)}$

Perform the indicated operation.

5. $\frac{3x}{2} + \frac{5x}{2}$ **6.** $\frac{3x}{8} + \frac{5x}{8}$ **7.** $\frac{5}{h} - \frac{3}{h}$ **8.** $\frac{6}{11p} - \frac{9}{11p}$ **9.** $\frac{3x}{5} - \frac{x}{2}$

10. $\frac{13}{2x} - \frac{13}{3x}$ **11.** $\frac{7x}{5} + \frac{5x}{7}$ **12.** $\frac{5a}{b} + \frac{3a}{5b}$ **13.** $\frac{7x}{8} \cdot \frac{32x}{35}$ **14.** $\frac{3x^2}{2} \cdot \frac{6}{x}$

15. $\frac{8x^2}{5} \cdot \frac{10}{x^3}$ **16.** $\frac{7x}{8} \cdot \frac{64}{14x}$ **17.** $\frac{16}{3x} \div \frac{5}{3x}$ **18.** $\frac{4x}{5} \div \frac{16}{15x}$ **19.** $\frac{x^3}{8} \div \frac{x^2}{16}$

Reference

Table 1 Measures

	United States Customary	Metric
Length	12 inches (in.) = 1 foot (ft)	10 millimeters (mm) = 1 centimeter (cm)
	36 in. = 1 yard (yd)	100 cm = 1 meter (m)
	3 ft = 1 yard	1000 mm = 1 meter
	5280 ft = 1 mile (mi)	1000 m = 1 kilometer (km)
	1760 yd = 1 mile	
Area	144 square inches (in.^{2}) = 1 square foot (ft^2)	100 square millimeters (mm^2) = 1 square centimeter (cm^2)
	9 ft^2 = 1 square yard (yd^2)	10,000 cm^2 = 1 square meter (m^2)
	43,560 ft^2 = 1 acre (a)	10,000 m^2 = 1 hectare (ha)
	4840 yd^2 = 1 acre	
Volume	1728 cubic inches (in.^{3}) = 1 cubic foot (ft^3)	1000 cubic millimeters (mm^3) = 1 cubic centimeter (cm^3)
	27 ft^3 = 1 cubic yard (yd^3)	1,000,000 cm^3 = 1 cubic meter (m^3)
Liquid Capacity	8 fluid ounces (fl oz) = 1 cup (c)	1000 milliliters (mL) = 1 liter (L)
	2 c = 1 pint (pt)	1000 L = 1 kiloliter (kL)
	2 pt = 1 quart (qt)	
	4 qt = 1 gallon (gal)	
Weight or Mass	16 ounces (oz) = 1 pound (lb)	1000 milligrams (mg) = 1 gram (g)
	2000 pounds = 1 ton (t)	1000 g = 1 kilogram (kg)
		1000 kg = 1 metric ton
Temperature	32°F = freezing point of water	0°C = freezing point of water
	98.6°F = normal human body temperature	37°C = normal human body temperature
	212°F = boiling point of water	100°C = boiling point of water

	Customary Units and Metric Units	
Length	1 in. = 2.54 cm	1 cm ≈ 0.39 in.
	1 ft ≈ 0.305 m	1 m ≈ 3.28 ft
	1mi ≈ 1.61 km	1 km ≈ 0.62 mi
Area	1 acre = 0.40 ha	1 ha = 2.47 acres
Capacity	1 qt ≈ 0.95 L	1 L ≈ 1.06 qt
Weight and Mass	1 oz ≈ 28.4 g	1 g ≈ 0.035 oz
	1 lb ≈ 0.45 kg	1 kg ≈ 2.205 lb

	Time	
60 seconds (s) = 1 minute (min)	4 weeks (approx.) = 1 month (mo)	12 months = 1 year
60 minutes = 1 hour (h)	365 days = 1 year (yr)	10 years = 1 decade
24 hours = 1 day (d)	52 weeks (approx.) = 1 year	100 years = 1 century
7 days = 1 week (wk)		

Table 2 **Reading Math Symbols**

Symbols	Words
$\cdot$, $\times$	multiplication sign, times
$\pm$	plus or minus positive or negative
$=$	equals
$\overset{?}{=}$	equals?
$\approx$	is approximately equal to
$\neq$	is not equal to
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\cong$	is congruent to
$\sim$	is similar to
( )	parentheses for grouping
[ ]	brackets for grouping
{ }	braces for a set
%	percent
$\lvert a \rvert$	absolute value of a
$-a$	opposite of a
$a : b$; $\frac{a}{b}$	ratio of a to b
$\frac{1}{a}$, a^{-1}, $a \neq 0$	reciprocal of a
a^n	nth power of a
a^{-n}	$\frac{1}{a^n}$, $a \neq 0$
$\sqrt{a}$	nonnegative square root of a
$\sqrt[n]{a}$	nth root of a (nonnegative if n even)
° as in $a°$	degree(s)
$\circ$ as in $f \circ g$	composition of functions
π	pi, an irrational number, approximately equal to 3.14
e	an irrational number approximately equal to 2.72
i	the imaginary number $\sqrt{-1}$
$a + bi$, $b \neq 0$	a complex number
∞	infinity
$\sum$	sigma, summation
$\overleftrightarrow{AB}$	line through points A and B
$\overline{AB}$	segment with endpoints A and B
AB	length of $\overline{AB}$; distance between points A and B
$\angle A$	angle A
$m\angle A$	measure of angle A
$\triangle ABC$	triangle ABC
(x, y)	ordered pair
$x_1, x_2, \ldots$	specific values of the variable x
$y_1, y_2, \ldots$	specific values of the variable y
$f(x)$	f of x; the function value at x
f^{-1}	function inverse
log	logarithm
ln	natural logarithm
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	matrix
^	raised to a power (in a spreadsheet formula)
*	multiply (in a spreadsheet formula)
/	divide (in a spreadsheet formula)
. . .	and so on

Properties and Formulas

Order of Operations

1. Perform any operation(s) inside grouping symbols.
2. Simplify any terms with exponents.
3. Multiply and divide in order from left to right.
4. Add and subtract in order from left to right.

The Pythagorean Theorem

In a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

$a^2 + b^2 = c^2$

The Distance Formula

The distance d between any two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The Midpoint Formula

The midpoint M of a line segment with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Chapter 1 Expressions, Equations, and Inequalities

Closure

For all real numbers a and b, $a + b$ and $a \cdot b$ are real numbers.

The Associative Properties

For all real numbers a, b, and c:

$(a + b) + c = a + (b + c)$
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

The Commutative Properties

For all real numbers a and b:

$a + b = b + a$ and $a \cdot b = b \cdot a$

The Identity Properties

For every real number a:

$a + 0 = a$ and $0 + a = a$	$a \cdot 1 = a$ and $1 \cdot a = a$
0 is the additive identity.	1 is the multiplicative identity.

The Inverse Properties

For every real number a:

$a + (-a) = 0$ and $a \cdot \frac{1}{a} = 1 \quad (a \neq 0)$

The Distributive Properties

For all real numbers a, b, and c:

$a(b + c) = ab + ac$	$(b + c)a = ba + ca$
$a(b - c) = ab - ac$	$(b - c)a = ba - ca$

Multiplication

Let a represent a real number.
Multiplication by 0: $0 \cdot a = 0$.
Multiplication by –1: $-1 \cdot a = -a$

Opposites

Let a and b represent real numbers.

Opposite of a Sum:	$-(a + b) = -a + (-b) = -a - b$
Opposite of a Difference:	$-(a - b) = -a + b = b - a$
Opposite of a Product:	$-(ab) = -a \cdot b = a \cdot (-b)$
Opposite of an Opposite:	$-(-a) = a$

Properties of Equality

Assume a, b, and c represent real numbers.

Reflexive:	$a = a$
Symmetric:	If $a = b$, then $b = a$.
Transitive:	If $a = b$ and $b = c$, then $a = c$.
Substitution:	If $a = b$, then you can replace a with b and vice versa.
Addition:	If $a = b$, then $a + c = b + c$.
Subtraction:	If $a = b$, then $a - c = b - c$.
Multiplication:	If $a = b$, then $ac = bc$.
Division:	If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

Properties of Inequality

Let a, b, and c represent real numbers.

Transitive:	If $a > b$ and $b > c$, then $a > c$.
Addition:	If $a > b$, then $a + c > b + c$.
Subtraction:	If $a > b$, then $a - c > b - c$.
Multiplication:	If $a > b$ and $c > 0$, then $ac > bc$. If $a > b$ and $c < 0$, then $ac < bc$.
Division:	If $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$. If $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$.

Chapter 2 Functions, Equations, and Graphs

Direct Variation

$y = kx$ or $\frac{y}{x} = k$, where $k \neq 0$.

Slope of a Line Containing (x_1, y_1) and (x_2, y_2)

$\text{slope} = \frac{\text{vertical change (rise)}}{\text{horizontal change (run)}} = \frac{y_2 - y_1}{x_2 - x_1}$,

where $x_2 - x_1 \neq 0$

Point-Slope Equation of a Line

The equation of the line through point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$.

Function Families

Assume a, k, and h are positive numbers.

Parent	$y = f(x)$
Reflection across x-axis	$y = -f(x)$
Vertical stretch ($a > 1$) Vertical shrink ($0 < a < 1$)	$y = af(x)$
Translation	
horizontal to left by h	$y = f(x + h)$
horizontal to right by h	$y = f(x - h)$
vertical up by k	$y = f(x) + k$
vertical down by k	$y = f(x) - k$

Chapter 4 Quadratic Functions and Equations

Quadratic Functions

Parent	$y = x^2$
Reflection across x-axis	$y = -x^2$
Stretch ($a > 1$) Shrink ($0 < a < 1$)	$y = ax^2$
Translation	
horizontal by h vertical by k	$y = (x - h)^2 + k$
Vertex Form	$y = a(x - h)^2 + k$
Standard Form	$y = f(x) = ax^2 + bx + c$

The graph is a parabola that opens up if $a > 0$ and down if $a < 0$.

The vertex is (h, k) (Vertex Form) and $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ (Standard Form).

The axis of symmetry is $x = h$ (Vertex Form) and $x = -\frac{b}{2a}$ (Standard Form).

Factoring Perfect-Square Trinomials

$a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

Factoring a Difference of Two Squares

$a^2 - b^2 = (a + b)(a - b)$

Multiplication Property of Square Roots

For any numbers $a \geq 0$ and $b \geq 0$, $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

Division Property of Square Roots

For any numbers $a \geq 0$ and $b > 0$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

Zero-Product Property

If $ab = 0$, then $a = 0$ or $b = 0$.

The Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Discriminant

The discriminant of a quadratic equation in the form $ax^2 + bx + c = 0$ is $b^2 - 4ac$.

$b^2 - 4ac > 0 \Rightarrow$ two real solutions
$b^2 - 4ac = 0 \Rightarrow$ one real solution
$b^2 - 4ac < 0 \Rightarrow$ two complex solutions

Square Root of a Negative Real Number

For any positive number a,
$\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$.
Example: $\sqrt{-5} = i\sqrt{5}$
Note that
$(\sqrt{-5})^2 = (i\sqrt{5})^2 = i^2(\sqrt{5})^2 = -1 \cdot 5 = -5$ (not 5).

Chapter 5 Polynomials and Polynomial Functions

End Behavior of a Polynomial Function

The end behavior of a polynomial function of degree n with leading term ax^n:

a	n	end behavior
positive	even	up and up
positive	odd	down and up
negative	even	down and down
negative	odd	up and down

Factor Theorem

The expression $x - a$ is a linear factor of a polynomial if and only if the value a is a zero of the related polynomial function.

Remainder Theorem

If you divide a polynomial $P(x)$ of degree $n \geq 1$ by $x - a$, then the remainder is $P(a)$.

Factoring a Sum or Difference of Cubes

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Rational Root Theorem

Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients.
Integer roots of $P(x) = 0$ must be factors of a_0.
Rational roots have reduced form $\frac{p}{q}$ where p is an integer factor of a_0 and q is an integer factor of a_n.

Conjugate Root Theorems

Suppose $P(x)$ is a polynomial with *rational* coefficients.
If $a + \sqrt{b}$ is an irrational root with a and b rational, then $a - \sqrt{b}$ is also a root.

Suppose $P(x)$ is a polynomial with *real* coefficients.
If $a + bi$ is a complex root with a and b real, then $a - bi$ is also a root.

Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly n roots, including multiple and complex roots.

Binomial Theorem

For every positive integer n, $(a + b)^n =$
$P_0a^n + P_1a^{n-1}b + P_2a^{n-2}b^2 + \cdots + P_{n-1}ab^{n-1} + P_nb^n$
where $P_0, P_1, \ldots, P_n$ are the numbers in the nth row of Pascal's Triangle.

Chapter 6 Radical Functions and Rational Exponents

Properties of Exponents

For any nonzero number a and any integers m and n,

$a^0 = 1$ $\qquad$ $(ab)^n = a^nb^n$

$\frac{a^m}{a^n} = a^{m-n}$ $\qquad$ $a^m \cdot a^n = a^{m+n}$

$a^{-n} = \frac{1}{a^n}$ $\qquad$ $(a^m)^n = a^{mn}$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

nth Roots of nth Powers

For any real number a,

$$\sqrt[n]{a^n} = \begin{cases} a \text{ if } n \text{ is odd} \\ |a| \text{ if } n \text{ is even} \end{cases}$$

Combining Radical Expressions: Products

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.

Combining Radical Expressions: Quotients

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $b \neq 0$, then $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.

Properties of Rational Exponents

If the nth root of a is a real number and m is an integer, then $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$. If m is negative, $a \neq 0$.

Composition of Inverse Functions

If f and f^{-1} are inverse functions, then $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$ for x in the domains of f and f^{-1}, respectively.

Radical Functions

	Square Root	nth Root
Parent	$y = \sqrt{x}$	$y = \sqrt[n]{x}$
Reflection across x-axis	$y = -\sqrt{x}$	$y = -\sqrt[n]{x}$
Stretch $(a > 1)$	$y = a\sqrt{x}$	$y = a\sqrt[n]{x}$
Shrink $(0 < a < 1)$		
Translation		
horizontal by h	$y = \sqrt{x - h} + k$	$y = \sqrt[n]{x - h} + k$
vertical by k		

Chapter 7 Exponential and Logarithmic Functions

Exponential Functions

Parent, $b > 0, b \neq 1$	$y = b^x$
Reflection across x-axis	$y = -b^x$
Stretch $(a > 1)$	$y = ab^x$
Shrink $(0 < a < 1)$	
Translation	
horizontal by h	$y = b^{x-h} + k$
vertical by k	

Continuously Compounded Interest

$A(t) = P \cdot e^{rt}$, where $A(t)$ represents the total, P represents the principal, r represents the interest rate, and t represents time in years.

Logarithmic Functions

	Base b	Base e
Parents, $b > 0, b \neq 1$	$y = \log_b x$	$y = \ln x$
Reflection across x-axis	$y = -\log_b x$	$y = -\ln x$
Stretch $(a > 1)$	$y = a \log_b x$	$y = a \ln x$
Shrink $(0 < a < 1)$		
Translation		
horizontal by h	$y = \log_b (x - h) + k$	$y = \ln (x - h) + k$
vertical by k		

Properties of Logarithms

For any positive numbers m, n, and b where $b \neq 1$

Product Property:	$\log_b mn = \log_b m + \log_b n$
Quotient Property:	$\log_b \frac{m}{n} = \log_b m - \log_b n$
Power Property:	$\log_b m^n = n \log_b m$

Change of Base Formula

For any positive numbers, m, b, and c, with $b \neq 1$ and $c \neq 1$, $\log_b m = \frac{\log_c m}{\log_c b}$.

Chapter 8 Rational Functions

Inverse Variation

$xy = k$, $y = \frac{k}{x}$, or $x = \frac{k}{y}$, where $k \neq 0$.

Combined Variation

z varies jointly with x and y: $z = kxy$
z varies jointly with x and y and inversely with w: $z = \frac{kxy}{w}$
z varies directly with x and inversely with the product wy: $z = \frac{kx}{wy}$

Reciprocal Functions

Parent	$y = \frac{1}{x}, x \neq 0$
Reflection across x-axis	$y = -\frac{1}{x}, x \neq 0$
Stretch $(a > 1)$	$y = \frac{a}{x}, x \neq 0$
Shrink $(0 < a < 1)$	
Translation	
horizontal by h	$y = \frac{a}{x-h} + k, x \neq h$
vertical by k	
Asymptotes	$y = k$ (horiz.), $x = h$ (vert.)

Chapter 9 Sequences and Series

Arithmetic Mean of Two Numbers

$\frac{x + y}{2}$

Arithmetic Sequence

A recursive definition for an arithmetic sequence with a starting value a and a common difference d has two parts:

$a_1 = a$: initial condition

$a_{n+1} = a_n + d$, for $n \geq 1$: recursive formula

An explicit definition for this sequence is the formula:

$a_n = a + (n - 1)d$ for $n \geq 1$.

Geometric Sequence

A recursive definition for a geometric sequence with a starting value a and a common ratio r has two parts:

$a_1 = a$: initial condition

$a_{n+1} = a_n \cdot r$, for $n \geq 1$: recursive formula

An explicit definition for this sequence is the formula:

$a_n = ar^{n-1}$, for $n \geq 1$.

Sum of a Finite Arithmetic Series

The sum S_n of a finite arithmetic series

$a_1 + a_2 + a_3 + \cdots + a_n$ is $S_n = \frac{n}{2}(a_1 + a_n)$

where a_1 is the first term, a_n is the nth term, and n is the number of terms.

Sum of a Finite Geometric Series

The sum S_n of a finite geometric series

$a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}$ is $S_n = \frac{a_1(1 - r^n)}{1 - r}$

where a_1 is the first term, r is the common ratio, and n is the number of terms.

Sum of an Infinite Geometric Series

An infinite geometric series with $|r| < 1$ converges to the sum S given by the following formula:

$S = \frac{a_1}{1 - r}$.

Chapter 10 Quadratic Relations and Conic Sections

Parabolas

Vertical	Vertex (0, 0)	Vertex (h, k)
Equation	$y = \frac{1}{4c}x^2$	$y = \frac{1}{4c}(x - h)^2 + k$
Focus	(0, c)	(h, $c + k$)
Directrix	$y = -c$	$y = -c + k$
Horizontal	Vertex (0, 0)	Vertex (h, k)
Equation	$x = \frac{1}{4c}y^2$	$x = \frac{1}{4c}(y - k)^2 + h$
Focus	(c, 0)	($c + h$, k)
Directrix	$x = -c$	$x = -c + h$

Circles, radius = r

	Center (0, 0)	Center (h, k)
Equation	$x^2 + y^2 = r^2$	$(x - h)^2 + (y - k)^2 = r^2$

Ellipses

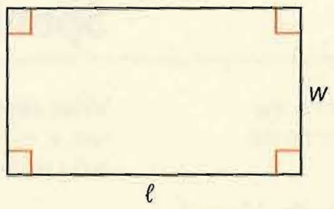
Horizontal, $a > b$	Center (0, 0)	Center (h, k)
Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$
Vertices	$(\pm a, 0)$	$(\pm a + h, k)$
Co-Vertices	$(0, \pm b)$	$(h, \pm b + k)$
Foci, $c^2 = a^2 - b^2$	$(\pm c, 0)$	$(\pm c + h, k)$
Major axis	$y = 0$	$y = k$
Minor axis	$x = 0$	$x = h$
Vertical, $a > b$	Center (0, 0)	Center (h, k)
Equation	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$	$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$
Vertices	$(0, \pm a)$	$(h, \pm a + k)$
Co-Vertices	$(\pm b, 0)$	$(\pm b + h, k)$
Foci, $c^2 = a^2 - b^2$	$(0, \pm c)$	$(h, \pm c + k)$
Major axis	$x = 0$	$x = h$
Minor axis	$y = 0$	$y = k$

Hyperbolas

Horizontal, $a > b$	Center (0, 0)	Center (h, k)
Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$
Vertices	$(\pm a, 0)$	$(\pm a + h, k)$
Foci, $c^2 = a^2 + b^2$	$(\pm c, 0)$	$(\pm c + h, k)$
Transverse axis	$y = 0$	$y = k$
Asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{b}{a}(x - h) + k$
Vertical, $a > b$	Center (0, 0)	Center (h, k)
Equation	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$
Vertices	$(0, \pm a)$	$(h, \pm a + k)$
Foci, $c^2 = a^2 + b^2$	$(0, \pm c)$	$(h, \pm c + k)$
Transverse axis	$x = 0$	$x = h$
Asymptotes	$y = \pm\frac{a}{b}x$	$y = \pm\frac{a}{b}(x - h) + k$

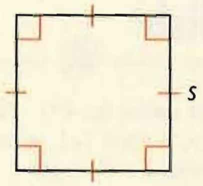
Formulas of **Geometry**

You will use a number of geometric formulas as you work through your algebra book. Here are some perimeter, area, and volume formulas.



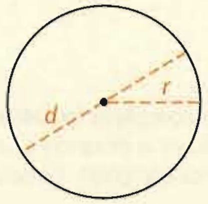
$P = 2\ell + 2w$
$A = \ell w$

Rectangle



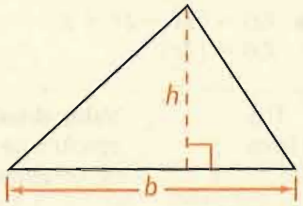
$P = 4s$
$A = s^2$

Square



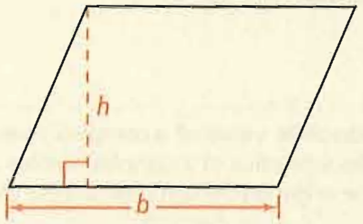
$C = 2\pi r$ or $C = \pi d$
$A = \pi r^2$

Circle



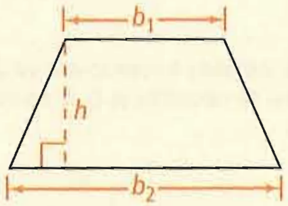
$A = \frac{1}{2}bh$

Triangle



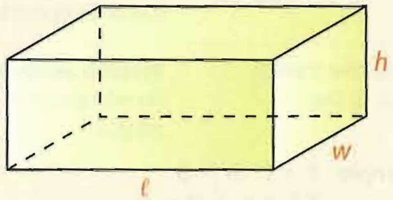
$A = bh$

Parallelogram



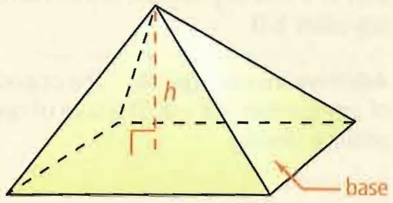
$A = \frac{1}{2}(b_1 + b_2)h$

Trapezoid



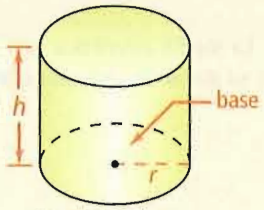
$SA = 2(\ell w + wh + h\ell)$
$V = Bh$
$V = \ell wh$

Right Prism



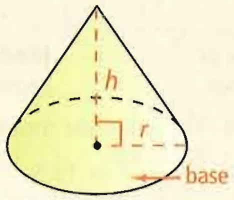
$V = \frac{1}{3}Bh$

Pyramid



$SA = 2\pi r(r + h)$
$V = Bh$
$V = \pi r^2 h$

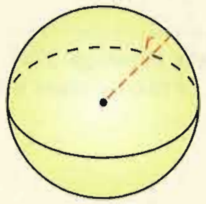
Right Cylinder



$V = \frac{1}{3}Bh$

$V = \frac{1}{3}\pi r^2 h$

Right Cone



$SA = 4\pi r^2$
$V = \frac{1}{3}\pi r^3$

Sphere

English/Spanish Illustrated **Glossary**

English | Spanish

A

Absolute value (p. 41) The absolute value of a real number, x, written $|x|$, is its distance from zero on the number line.

Valor absoluto (p. 41) El valor absoluto de un número real, x, escrito como $|x|$, es su distancia desde cero en la recta numérica.

Example $|3| = 3$
$|-4| = 4$

Absolute value function (p. 107) A function of the form $f(x) = |mx + b| + c$, where $m \neq 0$, is an absolute value function.

Función de valor absoluto (p. 107) Una función de la forma $f(x) = |mx + b| + c$, donde $m \neq 0$, es una función de valor absoluto.

Example $f(x) = |3x - 2| + 3$
$f(x) = |2x|$

Absolute value of a complex number (p. 249) The absolute value of a complex number is its distance from the origin on the complex number plane. In general, $|a + bi| = \sqrt{a^2 + b^2}$.

Valor absoluto de un número complejo (p. 249) El valor absoluto de un número complejo es la distancia a la que está del origen en el plano de números complejo. Generalmente, $|a + bi| = \sqrt{a^2 + b^2}$.

Example $|3 - 4i| = \sqrt{3^2 + (-4)^2} = 5$

Additive identity (p. 14) The additive identity is 0. The sum of 0 and any number is that number. The sum of opposites is 0.

Identidad aditiva (p. 14) La identidad aditiva es 0. La suma de 0 y cualquier número es ese mismo número. La suma de opuestos es 0.

Additive inverse (p. 14) The opposite or additive inverse of any number a is $-a$. The sum of opposites is 0, the additive identity.

Inverso aditivo (p. 14) El opuesto o inverso aditivo de un número a es $-a$. La suma de opuestos es 0, la identidad aditiva.

Example $3 + (-3) = 0$
$5.2 + (-5.2) = 0$

Algebraic expression (p. 5) An algebraic expression is a mathematical phrase that contains one or more variables.

Expresión algebraica (p. 5) Una expresión algebraica es una frase matemática.

Example $2x + 3$
$z - y$

Arithmetic mean (p. 574) The arithmetic mean, or average, of two numbers is their sum divided by two.

Media aritmética (p. 574) La media aritmética, o promedio, de dos números es su suma dividida por dos.

Example The arithmetic mean of 12 and 15 is $\frac{12 + 15}{2} = 13.5$.

English

Spanish

Arithmetic sequence (p. 572) An arithmetic sequence is a sequence with a constant difference between consecutive terms.

Progresión aritmética (p. 572) Una secuencia aritmética es una secuencia de números en la que la diferencia entre dos números consecutivos es constante.

Example The arithmetic sequence 1, 5, 9, 13, . . . has a common difference of 4.

Arithmetic series (p. 587) An arithmetic series is a series whose terms form an arithmetic sequence.

Serie aritmética (p. 587) Una serie aritmética es una serie cuyos términos forman una progresión aritmética.

Example $1 + 5 + 9 + 13 + 17 + 21$ is an arithmetic series with six terms.

Asymptote (p. 435) An asymptote is a line that a graph approaches as x or y increases in absolute value.

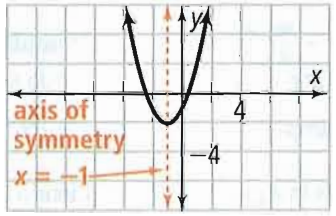
Asíntota (p. 435) Una asíntota es una recta a la cual se acerca una gráfica a medida que x o y aumentan de valor absoluto.

Example The function $y = \frac{x + 2}{x - 2}$ has $x = 2$ as a vertical asymptote and $y = 1$ as a horizontal asymptote.

Axis of symmetry (pp. 107, 194) The axis of symmetry is the line that divides a figure into two parts that are mirror images.

Eje de simetría (pp. 107, 194) El eje de simetría es la recta que divide una figura en dos partes que son imágenes una de la otra.

Example



$y = x^2 + 2x - 1$

B

Binomial Theorem (p. 327) For every positive integer n, $(a + b)^n = P_0a^n + P_1a^{n-1}b + P_2a^{n-2}b^2 + \cdots + P_{n-1}ab^{n-1} + P_nb^n$ where $P_0, P_1, \ldots, P_n$ are the numbers in the row of Pascal's Triangle that has n as its second number.

Teorema binomial (p. 327) Para cada número entero positivo n, $(a + b)^n = P_0a^n + P_1a^{n-1}b + P_2a^{n-2}b^2 + \cdots + P_{n-1}ab^{n-1} + P_nb^n$, donde $P_0, P_1, \ldots, P_n$ son los números de la fila del Triángulo de Pascal cuyo segundo número es n.

Example

$$\begin{aligned}(x + 1)^3 &= {}_3C_0(x)^3 + {}_3C_1(x)^2(1)^1 \\ &\quad + {}_3C_2(x)^1(1)^2 + {}_3C_3(1)^3 \\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

Boundary (p. 114) A boundary of the graph of a linear inequality is a line in the coordinate plane. It separates the solutions of the inequality from the nonsolutions. Points of the line itself may or may not be solutions.

Límite (p. 114) Un límite de la gráfica de una desigualdad lineal es una línea en el plano de coordenadas. Ésta separa las soluciones de la desigualdad de las no soluciones. Las soluciones pueden ser o no puntos de la línea.

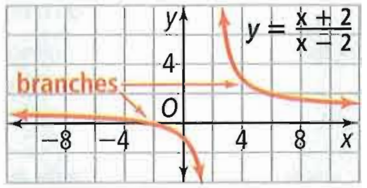
English

Spanish

Branch (p. 508) Each piece of a discontinuous graph is called a branch.

Rama (p. 508) Cada segmento de una gráfica discontinua se llama rama.

Example

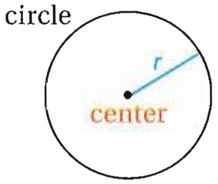


C

Center of a circle (p. 630) The center of a circle is the point that is the same distance from every point on the circle.

Centro de un círculo (p. 630) El centro de un círculo es el punto que está situado a la misma distancia de cada punto del círculo.

Example



Center of an ellipse (p. 639) The center of an ellipse is the midpoint of the major axis.

Centro de una elipse (p. 639) El centro de una elipse es el punto medio entre los dos ejes mayores.

Change of Base Formula (p. 464) $\log_b M = \frac{\log_c M}{\log_c b}$, where M, b, and c are positive numbers, and $b \neq 1$ and $c \neq 1$.

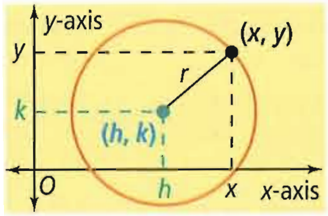
Fórmula de cambio de base (p. 464) $\log_b M = \frac{\log_c M}{\log_c b}$, donde M, b y c son números positivos y $b \neq 1$ y $c \neq 1$.

Example $\log_3 8 = \frac{\log 8}{\log 3} \approx 1.8928$

Circle (p. 630) A circle is the set of all points in a plane at a distance r from a given point. The standard form of the equation of a circle with center (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

Círculo (p. 630) Un círculo es el cojunto de todos los puntos situados en un plano a una distancia r de un punto dado. La forma normal de la ecuación cuyo centro es (h, k) y cuyo radio es r es $(x - h)^2 + (y - k)^2 = r^2$.

Example



Coefficient (p. 20) The numerical factor in a term.

Coeficiente (p. 20) El factor numérico de un término.

Example The coefficient of $-3k$ is -3.

English

Spanish

Combined variation (p. 501) A combined variation is a relation in which one variable varies with respect to each of two or more variables.

Variación combinada (p. 501) Una variación combinada es una relación en la que una variable varía con respecto a cada una de dos o más variables.

Example $y = kx^2\sqrt{z}$

$z = \frac{kx}{y}$

Common difference (p. 572) A common difference is the difference between consecutive terms of an arithmetic sequence.

Diferencia común (p. 572) La diferencia común es la diferencia entre los términos consecutivos de una progresión aritmética.

Example The arithmetic sequence 1, 5, 9, 13, . . . has a common difference of 4.

Common logarithm (p. 453) A common logarithm is a logarithm that uses base 10. You can write the common logarithm $\log_{10} y$ as $\log y$.

Logaritmo común (p. 453) El logaritmo común es un logaritmo de base 10. El logaritmo común $\log_{10} y$ se expresa como $\log y$.

Example $\log 1 = 0$

$\log 10 = 1$

$\log 50 = 1.698970004 \ldots$

Common ratio (p. 580) A common ratio is the ratio of consecutive terms of a geometric sequence.

Razón común (p. 580) Una razón común es la razón de términos consecutivos en una secuencia geométrica.

Example The geometric sequence 2.5, 5, 10, 20, . . . has a common ratio of 2.

Completing the square (p. 235) Completing the square is the process of finding a constant c to add to $x^2 + bx$ so that $x^2 + bx + c$ is the square of a binomial.

Completar el cuadrado (p. 235) Completar un cuadrado es el proceso mediante el cual se halla una constante c que se le pueda sumar a $x^2 + bx$, de manera que $x^2 + bx + c$ sea el cuadrado de un binomio.

Example $x^2 - 12x + ■$

$x^2 - 12x + \left(\frac{-12}{2}\right)^2$

$x^2 - 12x + 36$

Complex conjugates (p. 251) Number pairs of the form $a + bi$ and $a - bi$ are complex conjugates.

Conjugados complejos (p. 251) Los pares de números de la forma $a + bi$ y $a - bi$ son conjugados complejos.

Example The complex numbers $2 - 3i$ and $2 + 3i$ are complex conjugates.

Complex fraction (p. 536) A complex fraction is a rational expression that has a fraction in its numerator or denominator, or in both its numerator and denominator.

Fracción compleja (p. 536) Una fracción compleja es una expresión racional en la que el numerador, el denominador o ambos son una fracción.

Example $\frac{2}{\frac{1}{5}}$ $\frac{\frac{2}{7}}{\frac{3}{2}}$

English

Spanish

Complex number (p. 249) Complex numbers are the real numbers and the imaginary numbers.

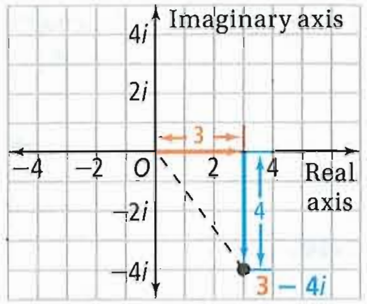
Número complejo (p. 249) Los números complejos son los números reales y los números imaginarios.

Example $6 + i$, 7, $2i$

Complex number plane (p. 249) The complex number plane is identical to the coordinate plane except each ordered pair (a, b) represents the complex number $a + bi$. The horizontal axis is the Real axis. The vertical axis is the Imaginary axis.

Plano de números complejos (p. 249) El plano de los números complejos es idéntico al plano de coordenadas, a excepción de que cada par ordenado (a, b) representa el número complejo $a + bi$. El eje horizontal es el eje real. El eje vertical es el eje imaginario.

Example



Composite function (p. 399) A composite function is a combination of two functions such that the output from the first function becomes the input for the second function.

Función compuesta (p. 399) Una función compuesta es la combinación de dos funciones. La cantidad de salida de la primera función es la cantidad de entrada de la segunda función.

Example

$$f(x) = 2x + 1 \quad g(x) = x^2 - 1$$

$$\begin{aligned}(g \circ f)(5) &= g(f(5)) = g(2(5) + 1)\\ &= g(11)\\ &= 11^2 - 1 = 120\end{aligned}$$

$$\begin{aligned}(f \circ g)(5) &= f(g(5)) = f((5)^2 - 1)\\ &= f(24)\\ &= 2(24) + 1 = 49\end{aligned}$$

Compound inequality (p. 36) You can join two inequalities with the word *and* or the word *or* to form a compound inequality.

Desigualdad compuesta (p. 36) Puedes unir dos desigualdades por medio de la palabra *y* o la palabra *o* para formar una desigualdad compuesta.

Example $-1 < x$ and $x \leq 3$

$x < -1$ or $x \geq 3$

Conic section (p. 614) A conic section is a curve formed by the intersection of a plane and a double cone.

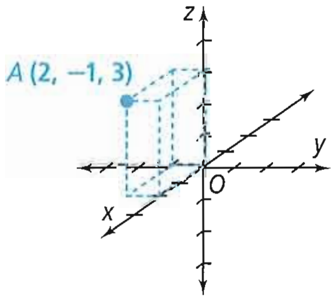
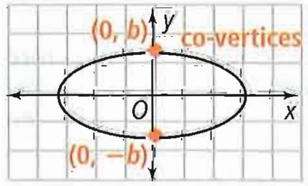
Sección cónica (p. 614) Una sección cónica es una curva que se forma por la intersección de un plano con un cono doble.

Example

ellipse

hyperbola

English	Spanish
Conjugate axis (p. 646) The conjugate axis for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a > b > 0$, is the segment from $(0, -b)$ to $(0, b)$. For $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, the conjugate axis is the segment from $(-b, 0)$ to $(b, 0)$.	**Eje conjugado (p. 646)** El eje conjugado de la hipérbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a > b > 0$, es el segmento desde el punto $(0, -b)$ hasta el punto $(0, b)$. Para $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, el eje conjugado es el segmento desde el punto$(-b, 0)$ hasta el punto $(b, 0)$.
Conjugate Root Theorem (p. 314) If $P(x)$ is a polynomial with rational coefficients, then the irrational roots of $P(x) = 0$ occur in conjugate pairs. That is, if $a + \sqrt{b}$ is an irrational root with a and b rational, then $a - \sqrt{b}$ is also a root. If $P(x)$ is a polynomial with real coefficients, then the complex roots of $P(x) = 0$ occur in conjugate pairs. That is, if $a + bi$ is a complex root with a and b real, then $a - bi$ is also a root.	**Teorema de raíces conjugadas (p. 314)** Si $P(x)$ es un polinomio con coeficientes racionales, entonces las raíces irracionales de $P(x) = 0$ ocurren en pares conjugados. Es decir, si $a + \sqrt{b}$ es una raíz irracional donde a y b son racionales, entonces $a - \sqrt{b}$ también es una raíz. Si $P(x)$ es un polinomio con coeficientes reales, entonces las raíces complejas de $P(x) = 0$ ocurren en los pares conjugados. Es decir, si $a + bi$ es una raíz compleja donde a y b son reales, entonces $a - bi$ también es una raíz.
Conjugates (p. 314) Number pairs of the form $a + \sqrt{b}$ and $a - \sqrt{b}$ are conjugates. **Example** $5 + \sqrt{3}$ and $5 - \sqrt{3}$ are conjugates.	**Conjugados (p. 314)** Los pares de números con la forma $a + \sqrt{b}$ y $a - \sqrt{b}$ son conjugados.
Consistent system (p. 137) A system of linear equations is consistent if it has at least one solution.	**Sistema consistente (p. 137)** Un sistema de ecuaciones lineales es consistente si tiene por lo menos una solución.
Constant (p. 5) A constant is a quantity whose value does not change.	**Constante (p. 5)** Una constante es una cantidad cuyo valor no cambia.
Constant of proportionality (p. 341) If $y = ax^b$ describes y as a power function of x, then y varies directly with, or is proportional to, the b^{th} power of x. The constant a is the constant of proportionality.	**Constante de proporcionalidad (p. 341)** Si $y = ax^b$ describe a y como una potencia de la función de x, entonces y varía directamente con, o es proporcional a, la b^{ma} potencia de x. La constante a es la constante de proporcionalidad.
Constant of variation (p. 68) The constant of variation is the ratio of the two variables in a direct variation and the product of the two variables in an inverse variation. **Example** In $y = 3.5x$, the constant of variation k is 3.5. In $xy = 5$, the constant of variation k is 5.	**Constante de variación (p. 68)** La constante de variación es la razón de dos variables en una variación directa y el producto de las dos variables en una variación inversa.
Constant term (p. 20) A constant term is a term with no variables.	**Término constante (p. 20)** Un término constante es un término que no tiene variables.
Constraint (p. 157) Constraints are restrictions on the variables of the objective function in a linear programming problem. *See* **Linear programming.**	**Restriccion (p. 157)** Las restricciones son limitaciones a las variables de una función objetiva en un problema de programación lineal. *Ver* **Linear programming.**
Continuous graph (p. 516) A graph is continuous if it has no jumps, breaks, or holes.	**Gráfica continua (p. 516)** Una gráfica es continua si no tiene saltos, interrupciones o huecos.

English	Spanish
Continuously compounded interest (p. 446) When interest is compounded continuously on principal P, the value A of an account is $A = Pe^{rt}$.	**Interés compuesto continuo (p. 446)** En un sistema donde el interés es compuesto continuamente sobre el capital P, el valor de A de una cuenta es $A = Pe^{rt}$.
Example Suppose that $P = \$1200$, $r = 0.05$, and $t = 3$. Then $A = 1200e^{0.05\cdot 3}$ $= 1200(2.718\ldots)^{0.15}$ ≈ 1394.20	
Converge (p. 598) An infinite series $a_1 + a_2 + \cdots + a_n + \cdots$ converges if the sum $a_1 + a_2 + \cdots + a_n$ get closer and closer to a real number as n increases.	**Convergir (p. 598)** Una serie infinita $a_1 + a_2 + \cdots a_n + \cdots$ es convergente si la suma $a_1 + a_2 + \cdots + a_n$ se aproxima cada vez más a un número real a medida que el valor de n incrementa.
Example $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converges.	
Coordinate space (p. 164) Coordinate space is a three-dimensional space where each point is described uniquely using an ordered triple of numbers.	**Espacio de coordenadas (p. 164)** Un espacio de coordenadas es un espacio tridimensional en el cual cada punto es definido de manera única por una tripleta ordenada de números.
Example 	
Correlation (p. 92) A correlation indicates the strength of a relationship between two data sets.	**Correlación (p. 92)** Una correlación indica la fuerza de una relación entre dos conjuntos de datos.
Correlation coefficient (p. 94) The correlation coefficient, r, indicates the strength of the correlation. The closer r is to 1 or −1, the more closely the data resembles a line and the more accurate your model is likely to be.	**Coeficiente de corrrelación (p. 94)** El coeficiente de correlación, r, indica la fuerza de la correlación. Mientras más cerca está r de 1 ó −1, más se parecen los datos a una línea y será más probable que tu modelo sea preciso.
Co-vertices (p. 639) The endpoints of the minor axis of an ellipse are the co-vertices of the ellipse.	**Covértices (p. 639)** Los puntos de intersección entre una elipse y los ejes menores son los covértices de la elipse.
Example 	

D

English	Spanish
Decay factor (p. 436) In an exponential function of the form $y = ab^x$, b is the decay factor if $0 < b < 1$. **Example** In the equation $y = 0.3^x$, 0.7 is the decay factor.	**Factor de decremento (p. 436)** En una función exponencial de la forma $y = ab^x$, b es el factor de decremento si, $0 < b < 1$.
Degree of a monomial (p. 280) The degree of a monomial in one variable is the exponent of the variable.	**Grado de un monomio (p. 280)** El grado de un monomio en una variable es el exponente de la variable.
Degree of a polynomial (p. 280) The degree of a polynomial is the greatest degree among its monomial terms. **Example** $P(x) = x^6 + 2x^3 - 3$ degree 6	**Grado de un polinomio (p. 280)** El grado de un polinomio es el grado mayor entre los términos de monomios.
Dependent system (p. 137) A system of equations that does not have a unique solution is a dependent system. **Example** $\begin{cases} y = 2x + 3 \\ -4x + 2y = 6 \end{cases}$ represents two equations for the same line, so it has many solutions. It is a dependent system.	**Sistema dependiente (p. 137)** Un sistema de ecuaciones es dependiente cuando no tiene una solución única.
Dependent variable (p. 63) If a function is defined by an equation using the variables x and y, where y represents output values, then y is the dependent variable. **Example** $y = 2x + 1$ y is the dependent variable.	**Variable dependiente (p. 63)** Si una función es definida por una ecuación que usa las variables x e y, donde y representa valores de salida, entonces y es la variable dependiente.
Descartes' Rule of Signs (p. 315) Let $P(x)$ be a polynomial with real coefficients written in standard form. - The number of positive real roots of $P(x) = 0$ is either equal to the number of sign changes between consecutive coefficients of $P(x)$ or is less than that by an even number; - The number of negative real roots of $P(x) = 0$ is either equal to the number of sign changes between consecutive coefficients of $P(-x)$ or is less than that by an even number. (Count multiple roots according to their multiplicity.)	**Regla de los signos de Descartes (p. 315)** Sea $P(x)$ un polinomio con coeficientes reales escritos en forma normal. - El número de raíces positivas reales de $P(x) = 0$ es igual al número de cambios de signos entre coeficientes consecutivos de $P(-x)$ o es menor que eso en un número par; - El número de raíces negativas reales de $P(x) = 0$ es igual al número de cambios de signos entre coeficientes consecutivos de $P(-x)$ o es menor que eso en un número par. (Cuenta las raíces múltiples según su multiplicidad).
Difference of cubes (p. 297) A difference of cubes is an expression of the form $a^3 - b^3$. It can be factored as $(a - b)(a^2 + ab + b^2)$. **Example** $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$	**Diferencia de dos cubos (p. 297)** La diferencia de dos cubos es una expresión de la forma $a^3 - b^3$. Se puede factorizar como $(a - b)(a^2 + ab + b^2)$.

English | Spanish

Difference of two squares (p. 220) A difference of two squares is an expression of the form $a^2 - b^2$. It can be factored as $(a + b)(a - b)$.

Diferencia de dos cuadrados (p. 220) La diferencia de dos cuadrados es una expresión de la forma $a^2 - b^2$. Se puede factorizar como $(a + b)(a - b)$.

Example $25a^2 - 4 = (5a + 2)(5a - 2)$
$m^6 - 1 = (m^3 + 1)(m^3 - 1)$

Direct variation (p. 68) A linear function defined by an equation of the form $y = kx$, where $k \neq 0$, represents direct variation.

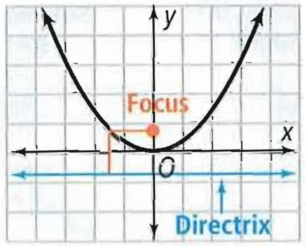
Variación directa (p. 68) Una función lineal definida por una ecuación de la forma $y = kx$, donde $k \neq 0$, representa una variación directa.

Example $y = 3.5x, y = 7x, y = -\frac{1}{2}x$

Directrix (p. 622) The directrix of a parabola is the fixed line used to define a parabola. Each point of the parabola is the same distance from the focus and the directrix.

Directriz (p. 622) La directriz de una parábola es la recta fija con que se define una parábola. Cada punto de la parábola está a la misma distancia del foco y de la directriz.

Example



Discontinuous graph (p. 516) A graph is discontinuous if it has a jump, break, or hole.

Gráfica discontinua (p. 516) Una gráfica es discontinua si tiene un salto, interrupción o hueco.

Discriminant (p. 242) The discriminant of a quadratic equation in the form $ax^2 + bx + c = 0$ is the value of the expression $b^2 - 4ac$.

Discriminante (p. 242) El discriminante de una ecuación cuadrática en la forma $ax^2 + bx + c = 0$ es el valor de la expresión $b^2 - 4ac$.

Example $3x^2 - 6x + 1$
discriminant $= (-6)^2 - 4(3)(1)$
$= 36 - 12 = 24$

Diverge (p. 598) An infinite series diverges if it does not converge.

Divergir (p. 598) Una serie infinita es divergente si no es convergente.

Example $1 + 2 + 4 + 8 + \cdots$ diverges.

Domain (p. 61) The domain of a relation is the set of all inputs, or x-coordinates, of the ordered pairs.

Dominio (p. 61) El dominio de una relación es el conjunto de todos los valores de entrada, o coordenadas x, de los pares ordenados.

Examples In the relation $\{(0, 1), (0, 2), (0, 3), (0, 4), (1, 3), (1, 4), (2, 1)\}$, the domain is $\{0, 1, 2\}$. In the function $f(x) = x^2 - 10$, the domain is all real numbers.

English

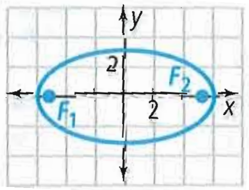
E

Spanish

Ellipse (p. 638) An ellipse is the set of points P in a plane such that the sum of the distances from P to two fixed points F_1 and F_2 is a given constant k. The standard form of the equation of an ellipse with its center at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if the major axis is horizontal and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ if the major axis is vertical, where $a > b$.

Elipse (p. 638) Una elipse es el conjunto de puntos P situados en un plano tal que la suma de las distancias entre P y dos puntos fijos F_1 y F_2 es una constante dada k. La forma normal de la ecuación de una elipse con su centro en el origen es $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ si el eje mayor es horizontal y $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ si el eje mayor es vertical, donde $a > b$.

Example



$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

$$F_1 = (-3\sqrt{3}, 0),\ F_2 = (3\sqrt{3}, 0)$$

End behavior (p. 282) End behavior of the graph of a function describes the directions of the graph as you move to the left and to the right, away from the origin.

Comportamiento extremo (p. 282) El comportamiento extremo de la gráfica de una función describe las direcciones de la gráfica al moverse a la izquierda y a la derecha, apartándose del origen.

Equation (p. 26) An equation is a statement that two algebraic expressions are equal.

Ecuación (p. 26) Una ecuación es un enunciado que describe dos expresiones algebraicas iguales.

Equivalent systems (p. 144) Equivalent systems are systems that have the same solution(s).

Sistemas equivalentes (p. 144) Sistemas equivalentes son sistemas que tienen la misma solución o las mismas soluciones.

Evaluate (p. 19) To evaluate an algebraic expression, substitute a number for each variable in the expression. Then simplify using the order of operations.

Evaluar (p. 19) Para evaluar una expresión algebraica, sustituye cada variable de la expresión con un número. Luego, simplifica usando el orden de operaciones.

Example When $x = 2$ and $y = -1$, $2x + 3y$ evaluates to 1.

Expand (p. 326) To expand the power of a binomial, multiply as needed, then write the polynomial in standard form.

Expandir (p. 326) Para expandir la potencia de un binomio, multiplica como sea necesario. Luego, escribe el polinomio
en forma normal.

Example

$$\begin{aligned}(x + 4)^3 &= (x + 4)(x + 4)^2\\ &= (x + 4)(x^2 + 8x + 16)\\ &= x^3 + 8x^2 + 16x + 4x^2 + 32x + 64\\ &= x^3 + 12x^2 + 48x + 64\end{aligned}$$

English

Spanish

Explicit formula (p. 565) An explicit formula expresses the nth term of a sequence in terms of n.

Fórmula explícita (p. 565) Una fórmula explícita expresa el n-ésimo término de una progresión en función de n.

Example Let $a_n = 2n + 5$ for positive integers n.
If $n = 7$, then $a_7 = 2(7) + 5 = 19$.

Exponential decay (p. 435) Exponential decay is modeled by a function of the form $y = ab^x$ with $0 < b < 1$.

Decaimiento exponencial (p. 435) El decaimiento exponencial se expresa con una función $y = ab^x$ donde $0 < b < 1$.

Exponential equation (p. 469) An exponential equation contains the form b^{cx}, with the exponent including a variable.

Ecuación exponencial (p. 469) Una ecuación exponencial tiene la forma b^{cx}, y su exponente incluye una variable.

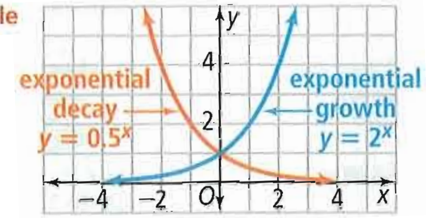
Example

$$\begin{aligned} 5^{2x} &= 270 \\ \log 5^{2x} &= \log 270 \\ 2x \log 5 &= \log 270 \\ 2x &= \frac{\log 270}{\log 5} \\ 2x &\approx 3.4785 \\ x &\approx 1.7392 \end{aligned}$$

Exponential function (p. 434) The general form of an exponential function is $y = ab^x$, where x is a real number, $a \neq 0$, $b > 0$, and $b \neq 1$. When $b > 1$, the function models exponential growth with growth factor b. When $0 < b < 1$, the function models exponential decay with decay factor b.

Función exponencial (p. 434) La forma general de una función exponencial es $y = ab^x$, donde x es un número real, $a \neq 0$, $b > 0$, y $b \neq 1$. Cuando $b > 1$, la función representa un incremento exponencial con factor de incremento b. Cuando $0 < b < 1$, la función representa el decremento exponencial con factor de decremento b.

Example



Exponential growth (p. 435) Exponential growth is modeled by a function of the form $y = ab^x$ with $b > 1$.

Crecimiento exponencial (p. 435) El crecimiento exponencial se expresa con una función de la forma $y = ab^x$ donde $b > 1$.

Extraneous solution (p. 42) An extraneous solution is a solution of an equation derived from an original equation but it is not a solution of the original equation.

Solución extraña (p. 42) Una solución extraña es una solución de una ecuación derivada de una ecuación dada, pero que no satisface la ecuación dada.

Example

$$\begin{aligned} \sqrt{x-3} &= x - 5 \\ x - 3 &= x^2 - 10x + 25 \\ 0 &= x^2 - 11x + 28 \\ 0 &= (x - 4)(x - 7) \\ x &= 4 \text{ or } 7 \end{aligned}$$

The number 7 is a solution, but 4 is not, since $\sqrt{4-3} \neq 4 - 5$.

F

English	Spanish
Factor Theorem (p. 289) The expression $x - a$ is a linear factor of a polynomial if and only if the value of a is a root of the related polynomial function.	**Teorema de factores (p. 289)** La expresión $x - a$ es un factor lineal de un polinomio si y sólo si el valor de a es una raíz de la función polinomial con la que se relaciona.
Example The value 2 makes the polynomial $x^2 + 2x - 8$ equal to zero. So, $x - 2$ is a factor of $x^2 + 2x - 8$.	
Factoring (p. 216) Factoring is rewriting an expression as the product of its factors.	**Descomposición factorial (p. 216)** Descomponer en factores es el proceso de escribir de nuevo una expresión como el producto de sus factores.
Example expanded form $x^2 + x - 56$ factored form $(x + 8)(x - 7)$	
Feasible region (p. 157) In a linear programming problem, the feasible region contains all the values that satisfy the constraints on the objective function.	**Región factible (p. 157)** En un problema de programación lineal, la región factible contiene todos los valores que satisfacen las restricciones de la función objetiva.
Finite Series (p. 587) A finite series is a series with a finite number of terms.	**Serie finite (p. 587)** Una serie finita es una serie con un número finito de términos.
Focal length (p. 622) The focal length of a parabola is the distance between the vertex and the focus.	**Distancia focal (p. 622)** La distancia focal de una parábola es la distancia entre el vértice y el foco.
Focus (plural: foci) of a hyperbola (p. 645) A hyperbola is the set of all points P in a plane such that the difference of the distances from P to two fixed points is constant. Each of the fixed points is a focus of the hyperbola.	**Foco de una hipérbola (p. 645)** Una hipérbola es el conjunto de puntos P en un plano tal que la diferencia de las distancias desde P hasta dos puntos fijos es constante. Cada uno de los puntos fijos es el foco de la hipérbola.
Focus (plural: foci) of a parabola (p. 622) A parabola is the set of all points in a plane that are the same distance from a fixed line and a fixed point not on the line. The fixed point is the focus of the parabola.	**Foco de una parabola (p. 622)** Una parábola es el conjunto de todos los puntos en un plano con la misma distancia desde una línea fija y un punto fijo que no permanece en la línea. El punto fijo es el foco de la parábola.
Focus (plural: foci) of an ellipse (p. 638) An ellipse is the set of all points P in a plane such that the sum of the distances from P to two fixed points is constant. Each of the fixed points is a focus of the ellipse.	**Foco de una elipse (p. 638)** Una elipse es el conjunto de todos los puntos P en un plano en el cual la suma de las distancias desde P hasta dos puntos fijos es constante. Cada uno de estos puntos fijos es un foco de la elipse.
Function (p. 62) A function is a relation in which each element of the domain corresponds with exactly one element in the range.	**Función (p. 62)** Una función es una relación en la que cada elemento del dominio corresponde exactamente con un elemento del rango.
Example The relation $y = 3x^3 - 2x + 3$ is a function. $f(x) = 3x^3 - 2x + 3$ is the same relation written in function notation.	

English	Spanish
Function notation (p. 63) If *f* is the name of a function, the function notation $f(x)$ shows the function name *f* and also represents the range value $f(x)$ for the domain value *x*. You read the function notation $f(x)$ as "*f* of *x*" or "a function of *x*." Note that $f(x)$ does *not* mean "*f* times *x*."	**Notación de una función (p. 63)** Si *f* es el nombre de una función, la notación de la función $f(x)$ indica el nombre de la función y también representa el valor del rango $f(x)$ para el valor del dominio *x*. La función de la notación $f(x)$ se lee "*f* de *x*" o "una función de *x*." Observa que $f(x)$ no significa "*f* por *x*".
Example When the value of *x* is 3, $f(3)$, read "*f* of 3," represents the value of the function at 3.	
Function rule (p. 63) A function rule represents an output value in terms of an input value.	**Regla de función (p. 63)** Una regla de función representa un valor de salida en función a un valor de entrada.
Fundamental Theorem of Algebra (p. 320) If $P(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $P(x) = 0$ has at least one complex root.	**Teorema fundamental de álgebra (p. 320)** Si $P(x)$ es un polinomio de grado $n \geq 1$ con coeficientes complejos, entonces $P(x) = 0$ tiene por lo menos una raíz compleja.
Example $P(x) = 3x^3 - 2x + 3$ is of degree 3, so $P(x) = 0$ has at least one complex root.	

G

English	Spanish
Geometric mean (p. 583) The geometric mean of any two positive numbers is the positive square root of the product of the two numbers.	**Media geométrica (p. 583)** La media geométrica de dos números positivos es la raíz cuadrada positiva del producto de los dos números.
Example The geometric mean of 12 and 18 is $\sqrt{12 \cdot 18} \approx 14.6969$.	
Geometric sequence (p. 580) A geometric sequence is a sequence with a constant ratio between consecutive terms.	**Secuencia geométrica (p. 580)** Una secuencia geométrica es una secuencia con una razón constante entre términos consecutivos.
Example The geometric sequence 2.5, 5, 10, 20, 40. . . , has a common ratio of 2.	
Geometric series (p. 595) A geometric series is the sum of the terms in a geometric sequence.	**Serie geométrica (p. 595)** Una serie geométrica es la suma de términos en una progresión geométrica.
Example One geometric series with five terms is $2.5 + 5 + 10 + 20 + 40$.	
Greatest common factor (p. 218) The greatest common factor (GCF) of an expression is the common factor of each term of the expression that has the greatest coefficient and the greatest exponent.	**Máximo factor común (p. 218)** El máximo factor común de una expresión es el factor común de cada término de la expresión que tiene el mayor coeficiente y el mayor exponente.
Example The GCF of $4x^2 + 20x - 12$ is 4.	

English

Spanish

Greatest integer function (p. 90) The greatest integer function corresponds each input x to the greatest integer less than or equal to x.

Función del entero mayor (p. 90) La función del entero mayor relaciona cada entrada x con el entero mayor que es menor o igual a x.

Growth factor (p. 436) In an exponential function of the form $y = ab^x$, b is the growth factor if $b > 1$.

Factor de incremento (p. 436) En una función exponencial de la forma $y = ab^x$, b es el factor de incremento si $b > 1$.

Example In the exponential equation $y = 2^x$, 2 is the growth factor.

H

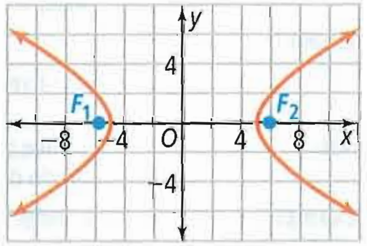
Half-plane (p. 114) A half-plane is the set of points in a coordinate plane that are on one side of the boundary of the graph of a linear inequality.

Semiplano (p. 114) Un semiplano es el conjunto de puntos de un plano de coordenadas que están a un lado del límite de la gráfica de desigualdad lineal.

Hyperbola (p. 645) A hyperbola is a set of points P in a plane such that the difference between the distances from P to the foci F_1 and F_2 is a given constant k. $|PF_1 - PF_2| = k$ The standard form of an equation of a hyperbola centered at (0, 0) is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if the transverse axis is horizontal and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ if the transverse axis is vertical.

Hipérbola (p. 645) Una hipérbola es un conjunto de puntos P en un plano tal que la diferencia entre las distancias de P a los focos F_1 y F_2 es una constante k dada. $|PF_1 - PF_2| = k$ La forma normal de la ecuación de una hipérbola centrada en (0, 0) es $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, si el eje transversal es horizontal, y $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, si el eje transversal es vertical.

Example



$$\frac{x^2}{5^2} - \frac{y^2}{3^2} = 1$$

I

i (p. 248) The imaginary number i is the principal square root of −1.

i (p. 248) El número imaginario i es la raíz cuadrada principal de −1.

Example $i = \sqrt{-1}$ and $i^2 = -1$.

Identity (p. 28) An equation that is true for every value of the variable is an identity.

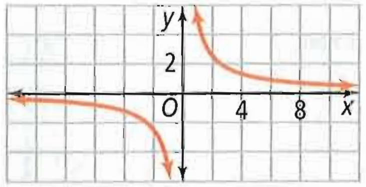
Identidad (p. 28) Una ecuación que es verdadera para cada valor de la variable es una identidad.

Imaginary number (p. 249) An imaginary number is any number of the form $a + bi$, where a and b are real numbers and $b \neq 0$.

Número imaginario (p. 249) Un número imaginario es cualquier número de la forma $a + bi$, donde a y b son números reales y $b \neq 0$.

Example $2 + 3i$, $7i$, i

English	Spanish
Imaginary unit (p. 248) The imaginary unit *i* is the complex number whose square is −1.	**Unidad imaginaria (p. 248)** La unidad imaginaria *i* es el número complejo cuyo cuadrado es −1.
Inconsistent system (p. 137) A system of equations that has no solution is an inconsistent system.	**Sistema incompatible (p. 137)** Un sistema incompatible es un sistema de ecuaciones para el cual no hay solución.
Example $\begin{cases} y = 2x + 3 \\ -2x + y = 1 \end{cases}$ is a system of parallel lines, so it has no solution. It is an inconsistent system.	
Independent system (p. 137) A system of linear equations that has a unique solution is an independent system.	**Sistema independiente (p. 137)** Un sistema de ecuaciones lineales que tenga una sola solución es un sistema independiente.
Example $\begin{cases} x + 2y = -7 \\ 2x - 3y = 0 \end{cases}$ has the unique solution (−3, −2). It is an independent system.	
Independent variable (p. 63) If a function is defined by an equation using the variables *x* and *y*, where *x* represents input values, then *x* is the independent variable.	**Variable independiente (p. 63)** Si una función es definida por una ecuación con las variables *x* e *y*, donde *x* representa los valores de entrada, entonces *x* es la variable independiente.
Example $y = 2x + 1$ *x* is the independent variable.	
Index (p. 362) With a radical sign, the index indicates the degree of the root.	**Índice (p. 362)** Con un signo de radical, el índice indica el grado de la raíz.
Example index 2 index 3 index 4 $\sqrt{16}$ $\sqrt[3]{16}$ $\sqrt[4]{16}$	
Infinite Series (p. 587) An infinite series is a series with infinitely many terms.	**Serie infinita (p. 587)** Una serie infinita es una serie con un número infinito de términos.
Inverse function (p. 405) If function *f* pairs a value *b* with *a* then its inverse, denoted f^{-1}, pairs the value *a* with *b*. If f^{-1} is also a function, then *f* and f^{-1} are inverse functions.	**Funcion inversa (p. 405)** Si la función *f* empareja un valor *b* con *a*, entonces su inversa, cuya notación es f^{-1}, empareja el valor *a* con *b*. Si f^{-1} también es una función, entonces *f* y f^{-1} son funciones inversas.
Example If $f(x) = x + 3$, then $f^{-1}(x) = x - 3$. $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	
Inverse operations (p. 27) Inverse operations are operations that undo each other.	**Operaciones inversas (p. 27)** Operaciones inversas son operaciones que se cancelan mutuamente.

English	Spanish
Inverse relation (p. 405) If a relation pairs element a of its domain with element b of its range, the inverse relation "undoes" the relation and pairs b with a. If (a, b) is an ordered pair of a relation, then (b, a) is an ordered pair of its inverse.	**Relación inversa (p. 405)** Si una relación empareja el elemento a de su dominio con el elemento b de su rango, la relación inversa "deshace" la relación y empareja b con a. Si (a, b) es un par ordenado de una relación, entonces (b, a) es un par ordenado de su inversa.
Inverse variation (p. 498) An inverse variation is a relation represented by an equation of the form $xy = k$, $y = \frac{k}{x}$, or $x = \frac{k}{y}$, where $k \neq 0$.	**Variación inversa (p. 498)** Una variación inversa es una relación representada por la ecuación $xy = k$, $y = \frac{k}{x}$, ó $x = \frac{k}{y}$, donde $k \neq 0$.
Example  $xy = 5, \text{or } y = \frac{5}{x}$	
J	
Joint variation (p. 501) A joint variation is a relation in which one variable varies directly with respect to each of two or more variables.	**Variación conjunta (p. 501)** Una variación conjunta es una relación en la cual el valor de una variable varía directamente con respecto a cada una de dos o más variables.
Example $z = 8xy$ $T = kPV$	
L	
Like radicals (p. 374) Like radicals are radical expressions that have the same index and the same radicand.	**Radicales semejantes (p. 374)** Los radicales semejantes son expresiones radicales que tienen el mismo índice y el mismo radicando.
Example $4\sqrt[3]{7}$ and $\sqrt[3]{7}$ are like radicals.	
Like terms (p. 21) Like terms have the same variables raised to the same powers.	**Términos semejantes (p. 21)** Los términos semejantes tienen las mismas variables elevadas a las mismas potencias.
Limits (p. 589) Limits in summation notation are the least and greatest integer values of the index n.	**Límites (p. 589)** Los límites en notación de sumatoria son el menor y el mayor valor del índice n en números enteros.
Example limits $\sum_{n=1}^{3}(3n + 5)$	
Line of best fit (p. 94) The trend line that gives the most accurate model of related data is the line of best fit.	**Recta de mayor aproximación (p. 94)** La línea de tendencia que representa con mayor precisión los datos relacionados es la recta de mayor aproximación.

English

Spanish

Linear equation (p. 75) A linear equation in two variables is an equation that can be written in the form $ax + by = c$. *See also* **Standard form of a linear equation.**

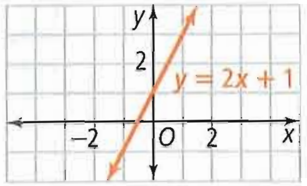
Ecuación lineal (p. 75) Una ecuación lineal de dos variables es una ecuación que se puede escribir de la forma $ax + by = c$. *Ver también* **Standard form of a linear equation.**

Example $y = 2x + 1$ can be written as $-2x + y = 1$.

Linear function (p. 75) A function whose graph is a line is a linear function. You can represent a linear function with a linear equation.

Función lineal (p. 75) Una función cuya gráfica es una recta es una función lineal. La función lineal se representa con una ecuación lineal.

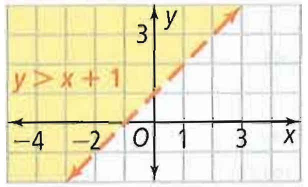
Example



Linear inequality (p. 114) A linear inequality is an inequality in two variables whose graph is a region of the coordinate plane that is bounded by a line.

Desigualdad lineal (p. 114) Una desigualdad lineal es una desigualdad de dos variables cuya gráfica es una región del plano de coordenadas delimitado por una recta.

Example

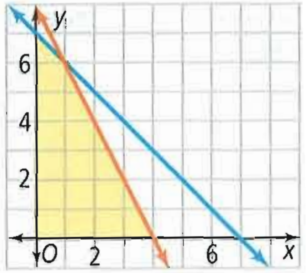


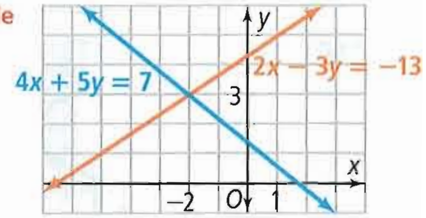
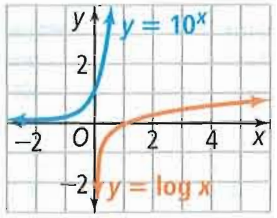
$y > x + 1$

Linear programming (p. 157) Linear programming is a method for finding a minimum or maximum value of some quantity, given a set of constraints.

Programación lineal (p. 157) Programación lineal es un método para hallar el valor mínimo y máximo de una cantidad que se expresa como un conjunto de limitaciones.

Example Restrictions: $x \geq 0$, $y \geq 0$, $x + y \leq 7$, and $y \leq -2x + 8$
Objective function: $B = 2x + 4y$
Evaluate $B = 2x + 4y$ at each vertex.
The minimum value of B occurs when $x = 0$ and $y = 0$. The maximum value of B occurs when $x = 0$ and $y = 7$.



English	Spanish
Linear system (p. 134) A linear system is a set of two or more linear equations that use the same variables. **Example** 	**Sistema lineal (p. 134)** Un sistema lineal es un conjunto de dos o más ecuaciones lineales con las mismas variables.
Literal equation (p. 29) A literal equation is an equation that uses more than one letter as a variable. **Example** $A = \frac{1}{2}bh$	**Ecuación literal (p. 29)** Una ecuación literal es una ecuación en la cual más de una letra expresa una variable.
Logarithm (p. 451) The logarithm base *b* of a positive number *x* is defined as follows: $\log_b x = y$, if and only if $x = b^y$. **Example** $\log_2 8 = 3$ $\log_{10} 100 = \log 100 = 2$ $\log_5 5^7 = 7$	**Logaritmo (p. 451)** El logaritmo con base *b* de un número positivo *x* se define como $\log_b x = y$, si y sólo si $x = b^y$.
Logarithmic equation (p. 471) A logarithmic equation is an equation that includes a logarithm involving a variable. **Example** $\log_3 x = 4$	**Ecuación logaritmica (p. 471)** Una ecuación logarítmica es una ecuación que incluye un logaritmo con una variable.
Logarithmic function (p. 454) A logarithmic function is the inverse of an exponential function. **Example** 	**Función logarítmica (p. 454)** Una función logarítmica es la inversa de una función exponencial.
Logarithmic scale (p. 453) A logarithmic scale is a scale that uses the logarithm of a quantity instead of the quantity itself.	**Escala logarítmica (p. 453)** Una escala logarítmica es una escala que usa el logaritmo de una cantidad en vez de la cantidad misma.

English

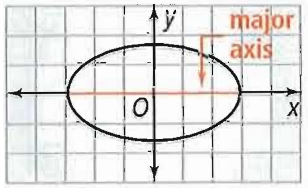
M

Spanish

Major axis (p. 639) The major axis of an ellipse is the segment that contains the foci of the ellipse and has endpoints on the ellipse.

Eje mayor (p. 639) En una elipsis, el eje mayor es el segmento que contiene los focos de la elipsis y tiene puntos extremos sobre la elipsis.

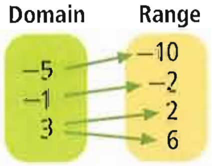
Example



Mapping diagram (p. 60) A mapping diagram describes a relation by linking elements of the domain with elements of the range.

Mapa (p. 60) Un mapa describe una relación al unir los elementos del dominio con los elementos del rango.

Example



Matrix (p. 174) A matrix is a rectangular array of numbers written within brackets.

Matriz (p. 174) Una matriz es un conjunto de números encerrados en corchetes y dispuestos en forma de rectángulo.

Example

$$A = \begin{bmatrix} 1 & -2 & 0 & 10 \\ 9 & 7 & -3 & 8 \\ 2 & -10 & 1 & -6 \end{bmatrix}$$

The number 2 is the element in the third row and first column.
A is a 3×4 matrix.

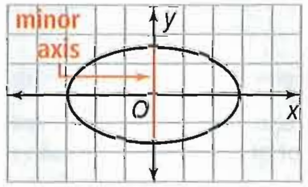
Matrix element (p. 174) Every item listed in a matrix is an element of the matrix. An element is identified by its position in the matrix.

Elemento matricial (p. 174) Cada cifra de una matriz es un elemento de la matriz. El elemento se identifica según la posición que ocupa en la matriz.

Example

$$A = \begin{bmatrix} 1 & -2 & 0 & 10 \\ 9 & 7 & -3 & 8 \\ 2 & -10 & 1 & -6 \end{bmatrix}$$

Element a_{21} is 9, the element in the second row and first column.

English	Spanish
Maximum value (p. 195) The maximum value of a function $y = f(x)$ is the greatest y-value of the function. It is the y-coordinate of the highest point on the graph of f.	**Valor máximo (p. 195)** El valor máximo de una función $y = f(x)$ es el valor más alto de y de la función. Es la coordenada y del punto más alto de la gráfica de f.
Minor axis (p. 639) The minor axis of an ellipse is the segment that is perpendicular to the major axis at its midpoint and has endpoints on the ellipse. **Example** 	**Eje menor (p. 639)** En una elipsis, el eje menor es el segmento perpendicular al eje mayor en su punto medio y que tiene puntos extremos sobre la elipsis.
Minimum value (p. 195) The minimum value of a function $y = f(x)$ is the least y-value of the function. It is the y-coordinate of the lowest point on the graph of f.	**Valor mínimo (p. 195)** El valor mínimo de una función $y = f(x)$ es el valor más bajo de y de la función. Es la coordenada y del punto más bajo de la gráfica de f.
Monomial (p. 280) A monomial is either a real number, a variable, or a product of real numbers and variables with whole number exponents. **Example** $1, x, 2z, 4ab^2$	**Monomio (p. 280)** Un monomio es un número real, una variable o un producto de números reales y variables cuyos exponentes son números enteros.
Multiple zero (p. 291) If a linear factor is repeated in the complete factored form of a polynomial, the zero related to that factor is a multiple zero. **Example** The zeros of the function $P(x) = 2x(x - 3)^2(x + 1)$ are 0, 3, and -1. Since $(x - 3)$ occurs twice as a factor, 3 is a multiple zero.	**Cero múltiplo (p. 291)** Si un factor lineal se repite en la forma factorizada completa de un polinomio, el cero relacionado con ese factor es un cero múltiplo.
Multiplicative identity (p. 14) The multiplicative identity is 1. The product of 1 and any number is that number. The product of reciprocals is 1.	**Identidad multiplicativa (p. 14)** La identidad multiplicativa es 1. El producto de 1 y cualquier otro número es ese número. El producto del recíproco es 1.
Multiplicative inverse (p. 14) The reciprocal or multiplicative inverse of any nonzero number a is $\frac{1}{a}$. The product of reciprocals is 1, the multiplicative identity. **Example** $5 \times \frac{1}{5} = 1$	**Inverso multiplicativo (p. 14)** El recíproco o inverso multiplicativo de cualquier número a, que no sea cero, es $\frac{1}{a}$. El producto de recíprocos es 1, la identidad multiplicativa.

English

Spanish

Multiplicity (p. 291) The multiplicity of a zero of a polynomial function is the number of times the related linear factor is repeated in the factored form of the polynomial.

Multiplicidad (p. 291) La multiplicidad de un cero de una función polinomial es el número de veces que el factor lineal relacionado se repite en la forma factorizada del polinomio.

Example The zeros of the function $P(x) = 2x(x - 3)^2(x + 1)$ are 0, 3, and -1. Since $(x - 3)$ occurs twice as a factor, the zero 3 has multiplicity 2.

N

nth root (p. 361) For any real numbers a and b, and any positive integer n, if $a^n = b$, then a is an nth root of b.

raíz n-ésima (p. 361) Para todos los números reales a y b, y todo número entero positivo n, si $a^n = b$, entonces a es la n-ésima raíz de b.

Example $\sqrt[5]{32} = 2$ because $2^5 = 32$.

$\sqrt[4]{81} = 3$ because $3^4 = 81$.

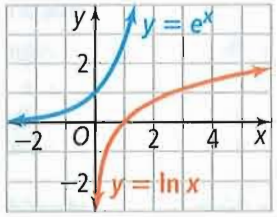
Natural base exponential function (p. 446) A natural base exponential function is an exponential function with base e.

Función exponencial con base natural (p. 446) Una función exponencial con base natural es una función exponencial con base e.

Natural logarithmic function (p. 478) A natural logarithmic function is a logarithmic function with base e. The natural logarithmic function, $y = \ln x$, is $y = \log_e x$. It is the inverse of $y = e^x$.

Función logarítmica natural (p. 478) Una función logarítmica natural es una función logarítmica con base e. La función logarítmica natural, $y = \ln x$, es $y = \log_e x$. Ésta es la función inversa de $y = e^x$.

Example



$\ln e^3 = 3$

$\ln 10 \approx 2.3026$

$\ln 36 \approx 3.5835$

Non-removable discontinuity (p. 516) A non-removable discontinuity is a point of discontinuity that is not removable. It represents a break in the graph of f where you cannot redefine f to make the graph continuous.

Discontinuidad irremovible (p. 516) Una discontinuidad irremovible es un punto de discontinuidad que no se puede remover. Representa una interrupción en la gráfica f donde no se puede redefinir f para volverla una gráfica continua.

Numerical expression (p. 5) A numerical expression is a mathematical phrase that contains numbers and operation symbols.

Expresión numérica (p. 5) Una expresión numérica es una expresión matemática compuesta de números y símbolos de operación.

English

Spanish

O

Objective function (p. 157) In a linear programming model, the objective function is a model of the quantity that you want to make as large or as small as possible. *See* **Linear programming.**

Función objetiva (p. 157) En un modelo de programación lineal, la función objetiva es un modelo de la cantidad que se quiere aumentar o disminuir cuanto sea posible. *Ver* **Linear programming.**

One-to-one function (p. 408) A one-to-one function is a function for which each y-value in the range corresponds to exactly one x-value in the domain. A one-to-one function f has an inverse f^{-1} that is also a function.

Función uno a uno (p. 408) Una función uno a uno es una función donde cada valor y que se encuentra en el rango corresponde exactamente a un valor x en el dominio. Una función uno a uno f tiene un inverso f^{-1} que también es una función.

Opposite (p. 14) The opposite or additive inverse of any number a is $-a$. The sum of opposites is zero, the additive identity.

Opuesto (p. 14) El opuesto o inverso aditivo de cualquier número a es $-a$. La suma de opuestos es cero, la identidad aditiva.

Example $3 + (-3) = 0$
$5.2 + (-5.2) = 0$

Ordered triples (p. 164) Ordered triples of the form (x, y, z) represent the location of a point in coordinate space.

Tripletas ordenadas (p. 164) Las tripletas ordenadas de la forma (x, y, z) representan la ubicación de un punto en el espacio de coordenadas.

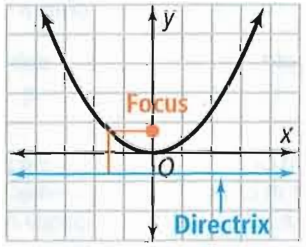
Example (2, 4, 5)
(0, 1, 2)
(0, 0, 0)

P

Parabola (p. 194) A parabola is the graph of a quadratic function. It is the set of all points P in a plane that are the same distance from a fixed point F, the focus, as they are from a line d, the directrix.

Parábola (p. 194) La parábola es la gráfica de una función cuadrática. Es el conjunto de todos los puntos P situados en un plano a la misma distancia de un punto fijo F, o foco, y de la recta d, o directriz.

Example



Parallel lines (p. 85) Parallel lines are coplanar lines that do not intersect. In the coordinate plane, parallel lines have the same slope.

Rectas paraleles (p. 85) Rectas paralelas son líneas coplanares que no se intersecan. En un plano de coordenadas, las rectas paralelas tienen la misma pendiente.

Parent function (p. 99) A parent function is the simplest form of a set of functions that form a family.

Función elemental (p. 99) Una función madre es la mínima expresión de un conjunto de funciones que forma una familia.

Example $y = x$ is the parent function for the functions of the form $y = x + k$.

English — Spanish

Pascal's Triangle (p. 327) Pascal's Triangle is a triangular array of numbers in which the first and last number is 1. Each of the other numbers in the row is the sum of the two numbers above it.

Triángulo de Pascal (p. 327) El Triángulo de Pascal es una distribución triangular de números en la cual el primer número y el último número son 1. Cada uno de los otros números en la fila es la suma de los dos números de encima.

Example **Pascal's Triangle**

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Perfect square trinomial (p. 219) A perfect square trinomial is a trinomial that is the square of a binomial.

Trinomio cuadrado perfecto (p. 219) Un trinomio cuadrado perfecto es un trinomio que es el cuadrado de un binomio.

Example

$$\overset{\text{perfect square trinominal}}{16x^2 - 24x + 9} = \overset{\text{binominal square}}{(4x - 3)^2}$$

Perpendicular lines (p. 85) Perpendicular lines are lines that intersect to form right angles. In the coordinate plane, perpendicular lines have slopes with product -1.

Rectas perpendiculares (p. 85) Rectas perpendiculares son rectas que se intersecan y forman ángulos rectos. En un plano de coordenadas, las rectas perpendiculares tienen pendientes cuyo producto es -1.

Piecewise function (p. 90) A piecewise function has different rules for different parts of its domain.

Función de fragmentos (p. 90) Una función de fragmentos tiene reglas diferentes para diferentes partes de su dominio.

Point of discontinuity (p. 516) A point of discontinuity is the x-coordinate of a point where the graph of $f(x)$ is not continuous.

Punto de discontinuidad (p. 516) Un punto de discontinuidad es la coordenada x de un punto donde la gráfica de $f(x)$ no es continua.

Example $f(x) = \frac{2}{x-2}$ has a point of discontinuity at $x = 2$.

Point-slope form (p. 81) The point-slope form of an equation of a line is $y - y_1 = m(x - x_1)$, where m is the slope of the line and (x_1, y_1) is a point on the line.

Forma punto-pendiente (p. 81) La forma punto-pendiente de una ecuación lineal es $y - y_1 = m(x - x_1)$, donde m es la pendiente de la recta y (x_1, y_1) es un punto de la recta.

Example

$$y - 3 = 2(x - 1)$$
$$y + 4 = 5(x - 2)$$

Polynomial (p. 280) A polynomial is a monomial or the sum of monomials.

Polinomio (p. 280) Un polinomio es un monomio o la suma de dos o más monomios.

Example $8x$, $3x^3 + 4x^2 - 2x + 5$

Polynomial function (p. 280) A polynomial in the variable x defines a polynomial function of x.

Función polinomial (p. 280) Un polinomio en la variable x define una función polinomial de x.

English

Spanish

Power function (p. 341) A power function is a function of the form $y = a \cdot x^b$, where a and b are nonzero real numbers.

Función de potencia (p. 341) Una función de potencia es una función de la forma $y = a \cdot x^b$, donde a y b son números reales diferentes de cero.

Principal root (p. 361) When a number has two real roots, the positive root is called the principal root. A radical sign indicates the principal root. The principal root of a negative number a is $i\sqrt{|a|}$.

Raíz principal (p. 361) Cuando un número tiene dos raíces reales, la raíz positiva es la raíz principal. El signo del radical indica la raíz principal. La raíz principal de un número negativo a es $i\sqrt{|a|}$.

Example The number 25 has two square roots, 5 and −5.
The principal square root, 5, is indicated by $\sqrt{25}$ or $25^{\frac{1}{2}}$.

Pure imaginary number (p. 249) If $a = 0$ and $b \neq 0$, the number $a + bi$ is a pure imaginary number.

Número imaginario puro (p. 249) Si $a = 0$ y $b \neq 0$, el número $a + bi$ es un número imaginario puro.

Q

Quadratic equation (p. 226) A quadratic equation is one that can be written in the standard form $ax^2 + bx + c = 0$, where $a \neq 0$.

Ecuación cuadrática (p. 226) Una ecuación cuadrática es una ecuación que se puede expresar en forma normal como $ax^2 + bx + c = 0$, donde $a \neq 0$.

Example $2x^2 + 3x + 1 = 0$

Quadratic Formula (p. 240) The Quadratic Formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. It gives the solutions to the quadratic equation $ax^2 + bx + c = 0$.

Fórmula cuadrática (p. 240) La fórmula cuadrática es $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Ésta da las soluciones a la ecuación cuadrática $ax^2 + bx + c = 0$.

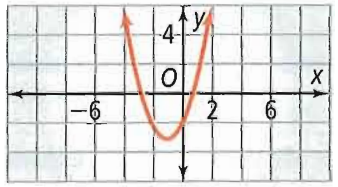
Example If $-x^2 + 3x + 2 = 0$, then

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(-1)(2)}}{2(-1)}$$
$$= \frac{-3 \pm \sqrt{17}}{-2}$$

Quadratic function (p. 194) A quadratic function is a function that you can write in the form $f(x) = ax^2 + bx + c$ with $a \neq 0$.

Función cuadrática (p. 194) Una función cuadrática es una función que puedes escribir como $f(x) = ax^2 + bx + c$ con $a \neq 0$.

Example



$y = x^2 + 2x - 2$

Quantity (p. 5) A mathematical quantity is anything that can be measured or counted.

Cantidad (p. 5) Una cantidad matemática es cualquier cosa que se puede medir o contar.

R

Radical equation (p. 390) A radical equation is an equation that has a variable in a radicand or has a variable with a rational exponent.

Ecuación radical (p. 390) La ecuación radical es una ecuación que contiene una variable en el radicando o una variable con un exponente racional.

Example $(\sqrt{x})^3 + 1 = 65$

$x^{\frac{3}{2}} + 1 = 65$

Radical function (p. 415) A radical function is a function that can be written in the form $f(x) = a\sqrt[n]{x - h} + k$, where $a \neq 0$. For even values of n, the domain of a radical function is the real numbers $x \geq h$. *See also* **Square root function.**

Función radical (p. 415) Una función radical es una función quepuede expresarse como $f(x) = a\sqrt[n]{x - h} + k$, donde $a \neq 0$. Para n par, el dominio de la función radical son los números reales tales que $x \geq h$. *Ver también* **Square root function.**

Example $f(x) = \sqrt{x - 2}$

Radicand (p. 362) The number under a radical sign is the radicand.

Radicando (p. 362) La expresión que aparece debajo del signo radical es el radicando.

Example The radicand in $3\sqrt[4]{7}$ is 7.

Radius (p. 630) The radius r of a circle is the distance between the center of the circle and any point on the circumference.

Radio (p. 630) El radio r de un círculo es la distancia entre el centro del círculo y cualquier punto de la circunferencia.

Example

Range (p. 61) The range of a relation is the set of all outputs or y-coordinates of the ordered pairs.

Rango (p. 61) El rango de una relación es el conjunto de todas las salidas posibles, o coordenadas y, de los pares ordenados.

Example In the relation {(0, 1), (0, 2), (0, 3), (0, 4), (1, 3), (1, 4), (2, 1)}, the range is {1, 2, 3, 4}.

In the function $f(x) = |x - 3|$, the range is the set of real numbers greater than or equal to 0.

Rational equation (p. 542) A rational equation is an equation that contains a rational expression.

Ecuación racional (p. 542) Una ecuación racional es una ecuación que contiene una expresión racional.

Rational exponent (p. 382) If the nth root of a is a real number and m is an integer, then $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$. If m is negative, $a \neq 0$.

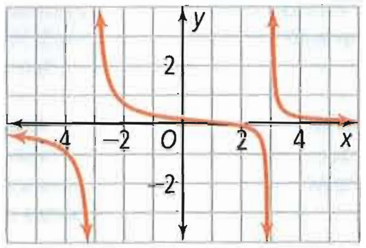
Exponente racional (p. 382) Si la raíz n-ésima de a es un número real y m es un número entero, entonces $a^{\frac{1}{n}} = \sqrt[n]{a}$ y $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$. Si m es negativo, $a \neq 0$.

Example $4^{\frac{1}{3}} = \sqrt[3]{4}$

$5^{\frac{3}{2}} = \sqrt{5^3} = (\sqrt{5})^3$

English	Spanish
Rational expression (p. 527) A rational expression is the quotient of two polynomials.	**Expresión racional (p. 527)** Una expresión racional es el cociente de dos polinomios.
Rational function (p. 515) A rational function $f(x)$ can be written as $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions.The domain of a rational function is all real numbers except those for which $Q(x) = 0$.	**Función racional (p. 515)** Una función racional $f(x)$ se puede expresar como $f(x) = \frac{P(x)}{Q(x)}$, donde $P(x)$ y $Q(x)$ son funciones de polinomios. El dominio de una función racional son todos los números reales excepto aquéllos para los cuales $Q(x) = 0$.

Example



The function $y = \frac{x-2}{x^2-9}$ is a rational function with three branches separated by asymptotes $x = -3$ and $x = 3$.

English	Spanish
Rational Root Theorem (p. 312) Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. Then there are a limited number of possible roots of $P(x) = 0$: - Integer roots must be factors of a_0; - Rational roots must have reduced form p/q where p is an integer factor of a_0 and q is an integer factor of a_n.	**Teorema de la Raíz Racional (p. 312)** Sea $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ un polinomio con enteros como coeficientes. Entonces hay un número limitado de raíces posibles para $P(x) = 0$: - Las raíces enteras deben ser factores de a_0; - Las raíces racionales deben ser de forma simplificada p/q, donde p es un factor entero de a_0 y q es un factor entero de a_n.

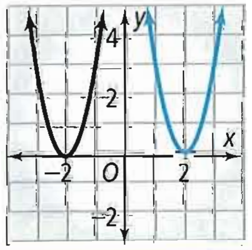
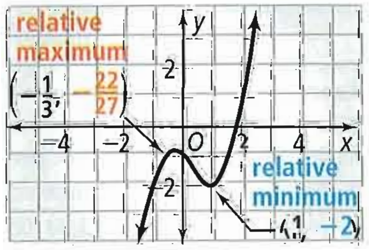
Example The polynomial equation $2x^2 + 6x - 3$ has leading coefficient 2 (with factors ± 1, ± 2) and constant term -113 (with factors ± 1 and ± 113). Its only possible rational roots are $\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{20}$.

English	Spanish
Rationalize the denominator (p. 369) To rationalize the denominator of an expression, rewrite it so there are no radicals in any denominator and no denominators in any radical.	**Racionalizar el denominador (p. 369)** Para racionalizar el denominador de una expresión, ésta se escribe de modo que no haya radicales en ningún denominador y no haya denominadores en ningún radical.

Example $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

English	Spanish
Reciprocal (p. 14) The reciprocal or multiplicative inverse of any nonzero number a is $\frac{1}{a}$. The product of reciprocals is 1, the multiplicative identity.	**Recíproco (p. 14)** El recíproco o inverso multiplicativo de un número distinto de cero a es $\frac{1}{a}$. El producto de recíprocos es 1, la identidad multiplicativa.

Example $5 \times \frac{1}{5} = 1$

English	Spanish
Reciprocal function (p. 507) A reciprocal function belongs to the family whose parent function is $f(x) = \frac{1}{x}$ where $x \neq 0$. You can write a reciprocal function in the form $f(x) = (\frac{a}{x} - h) + k$, where $a \neq 0$ and $x \neq h$.	**Función recíproca (p. 507)** Una función recíproca pertenece a la familia cuya función madre es $f(x) = \frac{1}{x}$ donde $x \neq 0$. Se puede escribir una función recíproca como $f(x) = (\frac{a}{x} - h) + k$, donde $a \neq 0$ y $x \neq h$.
Example $f(x) = \frac{1}{2x + 5}$ $p(v) = \frac{3}{v} + 5$	
Recursive formula (p. 565) A recursive formula defines the terms in a sequence by relating each term to the ones before it.	**Fórmula recursiva (p. 565)** Una fórmula recursiva define los términos de una secuencia al relacionar cada término con los términos que lo anteceden.
Reduced row echelon form (p. 177) A matrix that represents the solution of a system is in reduced row echelon form. The leading 1 in each row has 0's elsewhere in its column.	**Forma reducida fila-escalón (p. 177)** Una matriz que representa la solución de un sistema está en forma reducida fila-escalón. El 1 principal en cada fila tiene ceros en otras partes de la columna.
Reflection (p. 101) A reflection flips the graph of a function across a line, such as the *x*- or *y*-axis. Each point on the graph of the reflected function is the same distance from the line of reflection as is the corresponding point on the graph of the original function.	**Reflexión (p. 101)** Una reflexión voltea la gráfica de una función sobre una línea, como el eje de las *x* o el eje de las *y*. Cada punto de la gráfica de la función reflejada está *a* la misma distancia del eje de reflexión que el punto correspondiente en la gráfica de la función original.
Example 	
Relation (p. 60) A relation is a set of ordered pairs.	**Relación (p. 60)** Una relación es un conjunto de pares ordenados.
Example {(0, 1), (0, 2), (0, 3), (0, 4), (1, 3)}	
Relative maximum (minimum) (p. 291) A relative maximum (minimum) is the value of the function at an up-to-down (down-to-up) turning point.	**Máximo (mínimo) relativo (p. 291)** El máximo (mínimo) relativo es el valor de la función en un punto de giro de arriba hacia abajo (de abajo hacia arriba).
Example 	

English

Spanish

Remainder Theorem (p. 307) If you divide a polynomial $P(x)$ of degree $n > 1$ by $x - a$, then the remainder is $P(a)$.

Teorema del residuo (p. 307) Si divides un polinomio $P(x)$ con un grado $n > 1$ por $x - a$, el residuo es $P(a)$.

Example If $P(x) = x^3 - 4x^2 + x + 6$ is divided by $x - 3$, then the remainder is $P(3) = 3^3 - 4(3)^2 + 3 + 6 = 0$ (which means that $x - 3$ is a factor of $P(x)$).

Removable discontinuity (p. 516) A removable discontinuity is a point of discontinuity, a, of function f that you can remove by redefining f at $x = a$. Doing so fills in a hole in the graph of f with the point $(a, f(a))$.

Discontinuidad removible (p. 516) Una discontinuidad removible es un punto de discontinuidad a en una función f que se puede remover al redefinir f en $x = a$. Al hacer esto, se llena un hueco en la gráfica f con el punto $(a, f(a))$.

Root (p. 232) A root of a function is the input value for which the value of the function is zero. A root of an equation is a value that makes the equation true. *See also* **Zero of a function.**

Raíz (p. 232) La raíz de una función es el valor de entrada para el cual el valor de la función es cero. La raíz de una ecuación es un valor que hace verdadera la ecuación. *Ver también* **Zero of a function.**

Example -2 and 3 are roots of the function $f(x) = (x + 2)(x - 3)$ and the equation $(x + 2)(x - 3) = 0$.

Row operation (p. 176) A row operation on an augmented matrix is any of the following: switch two rows, multiply a row by a constant, add one row to another.

Operación de fila (p. 176) Una operación de fila en una matriz ampliada es cualquiera de las siguientes opciones: el intercambio de dos filas, la multiplicación de una fila por una constante o la suma de dos filas.

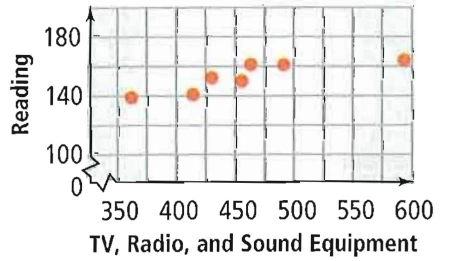
S

Scatter plot (p. 92) A scatter plot is a graph that relates two different sets of data by plotting the data as ordered pairs. You can use a scatter plot to determine a relationship between the data sets.

Diagrama de puntos (p. 92) Un diagrama de puntos es una gráfica que relaciona dos conjuntos de datos presentando los datos como pares ordenados. El diagrama de puntos sirve para definir la relación entre conjuntos de datos.

Example

Dollars Spent Per Capita on Entertainment



Source: U.S. Bureau of Labor Statistics

Sequence (p. 564) A sequence is an ordered list of numbers.

Progresión (p. 564) Una progresión es una sucesión de números.

Example 1, 4, 7, 10, . . .

English	Spanish
Series (p. 587) A series is the sum of the terms of a sequence.	**Serie (p. 587)** Una serie es la suma de los términos de una secuencia.
Example The series 3 + 6 + 9 + 12 + 15 corresponds to the sequence 3, 6, 9, 12, 15. The sum of the series is 45.	
Simplest form of a radical expression (p. 368) A radical expression with index *n* is in simplest form if there are no radicals in any denominator, no denominators in any radical, and any radicand has no *n*th power factors.	**Mínima expresión de una expresión radical (p. 368)** Una expresión radical con índice *n* está en su mínima expresión si no tiene radicales en ningún denominador ni denominadores en ningún radical y los radicandos no tienen factores de potencia.
Simplest form of a rational expression (p. 527) A rational expression is in simplest form if its numerator and denominator are polynomials that have no common divisor other than 1.	**Forma eimplificada de nua expresión racional (p. 527)** Una expresión racional se encuentra en su mínima expresión si su numerador y su denominador son polinomios que no tienen otro divisor aparte de 1.
Example $\frac{x^2 - 7x + 12}{x^2 - 9} = \frac{(x - 4)\cancel{(x - 3)}}{(x + 3)\cancel{(x - 3)}}$ $= \frac{x - 4}{x + 3}$, where $x \neq -3$	
Slope (p. 74) The slope of a non-vertical line is the ratio of the vertical change to the horizontal change between points. You can calculate slope by finding the ratio of the difference in the *y*-coordinates to the difference in the *x*-coordinates for any two points on the line. The slope of a vertical line is undefined.	**Pendiente (p. 74)** La pendiente de una línea no vertical es la razón del cambio vertical al cambio horizontal entre puntos. Puedes calcular la pendiente al hallar la razón de la diferencia de la coordenada *y* y la diferencia de la coordenada *x* para dos puntos cualesquiera de la línea. La pendiente de una línea vertical es indefinida.
Example The slope of the line through points (−1, −1) and (1, −2) is $\frac{-2 - (-1)}{1 - (-1)} = \frac{-1}{2} = -\frac{1}{2}$.	
Slope-intercept form (p. 76) The slope-intercept form of an equation of a line is $y = mx + b$, where m is the slope and b is the y-intercept.	**Forma pendiente-intercepto (p. 76)** La forma pendiente-intercepto de una ecuación lineal es $y = mx + b$, donde m es la pendiente y b es el intercepto en y.
Example $y = 8x + 2$ $y = -x + 1$	
Solution of a system (p. 134) A solution of a system is a set of values for the variables that makes all the equations true.	**Solución de un sistema (p. 134)** Una solución de un sistema es un conjunto de valores para las variables que hace que todas las ecuaciones sean verdaderas.
Solution of an equation (p. 27) A solution of an equation is a number that makes the equation true.	**Solución de una ecuación (p. 27)** Una solución de una ecuación es cualquier número que haga verdadera la ecuación.
Example The solution of $2x - 7 = -12$ is $x = -2.5$.	

English	Spanish
Square root equation (p. 390) A square root equation is a radical equation in which the radical has index 2.	**Ecuación de raíz cuadrada (p. 390)** Una ecuación de raíz cuadrada es una ecuación radical en la cual el radical tiene índice 2.
Example $\sqrt{x} = 4$	
Square root function (p. 415) A square root function is a function that can be written in the form $f(x) = a\sqrt{x - h} + k$, where $a \neq 0$. The domain of a square root function is all real numbers $x \geq h$.	**Función de raíz cuadrada (p. 415)** Una función de raíz cuadrada es una función que puede ser expresada como $f(x) = a\sqrt{x - h} + k$, donde $a \neq 0$. El dominio de una función de raíz cuadrada son todos los números reales tales que $x \geq h$.
Example $f(x) = 2\sqrt{x - 3} + 4$	
Standard form of a circle (p. 630) *See* **Circle.**	**Forma normal de un círculo (p. 630)** *Ver* **Circle.**
Example $(x - 3)^2 + (y - 4)^2 = 4$	
Standard form of a linear equation (p. 82) The standard form of a linear equation is $Ax + By = C$, where A, B, and C are real numbers, and A and B are not *both* zero.	**Forma normal de una ecuación lineal (p. 82)** La forma normal de una ecuación lineal es $Ax + By = C$, donde A, B y C son números reales, y A y B no son cero *ambos*.
Example In standard form, the equation $y = \frac{4}{3}x - 1$ is $4x + (-3)y = 3$.	
Standard form of a polynomial function (p. 281) The standard form of a polynomial function arranges the terms by degree in descending numerical order. A polynomial function, $P(x)$, in standard form is $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is a nonnegative integer and $a_n, \ldots, a_0$ are real numbers.	**Forma normal de una función polinomial (p. 281)** La forma normal de una función polinomial organiza los términos por grado en orden numérico descendiente. Una función polinomial, $P(x)$, en forma normal es $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, donde n es un número entero no negativo y $a_n, \ldots, a_0$ son números reales.
Example $2x^3 - 5x^2 - 2x + 5$	
Standard form of a quadratic function (p. 202) The standard form of a quadratic function is $f(x) = ax^2 + bx + c$ with $a \neq 0$.	**Forma normal de una función cuadrática (p. 202)** La forma normal de una función cuadrática es $f(x) = ax^2 + bx + c$ con $a \neq 0$.
Example $f(x) = 2x^2 + 5x + 2$	
Step function (p. 90) A step function pairs every number in an interval with a single value. The graph of a step function can look like the steps of a staircase.	**Función escalón (p. 90)** Una función escalón empareja cada número de un intervalo con un solo valor. La gráfica de una función escalón se puede parecer a los peldaños de una escalera.
Sum of cubes (p. 297) The sum of cubes is an expression of the form $a^3 + b^3$. It can be factored as $(a + b)(a^2 - ab + b^2)$.	**Suma de dos cubos (p. 297)** La suma de dos cubos es una expresión de la forma $a^3 + b^3$. Se puede factorizar como $(a + b)\,(a^2 - ab + b^2)$.
Example $x^3 + 27 = (x + 3)(x^2 - 3x + 9)$	

English | Spanish

Synthetic division (p. 306) Synthetic division is a process for dividing a polynomial by a linear expression $x - a$. You list the standard-form coefficients (including zeros) of the polynomial, omitting all variables and exponents. You use a for the "divisor" and add instead of subtract throughout the process.

División sintética (p. 306) La división sintética es un proceso para dividir un polinomio por una expresión lineal $x - a$. En este proceso, escribes los coeficientes de forma normal (incluyendo los ceros) del polinomio, omitiendo todas las variables y todos los exponentes. Usas a como "divisor" y sumas, en vez de restar, a lo largo del proceso.

Example

$$\begin{array}{r|rrrrr} -3 & 2 & 5 & 0 & -2 & -8 \\ & & -6 & 3 & -9 & 33 \\ \hline & 2 & -1 & 3 & -11 & 25 \end{array}$$

Divide $2x^4 + 5x^3 - 2x - 8$ by $x + 3$. $2x^4 + 5x^3 - 2x - 8$ divided by $x + 3$ gives $2x^3 - x^2 + 3x - 11$ as quotient and 25 as remainder.

System of equations (p. 134) A system of equations is a set of two or more equations using the same variables.

Sistema de ecuaciones (p. 134) Un sistema de ecuaciones es un conjunto de dos o más ecuaciones que contienen las mismas variables.

Example $\begin{cases} 2x - 3y = -13 \\ 4x + 5y = 7 \end{cases}$

T

Term of a sequence (p. 564) Each number in a sequence is a term.

Término de una progresión (p. 564) Cada número de una progresión es un término.

Example 1, 4, 7, 10, . . .
The second term is 4.

Term of an expression (p. 20) A term is a number, a variable, or the product of a number and one or more variables.

Término de una expresión (p. 20) Un término es un número, una variable o el producto de un número y una o más variables.

Example The expression $4x^2 - 3y + 7.3$ has 3 terms.

Test point (p. 115) A test point is a point that you pick on one side of the boundary of the graph of a linear inequality. If the test point makes the inequality true, then all points on that side of the boundary are solutions of the inequality. If the test point makes the inequality false, then all points on the other side are solutions.

Punto de prueba (p. 115) Un punto de prueba es un punto que escoges a un lado del límite de la gráfica de una desigualdad lineal. Si el punto de prueba hace que la desigualdad sea verdadera, entonces todos los puntos en ese límite son soluciones de la desigualdad. Si el punto de prueba hace que la desigualdad sea falsa, entonces todos los puntos del otro lado del límite son soluciones.

Tolerance (p. 44) The difference between a desired measurement and its maximum and minimum allowable values is the tolerance. The tolerance equals one half of the difference between the maximum and minimum values.

Tolerancia (p. 44) La diferencia entre una medida deseada y sus valores máximo y mínimo permitidos es la tolerancia. La tolerancia equivale a la mitad de la diferencia entre los valores máximo y mínimo.

Example A manufacturing specification calls for a dimension d of 10 cm with a tolerance of 0.1 cm. The allowable difference between d and 10 is less than or equal to 0.1.

English

Spanish

Transformation (p. 99) A transformation of a function $y = af(x - h) + k$ is a change made to at least one of the values a, h, and k. The four types of transformations are dilations, reflections, rotations, and translations.

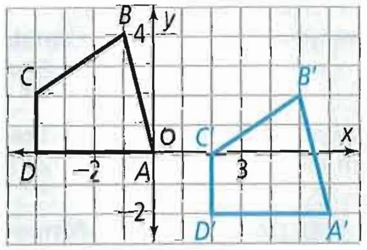
Transformación (p. 99) Una transformación de una función $y = af(x - h) + k$ es un cambio que se le hace a por lo menos uno de los valores a, h y k. Hay cuatro tipos de transformaciones: dilataciones, reflexiones, rotaciones y traslaciones.

Example $g(x) = 2(x - 3)^2$ is a transformation of $f(x) = x^2$.

Translation (p. 99) A translation shifts the graph of the parent function horizontally, vertically, or both without changing its shape or orientation.

Traslación (p. 99) Una traslación desplaza la gráfica de la función madre horizontalmente, verticalmente o en ambas direcciones, sin cambiar su forma u orientación.

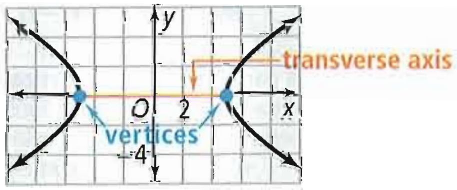
Example



Transverse axis (p. 646) The transverse axis of a hyperbola is the segment that is on the line containing the foci and has endpoints on the hyperbola.

Eje transversal (p. 646) El eje transversal de una hipérbola es el segmento que se encuentra sobre la línea que contiene los focos y tiene sus puntos extremos sobre la hipérbola.

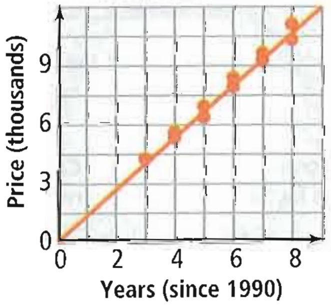
Example

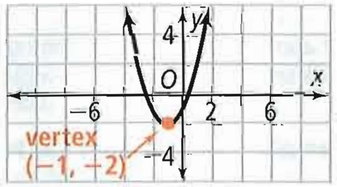


Trend line (p. 93) A trend line is a line that approximates the relationship between two variables, or data sets, of a scatter plot.

Línea de tendencia (p. 93) Una línea de tendencia es una línea que aproxima la relación entre dos variables o conjuntos de datos de un diagrama de dispersión.

Example


Used Car Prices

English	Spanish
Turning point (p. 282) A turning point of the graph of a function is a point where the graph changes direction from upwards to downwards or from downwards to upwards.	**Punto de giro (p. 282)** Un punto de giro de la gráfica de una función es un punto donde la gráfica cambia de dirección de arriba hacia abajo o vice versa.
V	
Value (p. 5) The value of a quantity is its measure or the number of items that you count.	**Valor (p. 5)** El valor de una cantidad as su medida o el número de datos que cuentas.
Variable (p. 5) A variable is a symbol, usually a letter, that represents one or more numbers. **Example** x, a, k	**Variable (p. 5)** Una variable es un símbolo, generalmente una letra, que representa uno o más valores.
Variable quantity (p. 5) A variable quantity can have values that vary.	**Cantidad variable (p. 5)** Una cantidad variable puede tener valores que varían.
Vertex (p. 107) A vertex of a function is a point where the function reaches a maximum or a minimum value.	**Vértice (p. 107)** El vértice de una función es el punto donde la función alcanza un valor máximo o mínimo.
Vertex form of a quadratic function (p. 194) The vertex form of a quadratic function is $f(x) = a(x - h)^2 + k$, where $a \neq 0$ and (h, k) is the coordinate of the vertex of the function. **Example** $f(x) = x^2 + 2x - 1 = (x + 1)^2 - 2$ The vertex is $(-1, -2)$.	**Forma del vértice de una función cuadrática (p. 194)** La forma vértice de una función cuadrática es $f(x) = a(x - h)^2 + k$, donde $a \neq 0$ y (h, k) es la coordenada del vértice de la función.
Vertex of a parabola (p. 194) The vertex of a parabola is the point where the function for the parabola reaches a maximum or a minimum value. The parabola intersects its axis of symmetry at the vertex. **Example**  The vertex of the quadratic function $y = x^2 + 2x - 1$ is $(-1, -2)$.	**Vértice de una parábola (p. 194)** El vértice de una parábola es el punto donde la función de la parábola alcanza un valor máximo o mínimo. La parábola y su eje de simetría se intersecan en el vértice.
Vertical compression (p. 102) A vertical compression reduces all y-values of a function by the same factor between 0 and 1.	**Compresión vertical (p. 102)** Una compresión vertical reduce todos los valores de y de una función por el mismo factor entre 0 y 1.
Vertical stretch (p. 102) A vertical stretch multiplies all y-values of a function by the same factor greater than 1.	**Estiramiento vertical (p. 102)** Un estiramiento vertical multiplica todos los valores de y por el mismo factor mayor que 1.

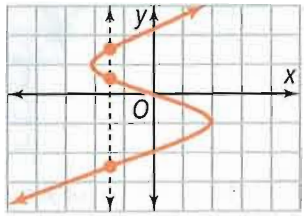
English

Spanish

Vertical-line test (p. 62) You can use the vertical-line test on the graph of a relation to tell whether the relation is a function. If a vertical line passes through more than one point on the graph of a relation, then the relation is *not* a function.

Prueba de la recta vertical (p. 62) Puedes usar la prueba de la línea vertical en la gráfica de una relación para saber si la relación es una función. Si una línea vertical pasa por más de un punto de la gráfica de la relación, entonces la relación *no* es una función.

Example

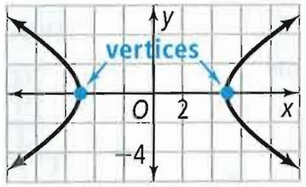


This relation is not a function.

Vertices of a hyperbola (p. 646) The endpoints of the transverse axis of a hyperbola are the vertices of the hyperbola.

Vértices de una hipérbola (p. 646) Los dos puntos de intersección de la hipérbola y su eje mayor son los vértices de la hipérbola.

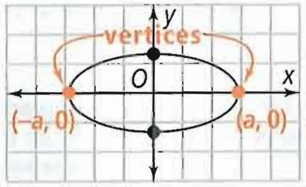
Example



Vertices of an ellipse (p. 639) The endpoints of the major axis of an ellipse are the vertices of the ellipse.

Vértices de una elipse (p. 639) Los dos puntos de intersección de la elipse y su eje mayor son los vértices de la elipse.

Example



English

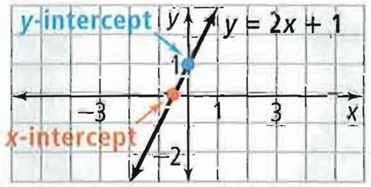
Spanish

X

***x*-intercept, *y*-intercept (p. 76)** The point at which a line crosses the *x*-axis (or the *x*-coordinate of that point) is an *x*-intercept. The point at which a line crosses the *y*-axis (or the *y*-coordinate of that point) is a *y*-intercept.

Intercepto en *x*, intercepto en *y* (p. 76) El punto donde una recta corta el eje *x* (o la coordenada *x* de ese punto) es el intercepto en *x*. El punto donde una recta cruza el eje *y* (o la coordenada *y* de ese punto) es el intercepto en *y*.

Example



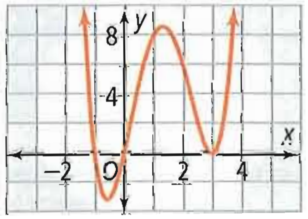
The *x*-intercept of $y = 2x + 1$ is $\left(-\frac{1}{2}, 0\right)$ or $-\frac{1}{2}$.
The *y*-intercept of $y = 2x + 1$ is (0, 1) or 1.

Z

Zero of a function (p. 226) A zero of a function $f(x)$ is any value of x for which $f(x) = 0$.

Cero de una función (p. 226) Un cero de una función $f(x)$ es cualquier valor de x para el cual $f(x) = 0$.

Example



The zeros of the function $P(x) = x(x - 3)^2(x + 1)$ are 0, 3, and −1.

Zero-Product Property (p. 226) If the product of two or more factors is zero, then one of the factors must be zero.

Propiedad del cero del producto (p. 226) Si el producto de dos o más factores es cero, entonces uno de los factores debe ser cero.

Example $(x - 3)(2x - 5) = 0$
$x - 3 = 0$ or $2x - 5 = 0$

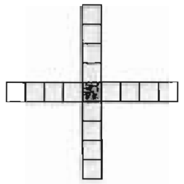
Selected Answers

Chapter 1

Get Ready! pp. 1 **1.** 0 **2.** −2 **3.** −2.09 **4.** 8.05 **5.** $-\frac{3}{4}$ **6.** $\frac{11}{12}$ **7.** $10\frac{7}{10}$ **8.** $3\frac{1}{2}$ **9.** −42 **10.** 72 **11.** 9 **12.** −9.8 **13.** $-3\frac{1}{3}$ **14.** $-5\frac{1}{2}$ **15.** $-4\frac{2}{3}$ **16.** $-\frac{3}{4}$ **17.** −21 **18.** 7.35 **19.** $-\frac{1}{6}$ **20.** $-\frac{3}{5}$ **21.** −20 **22.** 8 **23.** 0.97 **24.** −5 **25.** 55 **26.** 3 **27.** because the placement of the parentheses changes the order of operations **28.** 3 **29.** 3 terms **30.** Calculate the answer numerically. **31.** $\frac{3}{n-3}$

Lesson 1-1 pp. 4–10

Got It? **1.** The pattern shows a center square and a yellow square added to each side with the number of squares per side increasing by one.



2. 52 tiles **3. a.** \$12 **b.** \$20 **c.** The number of platys must be a whole number whereas the length of fish can be a fraction or a decimal.

Lesson Check **1.** add 35 **2.** rotate 90° clockwise

3.

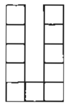
Input	Process Column	Output
1	2(1)	2
2	2(2)	4
3	2(3)	6
4	2(4)	8
⋮	⋮	⋮
n	2(n)	2n

4.

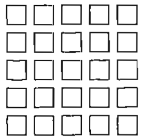
Input	Process Column	Output
1	3(1)	3
2	3(2)	6
3	3(3)	9
4	3(4)	12
⋮	⋮	⋮
n	3(n)	3n

5. Answers may vary. Sample: Look for the same type of change between consecutive figures.
6. Answers may vary. Sample: Tables of values and pictorial representations are both convenient ways to organize data and discover patterns. Tables give more detail. Pictorial representations are visual. **7.** No; the output is $\frac{1}{2}$ the input for all values except the first (Input: 3; Output: 2).

Exercises **9.** Base of 3 squares with the number of squares increasing vertically by one on each of the outer squares of the base.



11. One square, then 2^2 or 4 squares, then 3^2 or 9 squares, then 4^2 or 16 squares. In general, the number of squares is $n \times n$ or n^2.



13. 2n

Input	Process Column	Output
1	2(1)	2
2	2(2)	4
3	2(3)	6
4	2(4)	8
⋮	⋮	⋮
n	2(n)	2n

15. 4n − 1

Input	Process Column	Output
1	4(1) − 1	3
2	4(2) − 1	7
3	4(3) − 1	11
4	4(4) − 1	15
⋮	⋮	⋮
n	4(n) − 1	4n − 1

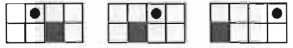
17. 7; 8; $n + 2$
19. Output = Input + 1

Input	Process Column	Output
1	(1) + 1	2
2	(2) + 1	3
3	(3) + 1	4
4	(4) + 1	5
5	(5) + 1	6
⋮	⋮	⋮
n	(n) + 1	n + 1

21. Output = Input − 1

Input	Process Column	Output
1	(1) − 1	0
2	(2) − 1	1
3	(3) − 1	2
4	(4) − 1	3
5	(5) − 1	4
⋮	⋮	⋮
n	(*n*) − 1	*n* − 1

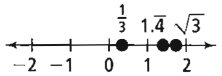
23. 40 **25.** add 6 or 6*n*; 30, 36, 42 **27.** add 3, then add 4, then add 5, and so on; 21, 28, 36 **29.** multiply by 3; 243, 729, 2187 **31.** The black square and dot each move clockwise one block



33. 9216 in.^{3} **35.** $n + 10$, where *n* is the number of months **37.** 21; $4n + 1$ **39.** −13; $7 - 4n$; or $-4n + 7$ **46.** 1.9 **47.** −3.8 **48.** 27 **49.** 0 **50.** −0.4 **51.** 7 **52.** 50% **53.** 25% **54.** 33.33% **55.** 140% **56.** 172% **57.** 123%

Lesson 1-2 pp. 11–17

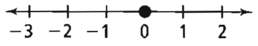
Got It? 1. rational numbers
2.



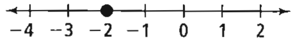
3. a. $\sqrt{26} < 6.25$ or $6.25 > \sqrt{26}$ **b.** $a < c$; *a* will be to the left of *c* on the number line.
4. a. Distr. Prop.
b. $a + [3 + (-a)]$

$$\begin{aligned} &= a + [(-a) + 3] && \text{Comm.}\\ &= [a + (-a)] + 3 && \text{Assoc.}\\ &= 0 + 3 && \text{Inverse}\\ &= 3 && \text{Identity} \end{aligned}$$

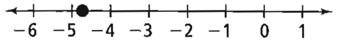
Lesson Check 1. Answers may vary. Sample: the number of times a cricket chirps **2.** Answers may vary. Sample: the change in number of people on a bus after a stop **3.** Answers may vary. Sample: the outdoor temperature in tenths of a degree **4.** Inv. Prop. of Add. **5.** Assoc. Prop. of Mult. **6.** multiplicative inverse **7.** Both properties result in the original term; 0 is the additive identity, whereas 1 is the multiplicative identity. **8.** The equation illustrates the Comm. Prop. of Add. **9.** Answers may vary. Sample: $\sqrt{2}$ is not a rational number because it cannot be written as a quotient of integers.
Exercises 11. *y*, natural numbers; *p*, rational numbers
13.



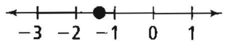
15.



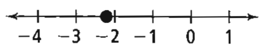
17.



19.



21.



23. > **25.** < **27.** > **29.** > **31.** > **33.** < **35.** Distr. Prop. **37.** Assoc. Prop. of Mult. **39.** Ident. Prop. of Add. **41–48.** Answers may vary. Samples are given. **41.** −5 **43.** $-1\frac{1}{4}$ **45.** $1\frac{2}{3}$ **47.** 4 **49.** $\sqrt{50}$ in. × $\sqrt{50}$ *in.* × $\sqrt{50}$ *in.* **51.** natural numbers **53.** irrational numbers **55.** irrational numbers **57.** 8, 1, $\frac{1}{3}$, $-\sqrt{2}$, −3 **59.** 5.73, $\frac{1}{4}$, −0.06, $-3\sqrt{3}$, −17 **61.** Answers may vary. Sample: 7 **63.** Answers may vary. Sample: $\sqrt{2}$ and $\sqrt{2}$ **75.** add 4; 20, 24, 28 **76.** add 1; 12, 13, 14 **77.** add 1; 0, 1, 2 **78.** $2\frac{1}{4}$ **79.** $11\frac{2}{3}$ **80.** $1\frac{1}{2}$ **81.** 5 **82.** 38 **83.** 15

Lesson 1-3 pp. 18–24

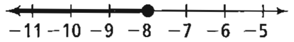
Got It? 1. H **2.** $150 - 2d$, with $d =$ the number of days **3. a.** 18 **b.** Yes; the numerator will become $2x^2 - y^2$, not $2x^2 - 2y^2$. **4.** Let $x =$ the number of two-point shots, $y =$ the number of three-point shots, $z =$ the number of one-point free throws, $2x + 3y + 1z$; 42 points **5. a.** $-3j^2 - 7k + 5j$ **b.** $12a - 53b$
Lesson Check 1. $\frac{2 + b}{3}$ **2.** $4k + m$ **3.** 12 **4.** 13 **5.** −5 **6.** −5 **7.** The student did not distribute the −1. $3p^2q + 2p - (5q + p - 2p^2q) = 3p^2q + 2p - 5q - p + 2p^2q = 5p^2q + p - 5q$ **8.** A constant is a term with no variables, whereas a coefficient is the numerical factor in a term. **9.** Answers may vary. Sample: Both algebraic expressions and numerical expressions represent a quantity using numbers, operations and grouping symbols. An algebraic expression includes variables when representing a quantity. Examples: numerical expression: $3 + 6(5 - 2)$; algebraic expression: $2z + 3z(6 + 5z)$.
Exercises 11. $8(x + 3)$ **13.** $130 - 10w$, with $w =$ number of weeks **15.** $250 - 60w$, with $w =$ number of weeks **17.** −16 **19.** −12 **21.** 4 ft **23.** 1600 ft **25.** $1331 **27.** $1610.51 **29.** Let $x =$ the number of 3-run home runs and $y =$ the number of 2-run hits; $3x + 2y$; 14 **31.** $2s + 5$ **33.** $6a + 3b$ **35.** $-0.5x$ **37.** $4g - 2$ **39.** 3 **41.** 37 **43.** 10 **45.** $\frac{\$84}{m}$ **47.** $\frac{5x^2}{2}$ **49.** *y* **51.** $-2x^2 + 2y^2$ **53.** $8.5x - 15$ **55.** No; John did not use the opposite of a sum correctly; $-(x + y) + 3(x - 4y)$; $-x - y + 3x - 12y$; $2x - 13y$ **57.** Distr. Prop. **59.** Opposite of a Difference **67.** −1.5, $-\sqrt{2}$, −1.4, −0.5 **68.** $-\frac{5}{6}$, $-\frac{3}{4}$, $-\frac{3}{8}$, $\frac{1}{2}$ **69.** −20, 0.2, $\frac{1}{2}$, $\sqrt{2}$ **70.** −3, −0.5, $-\frac{1}{4}$, $\frac{3}{4}$ **71.** $7x - 4$ **72.** $-p - \frac{2q}{3}$ **73.** $2b - 28$ **74.** $2k - 2m$

Lesson 1-4 pp. 26–32

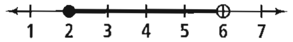
Got It? 1. $\frac{3}{2}$ **2.** −1 **3.** 40 m × 120 m **4. a.** never **b.** always **5. a.** $C = K - 273$ **b.** always
Lesson Check 1. 23.2 **2.** −90 **3.** 12
4. $k = \frac{1}{2}(r - 15)$ **5.** $k = \frac{1}{3}(z + 6)$
6. $k = -\left(\frac{1}{6}\right)(h + 14)$ **7.** To find a solution of an equation means to find the value of the variable that makes the equation true. **8.** Four buses are not enough. The number of buses must be a whole number, so round the number of buses to 5. **9.** The 2nd line is incorrect; subtract 10 from both sides: $12x = -12x = -1$
Exercises 11. −81 **13.** 14 **15.** 8 **17.** −5
19. $-\frac{1}{9}$ **21.** $\frac{3}{2}$ **23.** −6 **25.** 0 **27.** 300 mi/h; 600 mi/h
29. sometimes **31.** sometimes **33.** $h = \frac{2A}{b}$ **35.** $w = \frac{V}{\ell h}$
37. $x = \frac{c}{a + b}$, $a \neq -b$ **39.** $x = 2(m + n) + 2$ **41.** 1.5
43. $\frac{23}{3}$, or $7\frac{2}{3}$ **45.** 34° and 56° **47.** $b_2 = \frac{2A}{h} - b_1$
49. $v = \frac{h + 5t^2}{t}$ **51.** $r_2 = \frac{Rr_1}{r_1 - R}$ **53.** 40°, 140°
55. $x = \frac{3b + 2c - 5}{b - c}$, $b \neq c$ **57.** $x = \frac{4a - 3bc}{aq - 5bp}$, $5bp \neq aq$
59. $x = \frac{10c}{a}$, $a \neq 0$ **61.** Let c = number of swim days; 3c = 82 + c; 41 days **63.** No; $n = \frac{s}{1 - s}$ *not* $\frac{s}{s - 1}$
70. −7 **71.** $-\frac{16}{3}$ **72.** −20 **73.** $-\frac{11}{2}$ **74.** $x + 5$
75. $16x$ **76.** $3(12 - x)$ **77.** true **78.** false **79.** true

Lesson 1-5 pp. 33–40

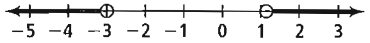
Got It? 1. $\frac{x}{3} \leq 15$
2. $x \leq -8$



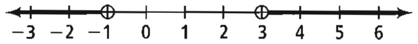
3. more than 32 songs **4.** always
5. a. $x \geq 2$ and $x < 6$



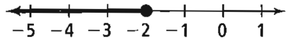
b. sometimes; The compound inequality is true when $x = 5$ and not true when $x = 7$.
6. a. $w < -3$ *or* $w > \frac{8}{7}$



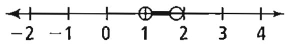
b. $x < -1$ or $x > 3$



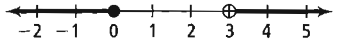
Lesson Check 1. $R \geq J$ **2.** $w \geq 40$ and $w < 74$
3. $x \leq -2$



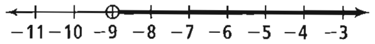
4. $1 < x < \frac{9}{5}$



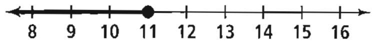
5. $x \leq 0$ or $x > 3$



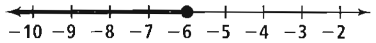
6. Answers may vary. Sample: 5 < 6, but −5 > −6.
7. The transitive, addition and subtraction properties of inequality are similar to the properties of equality. The multiplication and division properties differ. Multiplying or dividing each side of an inequality by a negative number reverses the direction of the inequality symbol.
8. Answers may vary. Sample: $3x + 5 < 3(x + 5)$
9. No; Answers may vary. Sample: $2x < x + 1$ and $x + 1 > 3$
Exercises 11. $8x \geq 25$ **13.** $\frac{x}{12} \leq 6$
15. $k > -9$



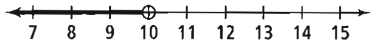
17. $t \leq 11$



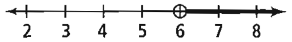
19. $y \leq -6$



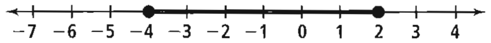
21. $m < 10$



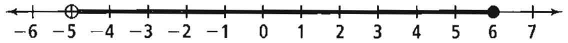
23. $w > 6$



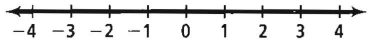
25. The longest side is less than 21 cm. **27.** at most 40 students **29.** always **31.** never **33.** sometimes
35. sometimes
37. $-4 \leq x \leq 2$



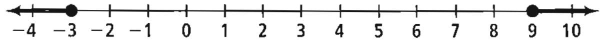
39. $-5 < x \leq 6$



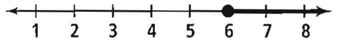
41. all real numbers



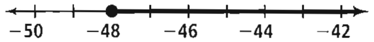
43. $x \leq -3$ or $x \geq 9$



45. $z \geq 6$

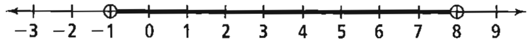


47. $x \geq -48$

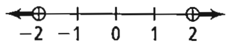


49. no solution **51.** 98 **53.** $2 < AB < 6$ **55.** The classmate reversed the direction of the ≥ symbol to ≤ incorrectly. The correct answer is $y \leq -20$. **57.** Distr. Prop.; arithmetic; Subtr. Prop. of Inequality; Mult. Prop. of Inequality

59. $-1 < x < 8$



61. $x < -2$ or $x > 2$

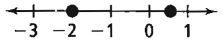


71. $7a + 5$ **72.** $-2x + 14y$ **73.** $\frac{b}{12} + 1$ **74.** $1.61 - 0.1k$ **75.** 4 **76.** no solution **77.** $\frac{9}{10}$ **78.** -20

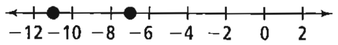
Lesson 1-6 pp. 41–48

Got It?

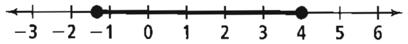
1. $\frac{2}{3}, -2$



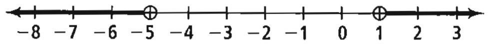
2. $-7, -11$



3. -1 **4.** $-\frac{4}{3} \leq x \leq 4$



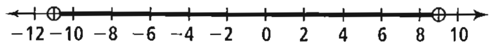
5. a. $x < -5$ or $x > 1$



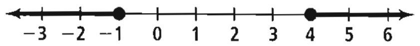
b. The graph will have two closed circles with an arrow extending to the left of one and to the right of the other. **6.** $|h - 52.5| \leq 0.5$

Lesson Check **1.** $-4, 4$ **2.** $-12, 4$ **3.** $-\frac{6}{5}$

4. $-11 < x < 9$



5. $x \leq -1$ or $x \geq 4$

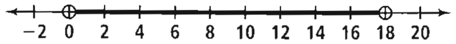


6. A solution of an eq. is extraneous if it is a solution to a derived eq., but is not a solution to the original eq.

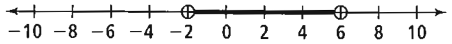
7. when the number is positive or 0 **8.** Answers may vary. Sample: $d < -5$ and $5d > 25$ **9.** Answers may vary. Sample: An absolute value equation or inequality represents two equations or inequalities; each equation or inequality is solved in the same manner as a linear equation or inequality.

Exercises **11.** $-8, 8$ **13.** $-\frac{5}{3}, 3$ **15.** no solution **17.** $-7, 17$ **19.** $-\frac{3}{2}$ **21.** $\frac{3}{2}$ **23.** $-1, \frac{3}{2}$

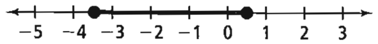
25. $0 < y < 18$



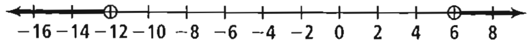
27. $-2 < x < 6$



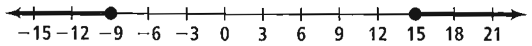
29. $-3\frac{1}{2} \leq w \leq \frac{1}{2}$



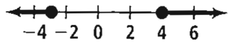
31. $x < -12$ or $x > 6$



33. $y \leq -9$ or $y \geq 15$

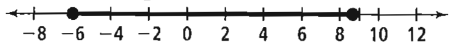


35. $x \leq -3$ or $x \geq 4$

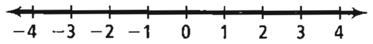


37. $|h - 1.4| \leq 0.1$ **39.** $|C - 27.5| \leq 0.25$ **41.** $|m - 1250| \leq 50$ **43.** no solution **45.** $-\frac{14}{3}, \frac{16}{3}$ **47.** no solution **49.** $\frac{11}{8}$ **51.** $-\frac{71}{36}$ **53.** $|c - 28.75| \leq 0.25$; 0.25; $28.50 \leq c \leq 29.00$ **55.** $|x| < 4$

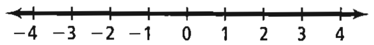
57. $-6 \leq x \leq 8\frac{2}{3}$



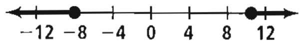
59. all real numbers



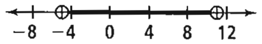
61. all real numbers



63. $x \leq -8.4$ or $x \geq 9.6$



65. $-5 < x < 11$

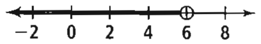


67. The graph of $|x| < a$ is the set of all points on the number line that lie between a and $-a$. The graph of $|x| > a$ has two parts; the left part consists of the points to the left of $-a$, and the right part consists of the points to the right of a. **69.** $|t - 350| \leq 5$ **71.** $|t - 15| \leq 30$ **73.** $|x - 9.55| \leq 0.02$; $9.53 \leq x \leq 9.57$

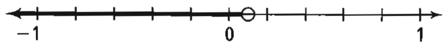
75. never; absolute value is nonnegative **77.** sometimes; $|5| = 5$ but $|-5| \neq -5$ **79.** sometimes; $|-4 + 2| \neq -4 + 2$ **81.** The "3" in the second set of equations should be "−3."

$$-4x + 1 < -3$$
$$-4x < -4$$
$$x > 1 \text{ } \textit{not } x > -\frac{1}{2}$$

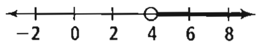
94. $y < 6$



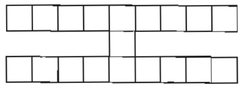
95. $s < \frac{2}{15}$



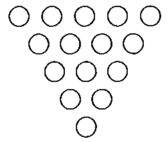
96. $a > 4$



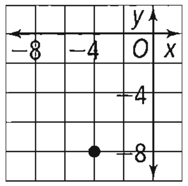
97. Each figure has 4 more squares than the previous figure.



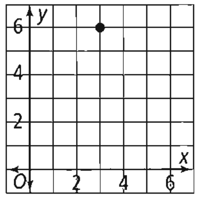
98. Each figure n has n more circles than the previous figure.



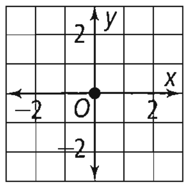
99.



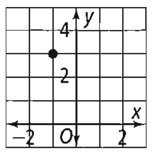
100.



101.



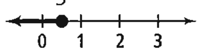
102.



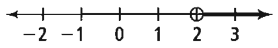
Chapter Review pp. 50–52

1. solution of the equation **2.** absolute value **3.** reciprocal **4.** compound inequality **5.** add 5; *25, 30, 35* **6.** add 1; 7, 8, 9 **7.** 12; $n + 8$ **8.** 76; $19n$ **9.** $\$20n$ **10.** irrational numbers **11.** rational numbers, integers **12.** rational numbers, integers, whole numbers, natural number **13.** real numbers, rational numbers **14.** $-\sqrt{60} > -8$ or $-8 < -\sqrt{60}$ **15.** $5 < \sqrt{32}$ or $\sqrt{32} > 5$ **16.** Inv. Prop. of Mult. **17.** Assoc. Prop. of Mult. **18.** 114 **19.** $5b$ **20.** 11 **21.** 6

22. $z \le \frac{2}{5}$

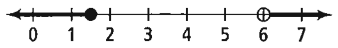


23. $x > 2$



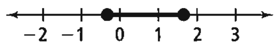
24. no solution

25. $x \le \frac{3}{2}$ or $x > 6$

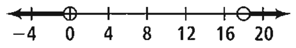


26. 10 cm, 6 cm **27.** 1 **28.** no solution **29.** $x = -8$ or $x = -12$ **30.** no solution

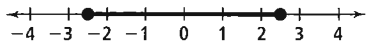
31. $-\frac{1}{3} \le x \le \frac{5}{3}$



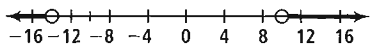
32. $y < 0$ or $y > 18$



33. $-\frac{18}{7} \le x \le \frac{18}{7}$



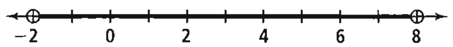
34. $x < -14$ or $x > 10$



35. $|x - 43.6| \le 0.1$

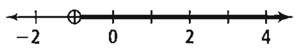
Chapter 2

Get Ready! p. 57 **1.** $6s$ **2.** $4a + b$ **3.** $xy - y + x$ **4.** $1.5g$ **5.** 0 **6.** $3b - 2c - 2$ **7.** $6f - 5d$ **8.** $3h + 3g$ **9.** $-2z + 5$ **10.** $2g - 4dg - 12d$ **11.** $8v - 6$ **12.** $7t - 3st - 5s$ **13.** −56 **14.** 80 **15.** −10 **16.** −24 **17.** 1075 **18.** 5 **19.** −1.75 **20.** 1.5 **21.** 20 **22.** 2 **23.** 5 **24.** 4 **25.** $-2 < x < 8$

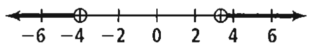


26. $a \le 0$

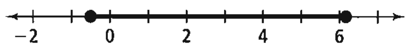
27. $x > -1$



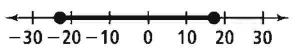
28. $x < -4$ or $x > \frac{10}{3}$



29. $-\frac{1}{2} \le d \le \frac{25}{4}$



30. $-24 \le f \le 18$

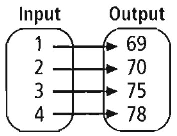


31. Answers may vary. Sample: the Civil War, the Great Depression, the Louisiana Purchase **32.** Answers may vary. Sample: From 1 to 2 years of age; a person has usually stopped growing by age 30, but a baby is still growing at age 1. **33.** Answers may vary. Sample: The image is a reflection, left to right, of what other people see; the size is the same. **34.** Answers may vary. Sample: An inequality determines the limit of a value, or a boundary, for the solution on the number line.

Lesson 2-1 pp. 60–67

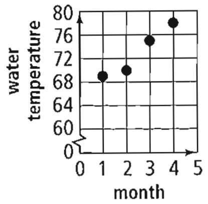
Got It?

1. Let Jan = 1, Feb = 2, Mar = 3, and Apr = 4.



{(1, 69), (2, 70), (3, 75), (4, 78)}

x Month	y Temperature (°F)
1	69
2	70
3	75
4	78

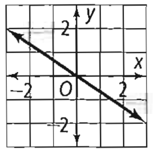


2. domain: $\{-3, 0, 2, 9, 23\}$, range: $\{-99, -18, 0, 7, 14\}$ **3. a.** no **b.** yes **c.** In a mapping diagram for a relation that is not a function, there is at least one element in the domain that has more than one arrow originating from it. In a mapping diagram for a function, each element in the domain has at most one arrow originating from it. **4.** b and c **5. a.** 9 **b.** 1 **c.** −19
6. Let x = number of bottles purchased and C = total cost; $C(x) = 1.19x$; $17.85
Lesson Check **1.** domain: $\{0, 3, 4\}$, range: $\{-2, 1, 2, 4\}$
2. domain: $\{-4, -3, 0, 4\}$, range: $\{-4, -3, 0, 4\}$
3. no **4.** yes **5.** Yes; a relation is any set of pairs of input and output values. No; a function is a relation in which each element of the domain is paired with exactly one element of the range. **6.** *Every* vertical line does not need to intersect a function. Rewrite as: "In a function, every vertical line must intersect the graph in *at most* one point." **7.** A horizontal-line test checks the pairing of one element of the range with one or more elements of the domain. A function can have a pairing of one element of the range with *one or more* elements of the domain. A horizontal-line test cannot determine whether a relation is a function.
Exercises **9.** domain: $\{1, 2, 3, 4, 5, 6\}$, range: $\{6, 7, 8, 9, 11\}$ **11.** yes **13.** yes **15.** yes **17.** 71; (4, 71)
19. −15; (9, −15) **21.** −2; (3, −2) **23.** −132; (−11, −132) **25.** $C(m) = 4.52 + 0.12m$; $34.52
27. 13.5 cm^2 **29.** domain: all real numbers, range: $y \geq 0$; yes **31.** $\approx 4849\ \text{cm}^3$ **33. a.** 109.4 **b.** 10.4
c. −11.1 **d.** −7.2 **44.** $\frac{2}{3}, -\frac{20}{3}$ **45.** −13, 15
46. $x > -3$ **47.** $x \leq \frac{3}{2}$ **48.** $-\frac{3}{2} < x < \frac{3}{2}$ **49.** $x \geq -3$
50. $\frac{1}{4}x$ **51.** $-\frac{1}{2}x$ **52.** $20x$

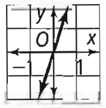
Lesson 2-2 pp. 68–73

Got It? **1. a.** yes; $-7, y = -7x$ **b.** no **2. a.** yes; $-\frac{5}{3}$
b. yes; $\frac{1}{9}$ **3.** 60 **4. a.** 280 **b.** No; if $y^2 = kx^2$, then $y = \pm\sqrt{kx}$. So, $\frac{y}{x}$ could be $+\sqrt{k}$ for one pair of values and $+\sqrt{k}$ for another pair. Then y would not vary directly with x.

5. a.

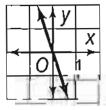


b.



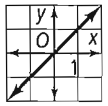
Lesson Check **1.** $y = -\frac{1}{2}x$ **2.** $\frac{3}{2}$ **3.** $\frac{5}{4}$ **4.** Answers may vary. Sample: Two variables are directly related when the ratio of the output to the input is a constant value.
5. For a direct variation, $y = kx$ where k is the constant of variation. If $x = 0$, then $y = 0$ and the graph of $y = kx$ passes through the origin. **6.** Answers may vary. Sample: $y = -8x$.
Exercises **7.** yes; $7, y = 7x$ **9.** no **11.** yes; 12
13. yes; −2 **15.** no **17.** yes; 6 **19.** −3 **21.** $\frac{6}{7}$ **23.** 21
25. 4 min **27.**

x	y
−1	3
1	−3
2	−6

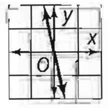


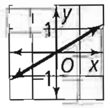
29.

x	y
−1	−1
1	1
2	2

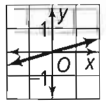


31. yes; $k = \frac{2}{3}, y = \frac{2}{3}x$ **33.** no
35. $y = 2x$

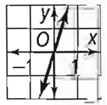
37. $y = -4.5x$

39. $y = \frac{3}{5}x$



41. $y = \frac{2}{7}x$

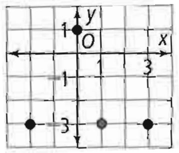


43. 0.625 **45.** 0.225 **47.** First, it does not say that y varies directly with x. Second, every direct variation includes the point (0, 0), so x cannot be determined because k could be any value.
49. Answers may vary. Sample: $y = 3.2x$



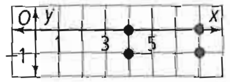
51. Answers may vary. Sample: If y varies directly with x^2, and $y = 2$ when $x = 4$, then $y = \frac{81}{8}$ when $x = 9$.
53. $c = 0, a \neq 0, b \neq 0$

62.



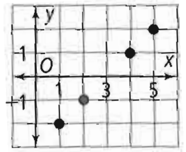
domain: {−2, 0, 1, 3};
range: {−3, 1}

63.



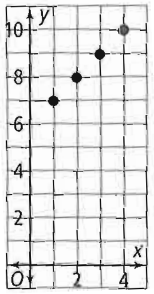
domain: {4, 7}; range:
{−1, 0}

64.



domain: {1, 2, 4, 5};
range: {−2, −1, 1, 2}

65.



domain: {1, 2, 3, 4};
range: {7, 8, 9, 10}

66. $8n$; 40, 48, 56 **67.** $7 - 2n$; −3, −5, −7
68. $12(13 - n)$; 96, 84, 72 **69.** $15(n + 1)$; 90, 105, 120 **70.** $\frac{17}{3}$; 7; $\frac{23}{3}$; $\frac{29}{3}$ **71.** −3.2; −2; −1.4; 0.4
72. −5; 1; 4; 13 **73.** −9; −8; −7.5; −6

Lesson 2-3 pp. 74–80

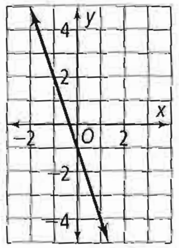
Got It? 1. a. −1 **b.** 1 **c.** undefined **d.** $\frac{1-4}{8-5} = \frac{-3}{3} = -1 = \frac{3}{-3} = \frac{4-1}{5-8}$ **2. a.** $y = 6x + 5$
b. $y = -\frac{1}{2}x - 3$ **c.** No; any two points on a line can be used to calculate the slope. **3. a.** $y = -\frac{3}{2}x + 9$; $-\frac{3}{2}$; (0, 9)
b. $y = -\frac{7}{5}x - 7$; $-\frac{7}{5}$; (0, −7)

4. $y = \frac{4}{7}x - 2$

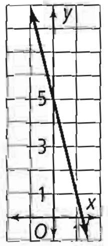
Lesson Check 1. $y = \frac{1}{2}x + 1$ **2.** $y = \frac{4}{3}x + \frac{1}{3}$ **3.** −1
4. 1 **5.** The y-intercept of a line is the point at which the line crosses the y-axis. The x-intercept is the point at which the line crosses the x-axis. **6.** Since division by zero is undefined, the slope of a vertical line that passes through (a, b) and (a, c), $\frac{c-b}{0}$, is undefined. **7.** She subtracted the x-coordinates in the wrong order. The x- and y-coordinates of each point must be subtracted consistently.

Exercises 9. −2 **11.** $\frac{4}{11}$ **13.** 1 **15.** 0 **17.** $y = 3x + 2$
19. $y = \frac{5}{6}x + 12$ **21.** $y = -5x - 7$ **23.** $y = \frac{3}{2}x + \frac{7}{2}$; $\frac{3}{2}$, $\left(0, \frac{7}{2}\right)$ **25.** $y = -\frac{4}{3}x + \frac{5}{6}$; $-\frac{4}{3}$, $\left(0, \frac{5}{6}\right)$ **27.** $y = 7$; 0, (0, 7)

29.



31.



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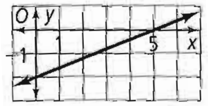

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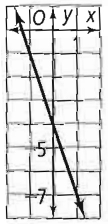

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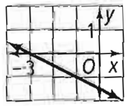

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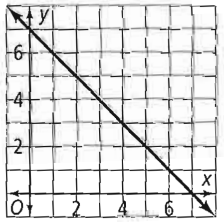

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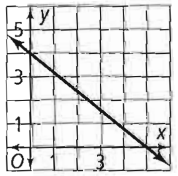
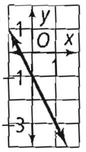
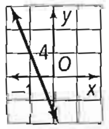


43.



45.

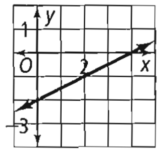


47. 0; (0, 3) **49.** $-\frac{1}{4}$; (0, 3) **51.** undefined slope; no y-intercept **53.** $-\frac{1}{2}$; $\left(0, -\frac{5}{2}\right)$ **55.** $-\frac{A}{B}$; $\left(0, \frac{C}{B}\right)$ **57. a.** 1
b. 1 **c.** 1 **d.** 1 **e.** Any two points on a line can be used to find the slope of the line. **59.** $-\frac{5}{13}$ **61.** $\frac{15}{2}$
67. domain: {−2, 1, 2, 3, 4}, range: {−2, −1, 2, 3}; no
68. domain: all real numbers, range: $y \geq -2$; no
69. domain: {−5, 0, 2, 9}, range: {−3, −1, 5, 15}; no
70. 2 **71.** 8 **72.** 13

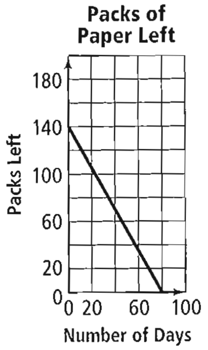
Lesson 2-4 pp. 81–88

Got It? 1. $y + 1 = -3(x - 7)$ **2. a.** $y - 7 = \frac{7}{5}x$
b. $y = \frac{7}{5}(x + 5)$; Either point can be used to put the equation of the line in point-slope form.
3. $-91x + 10y = 36$

4. (0, −2), (4, 0);

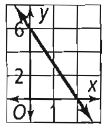


5. a.



b. $7x + 4y = 560$ **c.** 87.5 **6. a.** $y = -2x + 6$
b. $y = -\frac{3}{2}x + 6$

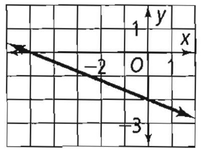
Lesson Check **1.** $y = -3x - 1$ **2.** $y = \frac{1}{2}x + 2$
3. (0, 6), (2, 0)



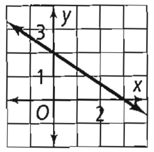
4. $3x + y = -1$ **5.** $3x + 2y = -9$ **6. a.** point-slope **b.** slope-intercept **c.** standard **d.** point-slope
7. Point-slope form; since the *x*-intercept is the point where *y* is zero, you know the point on the line, (*x*, 0), and you know the slope. **8.** $-\frac{b}{a}$ **9.** No; one line has a slope of −2 and the other line has a slope of $-\frac{1}{2}$. $-\frac{1}{2}$ is the reciprocal of −2, not the *negative* reciprocal.

Exercises **11.** $y - 12 = \frac{5}{6}(x - 22)$ **13.** $y + 2 = 0$
15. $y - 2 = 5x$ **17.** Answers may vary. Sample: $y = \frac{5}{4}(x - 1)$ or $y - 5 = \frac{5}{4}(x - 5)$ **19.** Answers may vary. Sample: $y + 1 = -\frac{4}{3}x$ or $y + 5 = -\frac{4}{3}(x - 3)$
21. Answers may vary. Sample: $y - 9 = -\frac{7}{5}(x - 1)$ or $y - 2 = -\frac{7}{5}(x - 6)$ **23.** $7x + y = -9$
25. $-42x + 10y = 79$

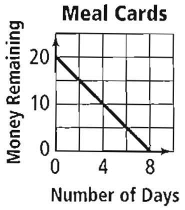
27. (0, −2), (−5, 0)



29. (0, 2), $\left(\frac{14}{5}, 0\right)$

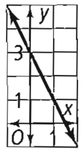


31. $y = -2.5x + 20$

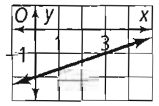


33. $y = \frac{5}{2}x + \frac{17}{2}$ **35.** $y = \frac{1}{3}x + \frac{5}{3}$

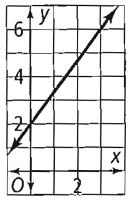
37.



39.

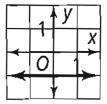


41.

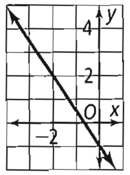


43. $y + 4 = \frac{7}{5}\left(x + 3\right)$ or $y + \frac{1}{2} = \frac{7}{5}\left(x + \frac{1}{2}\right)$
45. $y = \frac{3}{4}x + 3$ **47.** $y = -\frac{3}{2}x - \frac{1}{2}$
49. −1; (0, 1000),(1000, 0) **51.** 54; (0, −1), $\left(\frac{1}{54}, 0\right)$
53. 0; (0, 0), all points on *x*-axis

55. $y = -1$



57. $y = -\frac{3}{2}x - 1$

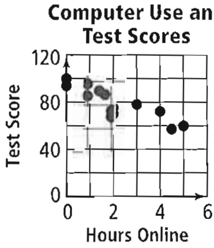


66. domain: {−3, −1, 0, 1, 2}, range: {−2, 0, 2, 4}; yes **67.** domain: all real numbers, range: {−2}; yes **68.** domain: {−18, −2, 0, 3, 39}, range: {−1, 3, 17, 28, 32}; yes **69.** Multiplicative Inv. **70.** Distr. Prop. **71.** Add. Inv., Add. Ident. **72.** $y = 3x - 5$
73. $y = \frac{1}{2}x$ **74.** $y = \frac{4}{5}x + 7$ **75.** $y = -\frac{3}{8}x + 12$

Lesson 2-5 pp. 92–98

Got It?

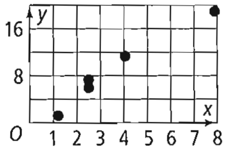
1. a. strong negative correlation



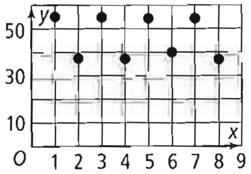
b. about $170 **2.** Answers may vary. Sample: $y = 3575x + 19354$
3. $y = 0.09x + 2.44$, where 1997 is year 0; $4.96

Lesson Check

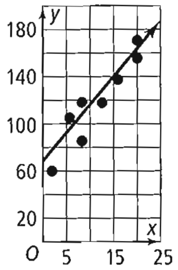
1. strong positive correlation



2. no correlation



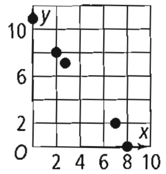
3. strong positive correlation



4. Plot the data points in a scatter plot to determine the correlation. The closer the data points fall along a line with a positive or negative slope, the stronger the correlation. **5.** No; answers may vary. Sample: A trend line is determined by using two pts. close to the line drawn through the data sets of the scatter plot. The line of best fit is the most accurate of the trend lines because it uses all the data pts. **6.** The slope of the trend line or line of best fit is positive for data pts. with positive correlation and negative for data pts. with negative correlation. The constant of variation for a direct variation is positive for data pts. with positive correlation.

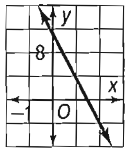
Exercises

7. strong negative correlation

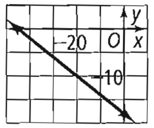


9. Answers may vary. Sample: $y = -0.7x - 4$
11. Answers may vary. Sample: $y = 4.47x + 33.31$
13. 6,055,359 tonnes **15.** positive correlation; no
17. positive correlation; yes

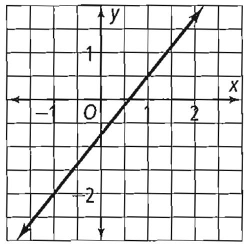
24.



25.

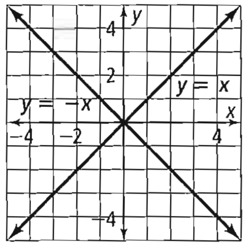


26.

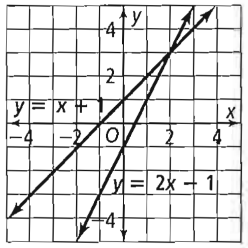


27. $-2x + y = 2$ **28.** $x + y = 0$ **29.** $y = 2$

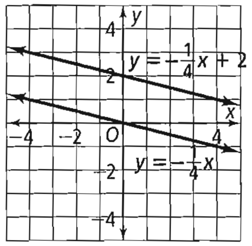
30.



31.



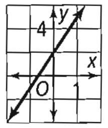
32.



Lesson 2-6 pp. 99–106

Got It? 1. a. Each output for $y = 2x - 3$ is three less than the corresponding output for $y = 2x$. The graph of $y = 2x - 3$ is the graph of $y = 2x$ translated down three units.

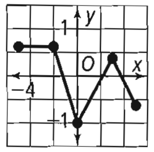
b.



2. $f\left(x + \frac{1}{2}\right)$ **3.** $h(x) = -3x - 3$

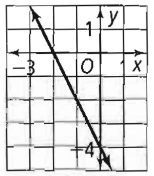
4. a.

x	y
−5	$\frac{2}{3}$
−2	$\frac{2}{3}$
0	−1
3	$\frac{1}{3}$
5	$-\frac{2}{3}$



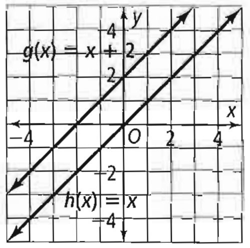
b. Sometimes; Answers may vary. Sample: switching the order of a horizontal translation and a reflection in the y-axis will change the resulting graph, but switching the order of a horizontal and vert. translation will not.
5. a. $g(x) = 2x - 3$ **b.** $g(x) = f(x + 4) - 2$; translated left 4 units and translated down 2 units
Lesson Check 1. translated 6 units up **2.** compressed vertically by a factor of 0.25 **3.** translated 4 units to the rt.
4. reflected over y-axis

5. translated 1 unit to the left and 2 units down



6. stretched vertically by a factor of 2 and translated 1 unit up

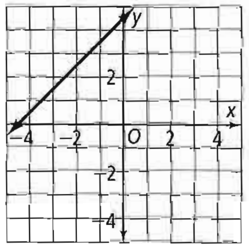
7. $g(x)$ is the graph of $h(x)$ translated 2 units up



8. Answers may vary. Sample: $f(x) = x$, $f(x - 2) = f(x) - 2$ **9.** $f(x) = -x - 2$; $g(x) = f(-x) = x - 2$

Exercises

11. The function is $y = x$ translated 4.5 units up.



13.

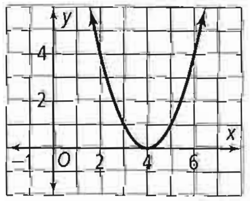
x	$f(x) + 3$
−2	6
0	4
1	1
3	2

15.

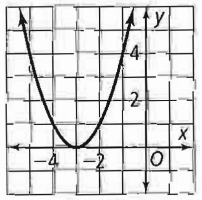
x	$f(x) + 4$
−3	5
−1	2
1	4
4	7

17. $y = f(x) + 4$

19. translated rt. 4 units



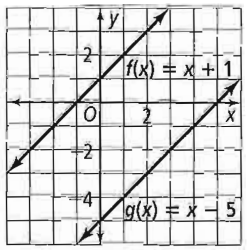
21. translated left 3 units



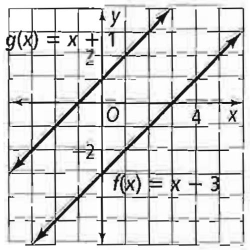
23. $g(x) = -x - 1$ **25.** $g(x) = -2x + 4$ **27.** $y = 2x$ **29.** $y = \frac{1}{4}x$ **31.** $g(x) = -0.5x$ **33.** vertically compressed by a factor of $\frac{1}{4}$ and translated down 2 units **35.** translate to the right 10 s

37. $f(x) = -\frac{1}{3}x - 1$; $g(x) = \frac{1}{3}x + 1$; $g(x) = -f(x)$

41. translated 6 units down



43. translated 4 units up



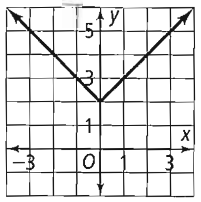
45. The first two steps are incorrect; the transformations should be: shift 1 unit left, vertically stretch by a factor of 2, and shift 3 units down. **55.** $y = -15.82x + 914.59$

56. −2, 8 **57.** −12, 11 **58.** $-3, \frac{21}{5}$

Lesson 2-7 pp. 107–113

Got It?

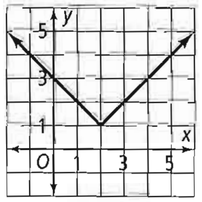
1. a.



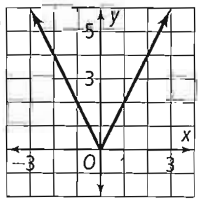
vertex at (0, 2); translated up 2 units from the parent function

b. No; transformations of this form move the vertex up or down along the axis of symmetry, so the axis stays the same.

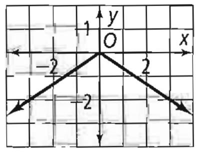
2.



3. a.



b.



4. $(1, -3)$; $x = 1$; translated 1 unit to the rt., vertically stretched by a factor of 2, then reflected over the x-axis and translated down 3 units **5.** $y = \frac{1}{4}|x - 2| - 1$

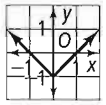
Lesson Check **1.** $(-4, -3)$; $x = -4$ **2.** $(-3, 9)$; $x = -3$ **3.** vert. stretch **4.** vert. stretch **5.** Yes; you can determine the position of a graph of an absolute value function by identifying the vertex, axis of symmetry and the transformation of the absolute value parent function. Answers may vary. Sample: $y = -\frac{1}{2}|x + 3| - 5$; vertex

(−3, −5); axis of symmetry, $x = -3$; translated 3 units to the left, vertically compressed by a factor of $\frac{1}{2}$, then reflected over the x-axis and translated down 5 units. **6.** Answers may vary. Sample: $y = |x + 1| - 2$ and $y = -|x + 1| - 2$ **7.** $y = |x|$ is the same as $y = x$ when $x \geq 0$ and is the reflection of $y = x$ across the x-axis when $x < 0$.

Exercises

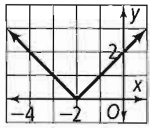
9.

x	y
−2	1
−1	0
0	−1
1	0
2	1



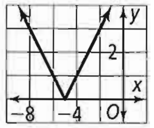
11.

x	y
−4	2
−3	1
−2	0
−1	1
0	2



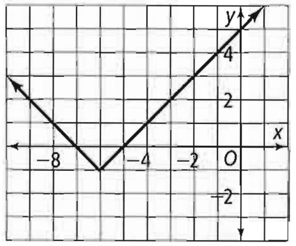
13.

x	y
−7	2
−6	1
−5	0
−4	1
−3	2

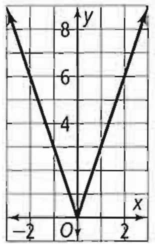


15.

x	y
−8	1
−7	0
−6	−1
−5	0
−4	1

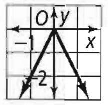


17.



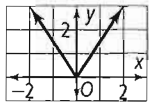
vertically stretched by a factor of 3

19.



vertically stretched by a factor of 2 and reflected across the x-axis

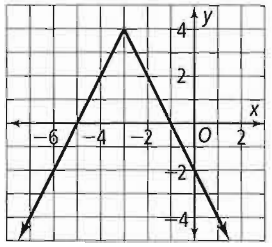
21.



vertically stretched by a factor of $\frac{3}{2}$

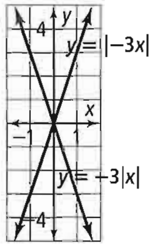
23. (−2, −4); $x = -2$; translate 2 units to the left and 4 units down **25.** (−6, 0); $x = -6$; translate 6 units to the left and vertically stretch by a factor of 3 **27.** (5, 0); $x = 5$; translate 5 units to the rt. and reflect across the x-axis **29.** $y = -2|x - 5| + 1$

31.



(−5, 0), (−1, 0), (0, −2)

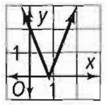
33.



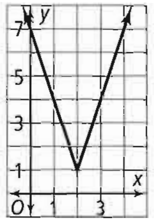
The graphs are not identical; one is the reflection across the x-axis of the other.

35. a. Answers may vary. Sample: reflection across the x-axis, vert. compression by a factor of $\frac{1}{2}$, translation down $\frac{1}{2}$ unit, translation rt. 6 units **b.** No; changing the order of the transformations can change the graph.

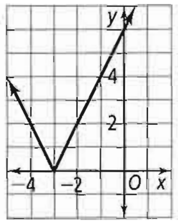
37.



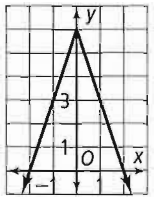
39.



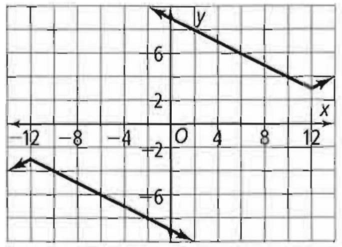
41.



43.

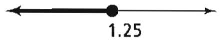


45. a.



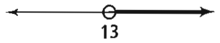
b. Answers may vary. Sample: same shape and size, different vertices, one opens down and one opens up. **55.** $y = x + 1$ **56.** $y = -\frac{1}{2}x + 2$ **57.** $g(x) = -x - 7$ **58.** $g(x) = -2x - 6$ **59.** $g(x) = -4 - x$ **60.** Answers may vary. Sample: $y = \frac{4}{5}x + 1$ **61.** Answers may vary. Sample: $y = -\frac{4}{5}x + 8$

62.



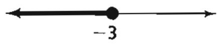
$p \leq 1.25$

63.



$t > 13$

64.

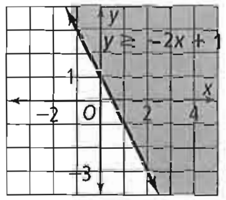


$t \le -3$

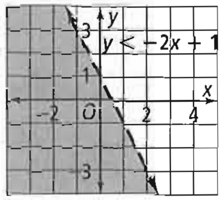
Lesson 2-8 pp. 114–120

Got It?

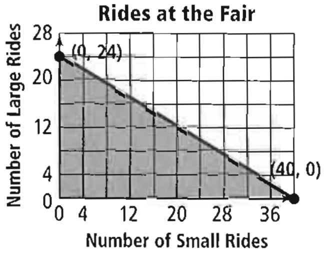
1. a.



b.

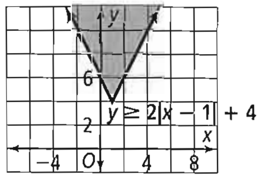


2. a.



b. The number of rides cannot be neg.

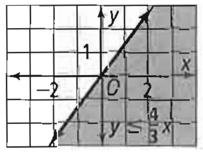
3.



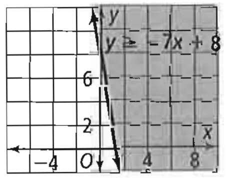
4. a. $y > -|x + 4| + 3$ **b.** No; when you solve the ineq. for y, you multiply both sides by −1. This changes the ineq. sign from > to <.

Lesson Check

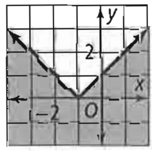
1.



2.



3.



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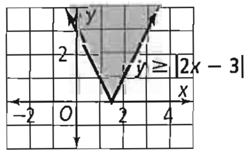
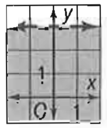


5. No; the boundary consists of points where *equality* holds, while the shaded region consists of points where the *inequality* holds. **6.** Graphing a linear inequality in two variables includes first graphing the boundary line, which is a linear eq. in two variables, and then shading the half-plane. **7.** No; $\left(\frac{3}{4}, 0\right)$ does not satisfy the inequality because 2.25 > 3 is false.

Exercises

9.



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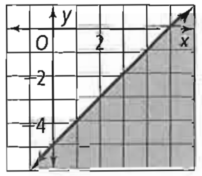

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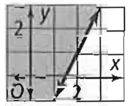

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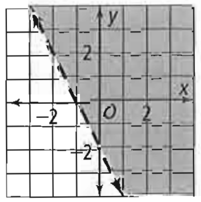
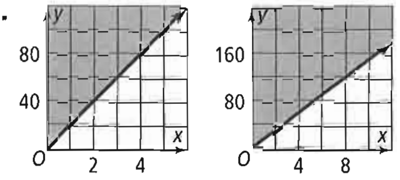


17. a. $y \ge 20x$ if $x \le 6$; $y \ge 15x$ if $x > 6$

b.



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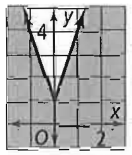

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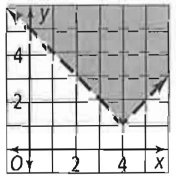

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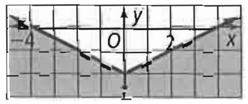

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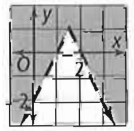
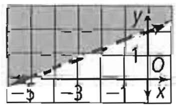


27. $y < -x - 2$ **29.** $2y \ge |2x + 6|$

31.



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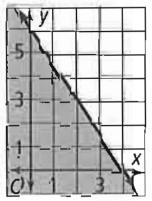

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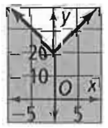

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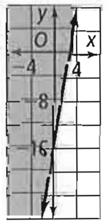
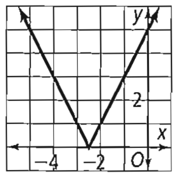
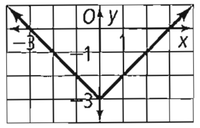


39. $x > -3$ **41.** $y \ge -2x + 4$
43. $y < -|x - 4|$ **45.** C

56.



57.



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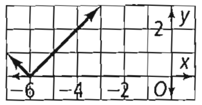
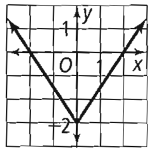


59.



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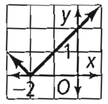
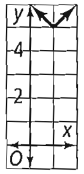
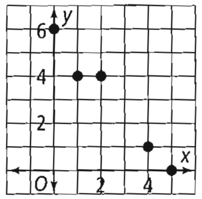


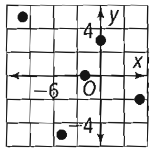
61.



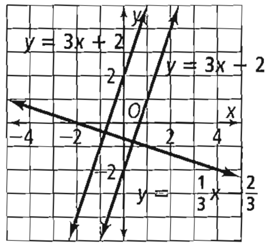
62. no **63.** yes; 100 **64.** yes; −5 **65.** no
66. yes; 3 **67.** no **68.** yes; −10 **69.** no
70. strong neg. correlation



71. no correlation



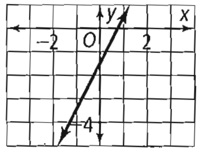
72–74.



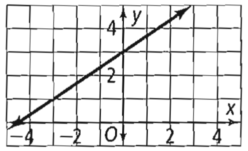
Chapter Review pp. 122–126

1. sometimes **2.** point-slope **3.** yes; domain: $\{-10, -6, 5, 6, 10\}$, range: $\{2, 3, 4, 7\}$ **4.** no; domain: $\{1, 3, 4, 10\}$, range: $\{5, 6, 8, 12\}$ **5.** no; domain: $\left\{-2, -\frac{3}{2}, -1, \frac{1}{2}, 1, 2, 3\right\}$, range: $\left\{-\frac{7}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}\right\}$
6. yes; domain: $\{-2, -1, \frac{1}{2}, 3\}$, range: $\{2\}$ **7.** 6, 4.5, 1
8. $-3\frac{3}{4}, -3\frac{3}{16}, -1\frac{7}{8}$ **9.** no **10.** no **11.** yes; 1; $y = x$
12. −4; 1.2 **13.** $\frac{10}{3}$; −1 **14.** $\frac{7}{2}$; $-1\frac{1}{20}$ **15.** $-\frac{4}{3}$; 0.4
16. $-\frac{2}{5}$ **17.** $\frac{7}{6}$ **18.** $\frac{2}{3}$ **19.** $-\frac{4}{9}$ **20.** $y = -3x + 4$
21. $y = \frac{1}{2}x + 6$

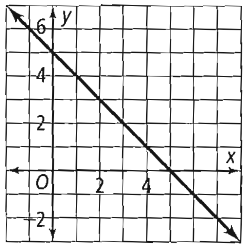
22. $y = 2x - \frac{3}{2}$



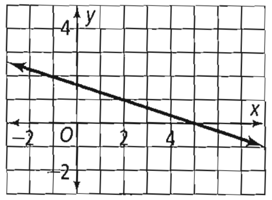
23. $y = \frac{2}{3}x + 3$



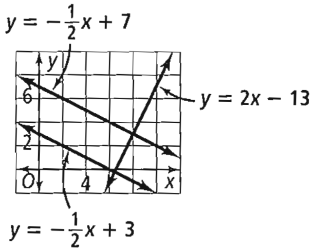
24. $y = -x + 5$



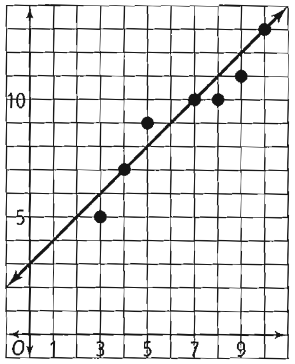
25. $y = -\frac{1}{3}x + \frac{5}{3}$



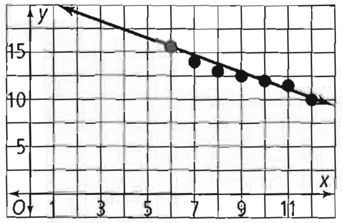
26. $y = -3(x - 4)$; $3x + y = 12$
27. $y + 1 = 5(x - 1)$; $5x - y = 6$
28. $y + 7 = -\frac{7}{3}(x - 3)$; $7x + 3y = 0$
29. $y - 3 = 2(x - 2)$; $2x - y = 1$
30. a. $y = -\frac{1}{2}x + 7$ **b.** $y = 2x - 13$
c.



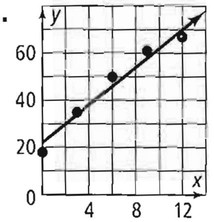
31.



strong pos. correlation; Answers may vary. Sample: $y = x + 3$; 18

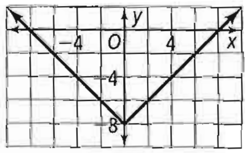
32. 

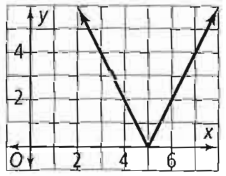
strong neg. correlation; Answers may vary. Sample: $y = -0.9x + 21$; 7.5

33. 

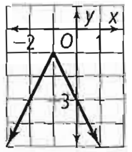
strong pos. correlation; Answers may vary. Sample: $y = 4x + 22$; 82

34. $y = f(x + 2) - 7$ **35.** $y = -f(x - 5)$
36. $y = f(-x) + 3$ **37.** translated 4 units down
38. vertically stretched by a factor of 12, translated 2 units up **39.** vertically stretched by a factor of 2, reflected across the *y*-axis, reflected across the *x*-axis
40. $y = |x - 2| + 4$ **41.** $y = |x + 3|$
42. $y = |x - 5| + 2$ **43.** $y = |x - 4| + 1$

44. 

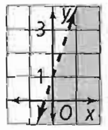
45. 

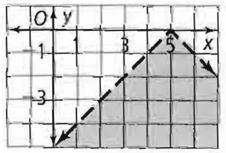
46. 

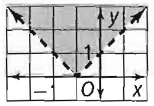
47. 

48. (4, 0); $x = 4$ **49.** (0, 2); $x = 0$

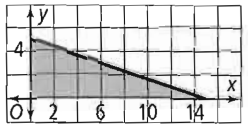
50. 

51. 

52. 

53. 

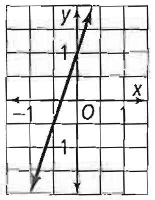
54. a. Answers may vary. Sample: $x + 3y \leq 15$
b. Answers may vary. Sample: domain: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}, range: {0, 1, 2, 3, 4, 5}
c.

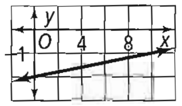


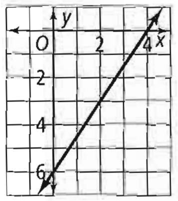
55. Answers may vary. Sample: $y \leq -|x| - 1$

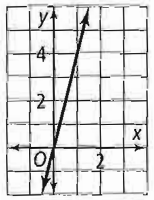
Chapter 3

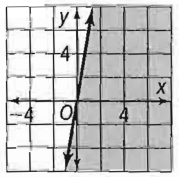
Get Ready! p. 131 **1.** 28 **2.** 33 **3.** $-\frac{15}{2}$ **4.** 15
5. $y = \frac{1}{2}x - \frac{5}{2}$ **6.** $y = -2x - 3$ **7.** $y = 5x + 16$
8. $y = 3x - 7$ **9.** $y = \frac{2}{5}x - \frac{3}{2}$ **10.** $y = 4x + 11$
11. $y = -6x - 8$ **12.** $y = -3x + 18$

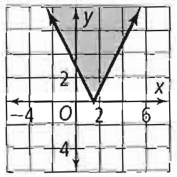
13. 

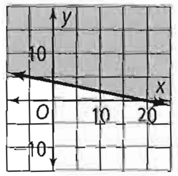
14. 

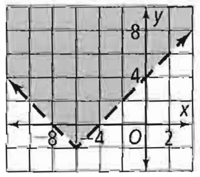
15. 

16. 

17. 

18. 

19. 

20. 

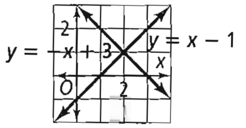
21. Answers may vary. Samples: Rocky Mountains, Appalachian Mountains **22.** Answers may vary. Sample: Your actions are consistent with your words when your actions show what you are saying. For example, if you say you are happy and you are laughing or smiling. **23.** 15 books or more; 15 books or more but less than 19, i.e., 15, 16, 17, or 18 books

Lesson 3-1 pp. 134–141

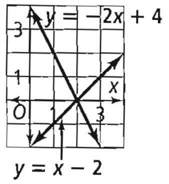
Got It? 1. $(2, -1)$ **2. a.** Spiny Dogfish: 59.5 cm; Greenland: 55.75 cm **b.** Each species of shark has a maximum total length; growth rates decrease with increase in age. **3.** in the yr 1990; about 1,100,000 **4. a.** inconsistent **b.** independent **c.** dependent

Lesson Check

1. $(2, 1)$



2. $(2, 0)$



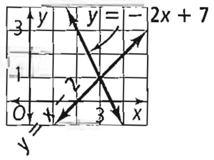
3. 2 pens; 4 pencils **4.** No; an independent system has a unique solution whereas an inconsistent system has no solution.

5. Answers may vary. Sample: $\begin{cases} y = 2x + 1 \\ y = 2x - 3 \end{cases}$

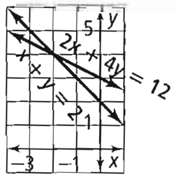
6. Independent; if the slope of one equation is the negative reciprocal of the slope of the other equation, the lines are perpendicular and intersect at a unique point.

Exercises 7–11. How solutions are determined may vary (graphing or using a table).

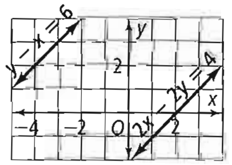
7. $(3, 1)$



9. $(-2, 4)$



11. no solution



13. 2 small; 4 large

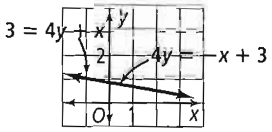
15. Models may vary. Sample: Use 0 for 1970.

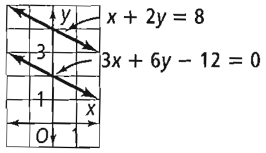
$\begin{cases} y = 0.22x + 67.5 \\ y = 0.15x + 75.507 \end{cases}$

Around 2085, the quantities will be equal.

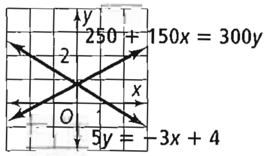
17. dependent **19.** inconsistent **21.** independent **23.** dependent **25.** independent **27.** dependent

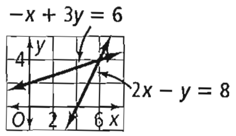
29. infinitely many solutions **31.** no solution



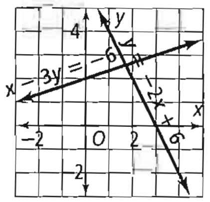


33. $\left(-\frac{1}{33}, \frac{9}{11}\right)$ **35.** $(6, 4)$



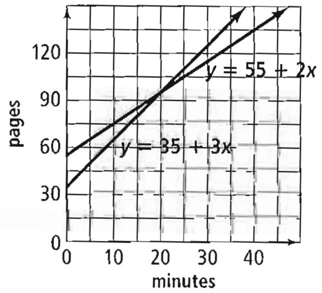


37. $\left(\frac{12}{7}, \frac{18}{7}\right)$



39. dependent

41.

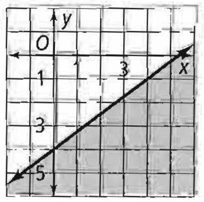


After 20 min you and your friend will have read the same number of pages.

43. My friend did not extend the table of values far enough. Scrolling down will show that when $x = 4$, then $Y1 = Y2 = -2$. So, the system has a solution, $(4, -2)$.

45. An independent system has one solution. The slopes are different, but the y-intercepts could be the same. An inconsistent system has no solution. The slopes are the same and the y-intercepts are different. A dependent system has an infinite number of solutions. The slopes and y-intercepts are the same. **47.** sometimes **49.** never

59.



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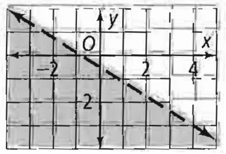
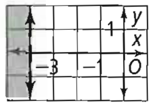


61.



62. $n < -\frac{8}{7}$ **63.** $x \geq -\frac{29}{2}$ or −14.5 **64.** $x > \frac{5}{8}$
65. 2 **66.** $-\frac{3}{5}$ **67.** 2 **68.** 10 **69. a.** −3 **b.** 8 **c.** −10

Lesson 3-2 pp. 142–148

Got It? 1. (−2.5, 2.5) **2.** \$.95 per download; \$5.50 one-time registration fee **3.** (4, 0) **4. a.** (−2, 3) **b.** Yes; the solution (−5, 2) is a solution to both eqs. in the system, so substituting $y = 2$ into either equation will result in $x = -5$. **5. a.** no solution; The eq. is always false. **b.** infinite number of solutions; The eq. is always true.

Lesson Check 1. (1, 2) **2.** (−6, −6) **3.** (2, 1)
4. (5, −3) **5.** $\left(-\frac{1}{5}, \frac{19}{5}\right)$ **6.** (2, 1)
7. Answers may vary. Sample:

$$\begin{cases} 4x - 3y = -2 \\ 3x - 2y = -1 \end{cases}$$

$$\begin{cases} -8x + 6y = 4 \\ 9x - 6y = -3 \end{cases}$$

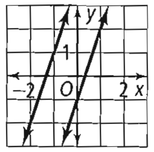
8. In the substitution method of solving a system of equations, you first solve one equation for one of the variables. Then substitute for this variable in the other equation and solve for the other variable. In the elimination method, you create an equivalent system of equations that contain a pair of additive inverses so that you can eliminate one variable and solve for the remaining variable.
9. Let r = number of regular cups of coffee and c = number of large cups of coffee. First, $r + c = 5$: because a total of 5 cups of coffee were purchased. Second, $r + 1.5c = 6$: because each regular cup of coffee is \$1, each large cup is \$1.50, and the total spent is \$6. Then, solve the system of equations using elimination by subtracting the first equation from the second to eliminate r and solve for c. $c = 2$; 2 large cups

Exercises 11. (−2, 4) **13.** (0.75, 2.5) **15.** (8, −1)
17. (−2, −5) **19.** seven \$1-bills; eight \$5-bills
21. 3 vans and 2 sedans **23.** (2, 4) **25.** (2, −2)
27. (4, 1) **29.** (1, 1) **31.** infinite number of solutions; $\{(x, y) | -2x + 3y = 13\}$ **33.** (3, 2) **35.** (5, 4)
37. $\left(\frac{20}{17}, \frac{19}{17}\right)$ **39.** (4, 1) **41.** no solution
43. 10 deliveries **45.** (4, −3) **47.** (−3, 4)
49. (300, 150) **51.** (0.5, 0.25)
53. Error in 5th line: $-4(-7 - x) = 28 + 4x$ not $-28 - 4x$; Lines 5–9 should be: $3x + 28 + 4x = 14$; $7x = -14$; $x = -2$; $y = -7 - (-2)$; $y = -5$
55. Answers may vary. Sample:

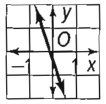
$$\begin{cases} -3x + 4y = 12 \\ 5x - 3y = 13 \end{cases}; (8, 9)$$

57. In determining whether to use substitution or elimination to solve an equation, look at the equations to determine if one is solved or can be easily solved for a particular variable. If that is the case, substitution can easily be used. Otherwise, elimination might be easier.
59. Substitution; the second equation is solved for y; (−7, −26) **61.** Elimination; substitution would be difficult since no coefficient is 1 in the original system. Dividing the first equation by 3 and dividing the second equation by 5 results in an equivalent system where y would be eliminated from the system if the equations were subtracted; (−1, −3)

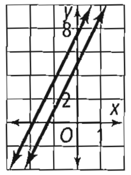
72. no solution



73. infinite number of solutions, $\{(x, y) | -9x - 3y = 1\}$

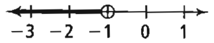


74. no solution

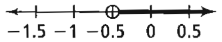


75. function **76.** function **77.** not a function

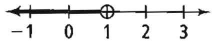
78. $x < -1$



79. $x > -\frac{1}{2}$



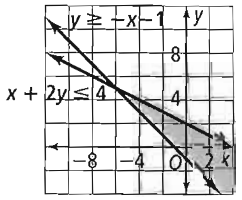
80. $y < 1$



Lesson 3-3 pp. 149–155

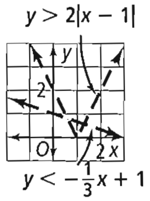
Got It? 1. (4, 1), (5, 0), (6, 0), (7, 0)

2.



3. 5 meats and no vegetables; 4 meats and 1 or 2 vegetables; 3 meats and 2, 3, or 4 vegetables; 2 meats and 3, 4, 5, or 6 vegetables; 1 meat and 4 − 8 vegetables; no meat 5 − 10 vegetables

4.



Lesson Check

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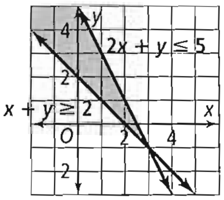

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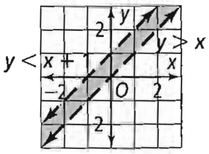

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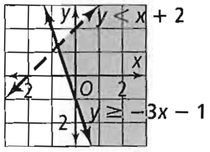
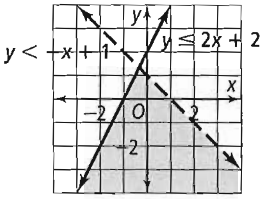


4. 0 h TV and 1, 2, or 3 h football; 1 h TV and 1 or 2 h football; or 2 h TV and 1 h football **5.** Intersection; the solution of two inequalities is the overlap or the intersection of the graphs of the individual inequalities. **6.** The graphical solution of a system of inequalities consists of the overlap or intersection of the individual half-planes and corresponding boundary lines (either dotted or solid). The graphical solution of a system of equations includes only the intersection of the lines, not the half-planes. **7.** For each inequality, the wrong half-plane has been shaded. The half-plane below $y = \frac{1}{2}x - 1$ should be shaded and the half-plane above $y = -3x + 3$ should be shaded. Also, both boundary lines should be dashed because the inequalities are < and >.

Exercises 9. (0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (1,5), (1, 6), (2, 0), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4)

11.



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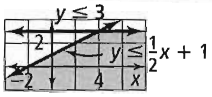

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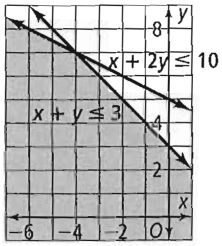

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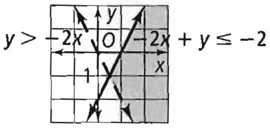
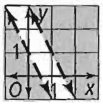


19. no solution

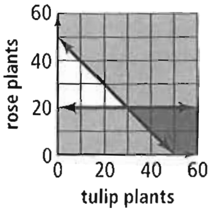


21. Let r = number of rose plants and t = number of tulip plants.

$t + r \geq 50$

$r \leq 80$

Because the number of plants must be a whole number, only the points in the overlap that represent whole numbers are solutions of the problem.



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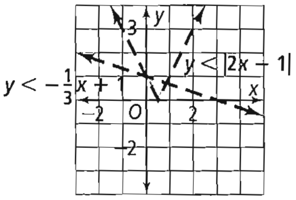

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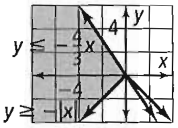

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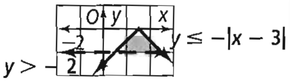

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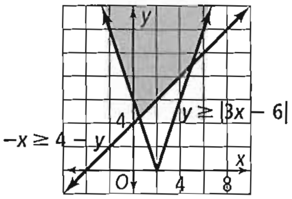
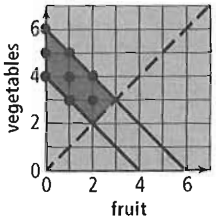
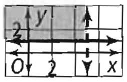


31. (0, 4), (0, 5), (0, 6), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4); the sum of the servings must be greater than or equal to 4 and less than or equal to 6.

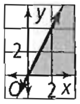


33. Answers may vary. Sample: $\begin{cases} x < 5 \\ y \geq 1 \end{cases}$



35. Use test pts. that are not on either of the boundary lines and that make the calculations as easy as possible (e.g., the origin). **37.** A, B **39.** A, B **41.** B, C **43.** A **45.** A

47.



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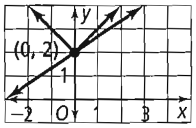

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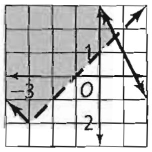

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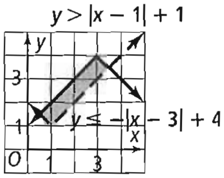


63. (−9, −26) **64.** $\left(\frac{23}{14}, -\frac{13}{14}\right)$ **65.** no solution

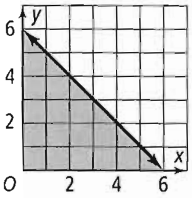
66. $(-2, -1)$ **67.** $(-1, 2)$ **68.** $\left(-\frac{4}{7}, \frac{1}{14}\right)$
69–72. Answers may vary. Samples are given for each exercise. **69.** $(0, 3)$ **70.** $(0, 3)$ **71.** $(2, -1)$ **72.** $(1, -1)$

Lesson 3-4 pp. 157–162

Got It? 1. a. P has a maximum value of 7.5 at $(0, 2.5)$. **b.** Yes; Answers may vary. Sample: $P = 5$ has the same (maximum) value at all four vertex points. $P = x + 2y$ has maximum value 5 at R and S. **2.** 100 T-shirts and 10 sweatshirts.

Lesson Check

1.



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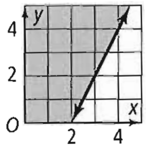

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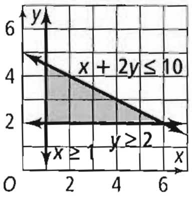

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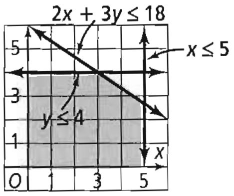
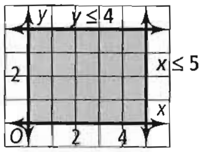
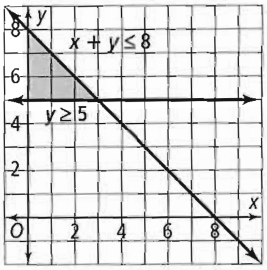


5. $(0, 0), (0, 4), (5, 0), (5, 4)$



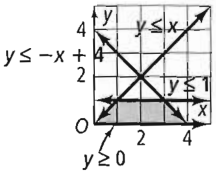
6. $(0, 8), (3, 5), (0, 5)$



7. Constraints are limits or restrictions on the variables in the objective function in a linear programming problem. These constraints are written as linear inequalities.
8. Linear programming is an extension of solving linear inequalities. For each, you are given constraints represented by linear inequalities that are graphed. All the points in the overlapping region are solutions, but linear programming problems are usually looking for maximum or minimum values of some quantity modeled with an objective function.

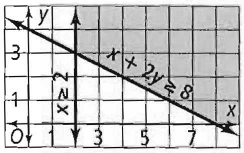
9. Answers may vary. Sample: $\begin{cases} y \le x \\ y \le -x + 4 \\ 0 \le y \le 1 \end{cases}$

$P = 2x + 3y$, $P(0, 0) = 0$, $P(1, 1) = 5$, $P(3, 1) = 9$, $P(4, 0) = 8$; maximum value of P is 9 at $(3, 1)$



Exercises

11.



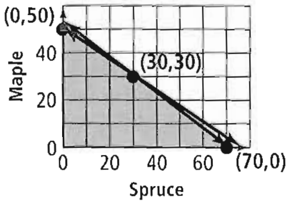
vertices: $(8, 0)$, $(2, 3)$; minimized at $(8, 0)$

13. Let s = number of Spruce trees and m = number of Maple trees.

a. $\begin{cases} 30s + 40m \le 2100 \\ 600s + 900m \le 45{,}000 \\ s \ge 0, m \ge 0 \end{cases}$

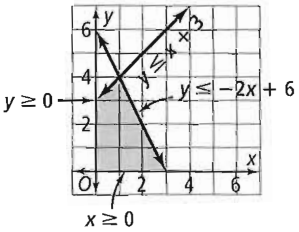
b. $P = 650s + 300m$

c.

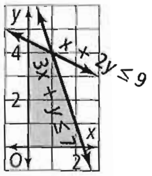


vertices: $(0, 0)$, $(0, 50)$, $(30, 30)$, $(70, 0)$

d. 70 spruce trees and 0 maple trees
15. He is not considering the constraint $y \le x + 3$; maximize when $P = 11$ at $(1, 4)$

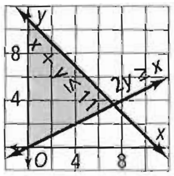


17.



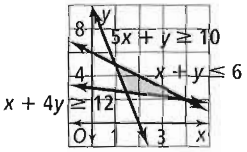
vertices: $(0, 0)$, $(1, 4)$, $(0, 4.5)$, $\left(\frac{7}{3}, 0\right)$; maximized when $P = 6$ at $(1, 4)$

19.



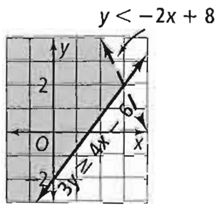
vertices: $(0, 0)$, $\left(7\frac{1}{3}, 3\frac{2}{3}\right)$, $(0, 11)$; maximized when $P = 29\frac{1}{3}$ at $\left(7\frac{1}{3}, 3\frac{2}{3}\right)$.

21.



vertices: $\left(\frac{28}{19}, \frac{50}{19}\right)$, $(4, 2)$ $(1,5)$; minimized when $C = 67{,}370$ at $\left(\frac{28}{19}, \frac{50}{19}\right)$

28.



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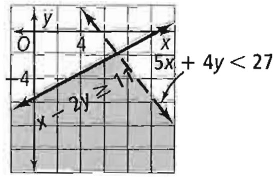
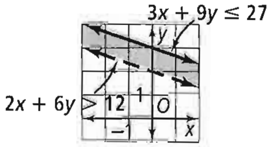


30.



31. 1 **32.** −34 **33.** 24 **34.** 65 **35.** (0, 6), (−3, 0) **36.** (0, 4), (18, 0) **37.** (0, −1), (1, 0)

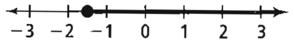
Lesson 3-5 pp. 166–173

Got It? 1. (4, 2, −3) **2. a.** (4, −1, 2) **b.** Answers may vary. Sample: Yes; you can choose to eliminate either *x*, *y*, or z resulting in a system of equations in 2 variables. **3. a.** (2, 1, −4) **b.** No; in Step 1 we solved for *x* in terms of *y* only. Therefore, once we found the value of *y* we could have substituted that value into the equation we wrote in Step 1 and solved for *x* without ever finding the *z*-value. **4.** 50 T-shirts, 50 polo shirts, and 100 rugby shirts

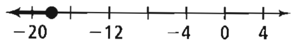
Lesson Check 1. (5, −3, −2) **2.** (6, 0, −2) **3.** (0, 3, 4) **4.** (2, 1, −5) **5.** Answers may vary. Sample: Substitution is the best method to use when one of the equations can be solved easily for one variable. **6.** Answers may vary. Sample: (0, 0, 0) is a unique solution to a system of three variables. The planes intersect at one common point. When a system has no solution, no point lies in all three planes. **7.** No solution since no point lies in all three planes. The three planes are parallel. **8.** infinitely many solutions

Exercises 9. (4, 2, −3) **11.** (2, 1, −5) **13.** $\left(\frac{1}{2}, -3, 1\right)$ **15.** (1, −4, 3) **17.** (4, −1, 2) **19.** $\left(-\frac{10}{13}, -\frac{2}{13}, \frac{4}{13}\right)$ **21.** (8, −4, 2) **23.** (−2, −1, −3) **25.** (0, 1, 7) **27.** (5, −2, 0) **29.** (1, 3, 2) **31.** $m\angle P = 32°$; $m\angle Q = 96°$; $m\angle R = 52°$ **33.** (8, 1, 3) **35.** $\left(\frac{1}{2}, 2, -3\right)$ **37.** no solution **39.** (2, 4, 6) **41.** (0, 2, −3) **50.** $P = 12$ is maximized at (0, 4).

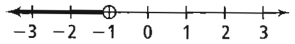
51. $x \geq -\frac{3}{2}$;



52. $x \leq -18$;



53. $x < -1$;



54. $\left(7, \frac{5}{4}\right)$

55. dependent system; infinite number of solutions, $\left\{(x, y) \mid y = -\frac{1}{2}x - \frac{3}{4}\right\}$

56. inconsistent system; no solution

Lesson 3-6 pp. 174–181

Got It? 1. 17

2. a. $\left[\begin{array}{cc|c} -4 & -2 & 7 \\ 3 & 1 & -5 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 4 & -1 & 2 & 1 \\ 0 & 1 & 5 & 20 \\ 2 & 1 & 0 & 7 \end{array}\right]$

3. $\begin{cases} 2x = 6 \\ 5x - 2y = 1 \end{cases}$

4. a. (1, 2) **b.** elimination; you use the same steps to solve **5.** $\left(1, \frac{1}{2}, 3\right)$

Lesson Check 1. 2 × 1 **2.** 2 × 4

3. $\left[\begin{array}{cc|c} 3 & 5 & 0 \\ 1 & 1 & 2 \end{array}\right]$ **4.** $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 1 & 0 & 2 & 8 \\ 0 & 2 & -1 & 1 \end{array}\right]$

5. 16 **6.** a_{21} is 0, the element in row 2, column 1. a_{12} is −9, the element in row 1 and column 2. **7.** Answers may vary. Sample: The entry fee to a school play is $2 for adults. Jamie paid a total of $8 for 4 student entry fees and 2 adult entry fees. What is the student entry fee?

Exercises 9. 1 **11.** 8

13. $\left[\begin{array}{cc|c} 3 & 2 & 16 \\ 0 & 1 & 5 \end{array}\right]$ **15.** $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 150 \\ 2 & 0 & 1 & 425 \\ 0 & 1 & 3 & 0 \end{array}\right]$

17. $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & -2 & -1 & 5 \\ 2 & -1 & 2 & 8 \end{array}\right]$ **19.** $\begin{cases} 5x + y = -3 \\ -2x + 2y = 4 \end{cases}$

21. $\begin{cases} 2x + y + z = 1 \\ x + y + z = 2 \\ x - y + z = -2 \end{cases}$ **23.** $\begin{cases} 5x + 2y + z = 5 \\ 4x + y + 2z = 8 \\ x + 3y - 6z = 2 \end{cases}$

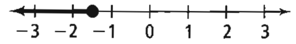
25. (−1, 0) **27.** (4, 6) **29.** (2, 3) **31.** $10,000 at 4% and $15,000 at 6%; Let x = amount invested at 4% and y = amount invested at 6%.

$$\begin{cases} x + y = 25{,}000 \\ 0.04x + 0.06y = 1300 \end{cases}$$

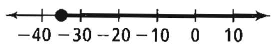
$$\left[\begin{array}{cc|c} 1 & 1 & 25000 \\ 0.04 & 0.06 & 1300 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 10000 \\ 0 & 1 & 15000 \end{array}\right]$$

33. (3, 1,1) **35.** (35, −22, −16) **37.** (1, 1, 1, 1) **39.** (2, 3) **41.** 1 qt. of red paint: $7.75; 1 qt. of yellow paint: $5.75

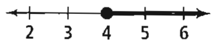
51. $x \leq -\frac{3}{2}$;



52. $x \geq -35$;



53. $x \geq 4$;



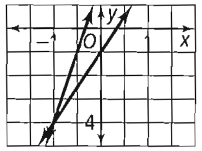
54. $\frac{15}{2}, -\frac{9}{2}$ **55.** 10, −10 **56.** 10, −6 **57.** $y = 2x$

58. $y = \frac{1}{3}x$

Chapter Review pp. 183–186

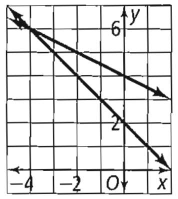
1. independent system **2.** Linear programming; constraints

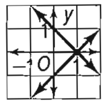
3. independent; (−1, −4)



4. dependent **5.** inconsistent **6.** dependent

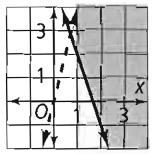
7. independent; (−4, 6) **8.** independent; (1, 0)



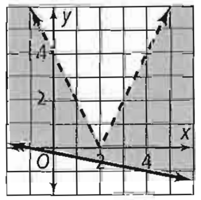


9. 3 pens **10.** (−1, −2) **11.** (0, −5) **12.** (−2, 3)
13. inconsistent; no solution **14.** 1 serving of roast beef and 2 servings of mashed potatoes

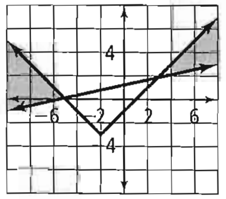
15.



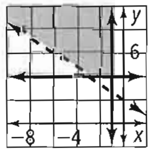
16.



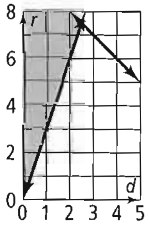
17.



18.



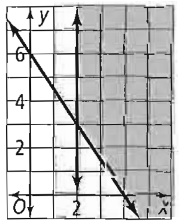
19.



Let r = amount of regular coffee and d = amount of decaffeinated coffee

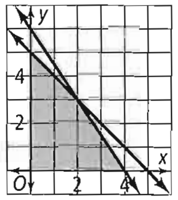
$$\begin{cases} r + d \le 10 \\ r \ge 3d, r \ge 0, d \ge 0 \end{cases}$$

20.



vertices: (4, 0) and (2, 3); $C = 4$ is minimized at (4, 0).

21.



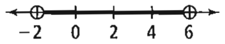
vertices: (0, 0), (4, 0), (2, 3), (0, 5); $P = 25$ is maximized at (0, 5).

22. 50 chef's salads and 50 Caesar salads **23.** (1, 3, −2)
24. (−4, 1, −5) **25.** (6, 0, −2) **26.** no solution
27. $\left(\frac{1}{2}, \frac{1}{4}\right)$ **28.** (1, −1) **29.** (2, −4, 6) **30.** (5, 2, −3)

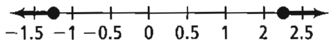
Chapter 4

Get Ready! p. 191 **1.** 6 **2.** 4

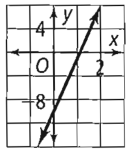
3. $-2 < x < 6$

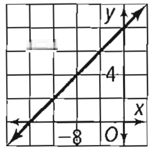


4. $y \le -\frac{5}{4}$ or $y \ge \frac{9}{4}$



5. $y = 9x - 10$ **6.** $y = \frac{1}{2}x + 8$



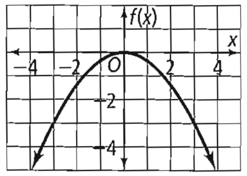


7. translated 4 units to the right and 2 units up
8. translated 10 units to the left and 3 units down
9. (10, −1) **10.** (−6, −6) **11.** Answers may vary. Sample: application forms, registration forms, tests
12. Answers may vary. Sample: monsters, ghosts, tooth fairy **13.** writing

Lesson 4-1 pp. 194–201

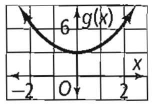
Got It?

1. a.

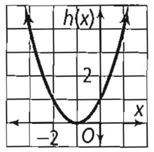


b. If a is a negative number, the parabola will open downward. There will be a maximum value for y at the vertex of the parabola.

2. a. translated 3 units up



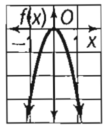
b. translated 1 unit to the left



3. vertex: (−1, 4); axis of symmetry: $x = -1$; maximum: 4; domain: all real numbers, range: $y \le 4$ **4.** stretch by the factor 2, translate 2 units to the left and 5 units down.
5. $f(x) = -\frac{2}{9}(x - 2)^2 + 7$

Lesson Check

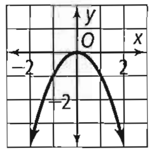
1.



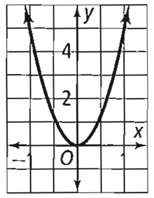
2. minimum **3.** $y = -2(x - 0)^2 + 35$ **4.** when $a > 0$ **5.** No; a must be > 0 or < 0. **6.** $y = (x + 6)^2$ is the graph of $y = x^2$ translated 6 units to the left and has a minimum at $(-6, 0)$; $y = (x - 6)^2 + 7$ is the graph of $y = x^2$ translated 6 units to the right and 7 units up, with a minimum at $(6, 7)$.

Exercises

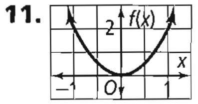
7.



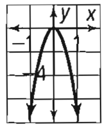
9.



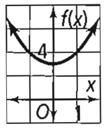
11.



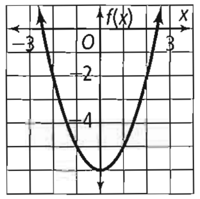
13.



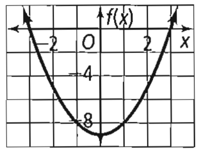
15. translated 3 units up



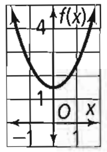
17. translated 6 units down



19. translated 9 units down

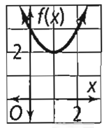


21. translated 1.5 units up

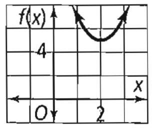


23. vertex: $(-20, 0)$; axis of symmetry: $x = -20$; maximum: 0; domain: all real numbers, range: $y \leq 0$ **25.** vertex: $(-5.5, 0)$; axis of symmetry: $x = -5.5$; minimum: 0; domain: all real numbers, range: $y \geq 0$ **27.** vertex: $(4, -25)$; axis of symmetry: $x = 4$; maximum: -25; domain: all real numbers, range: $y \leq -25$

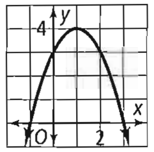
29. $x = 1$



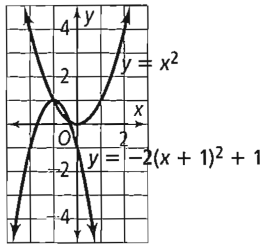
31. $x = 2$



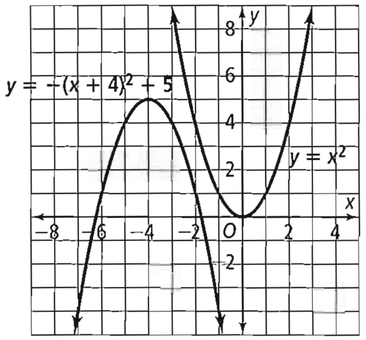
33. $x = 1$



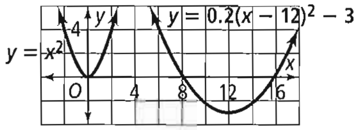
35. $y = (x - 2)^2 + 5$ **37.** $y = -\frac{1}{2}(x + 4)^2 + 2$ **39.** 1000 chips; \$20 **41.** stretch vertically by a factor of 2, reflect across the x-axis, translate 1 unit to the left and 1 unit up



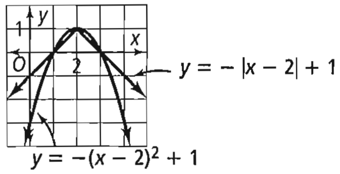
43. reflect across the x-axis, translate 4 units to the left and 5 units up



45. stretch vertically by a factor of 0.2, translate 12 units to the right and 3 units down

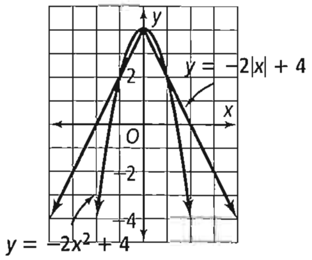


47. parabola $y = a(x - 3)^2 + 4$ with vertex $(3, 4)$ and $a > 0$ or $a < 0$ **49.** $y = -7(x - 1)^2 + 2$ **51.** $y = -7x^2 + 5$ **53.** similar: same vertex $(2, 1)$ and open downward, same domain (all real numbers), same range $(y \leq 1)$, same x-intercepts, $(1, 0)$, $(3, 0)$; different: $y = -|x - 2| + 1$ is an absolute value function with y-intercept $(0, -1)$ and $y = -(x - 2)^2 + 1$ is a quadratic function with y-intercept of $(0, -3)$.



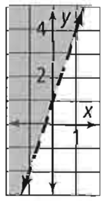
55. similar: same vertex $(0, 4)$ and open downward, same domain (all real numbers), same range $(y \leq 4)$, same y-intercept, $(0, 4)$; different: $y = -2|x| + 4$ is an absolute value function with x-intercepts $(\pm 2, 0)$ and

$y = -2(x)^2 + 4$ is a quadratic function with x-intercepts $(\pm\sqrt{2}, 0)$

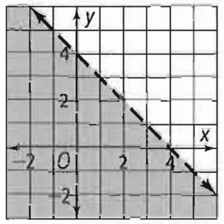


57. Answers may vary. Sample: $y = (x + 10)^2 - 4$
74. (3, 2) **75.** (−10, 6) **76.** (1, 0, 3)

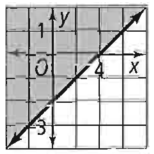
77.



78.



79.

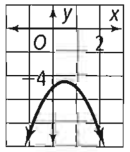


80. yes **81.** no **82.** no **83.** (0, 0) **84.** (−1, 0) **85.** (5, 0)

Lesson 4-2 pp. 202–208

Got It? 1. vertex: $\left(-\frac{2}{3}, 7\frac{1}{3}\right)$; axis of symmetry: $x = -\frac{2}{3}$; maximum: $7\frac{1}{3}$; range: $y \leq 7\frac{1}{3}$

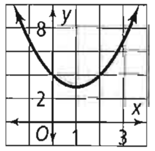
2.



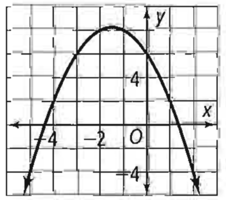
3. $y = -(x - 2)^2 - 1$ **4. a.** 4 ft **b.** because the y-intercept is (0, 0)

Lesson Check 1. vertex: (0, −4); axis of symmetry: $x = 0$; minimum: −4

2.



3.



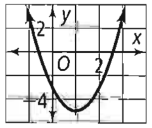
4. $y = (x - 1)^2 + 8$ **5.** $y = -\left(x - \frac{3}{2}\right)^2 + \frac{5}{4}$

6. Error in calculation of x. The correct calculation is:

$$x = \frac{-(-4)}{2(2)} = 1$$

$$\begin{aligned} y &= 2(1) - 4(1) - 3 \\ &= 2 - 4 - 3 \\ &= -5 \end{aligned}$$

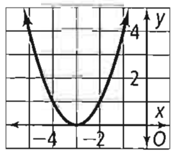
Vertex: (1, −5)



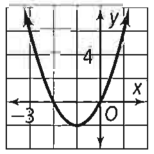
7. The vertex of a function written in vertex form can easily be determined. It is (h, k) where $f(x) = a(x - h)^2 + k$. The vertex of a function in standard form is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ where $f(x) = ax^2 + bx + c$.

Exercises 9. vertex: (1, 2); axis of symmetry: $x = 1$; maximum: 2; range: $y \leq 2$ **11.** vertex: (1, 6); axis of symmetry: $x = 1$; maximum: 6; range: $y \leq 6$ **13.** vertex: $\left(-\frac{3}{4}, 5\frac{1}{8}\right)$; axis of symmetry: $x = -\frac{3}{4}$; maximum: $5\frac{1}{8}$; range: $y \leq 5\frac{1}{8}$ **15.** vertex: $\left(-\frac{1}{2}, \frac{1}{4}\right)$; axis of symmetry: $x = -\frac{1}{2}$; maximum: $\frac{1}{4}$; range: $y \leq \frac{1}{4}$

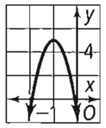
17.



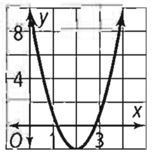
19.



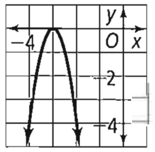
21.



23.

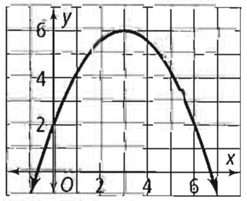


25.

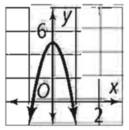


27. $y = (x + 1)^2 + 4$ **29.** $y = 2\left(x - \frac{5}{4}\right)^2 + \frac{71}{8}$ **31.** $y = \frac{9}{4}\left(x + \frac{2}{3}\right)^2 - 2$

33.



35.



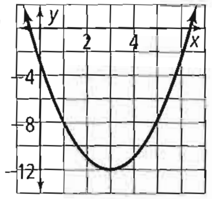
37. 4 cm × 9 cm × 9 cm **39.** $b = -6$; $c = 5$ **41.** $a = 1$; $c = -2$ **43.** 2 s; 64 ft **45.** (0, 3) **47.** (0, −54) **57.** $\frac{1}{4}$ **58.** −2.5 **59–61.** Explanations may vary. Samples are given. **59.** Elimination because the x-terms have opposite signs; (7, −1) **60.** Substitution because the first equation is already solved for y; (27, 15) **61.** Elimination because you can multiply the second equation by −3 to eliminate n; (3, 2) **62.** vertex: (−2, −1); axis of symmetry: $x = -2$; minimum: −1; domain: all real numbers, range: $y \geq -1$ **63.** vertex: (1, 3); axis of symmetry: $x = 1$; maximum: 3; domain: all real numbers, range: $y \leq 3$ **64.** vertex: (3, −2); axis of symmetry: $x = 3$; minimum: −2; domain: all real numbers, range: $y \geq -2$ **65.** vertex: (−3, 5); axis of symmetry: $x = -3$; maximum: 5; domain: all real numbers, range: $y \leq 5$ **66.** vertex: (−4, 0); axis of symmetry: $x = -4$; minimum: 0; domain:

all real numbers, range: $y \geq 0$ **67.** vertex: (4, 6); axis of symmetry: $x = 4$; maximum: 6; domain: all real numbers, range: $y \leq 6$ **68.** $y = -(x - 1)^2 + 5$ **69.** $y = x^2 - 4$ **70.** $y = (x + 1)^2 - 2$

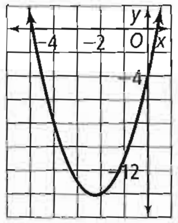
Lesson 4-3 pp. 209–214

Got It? 1. $y = -3x^2 + x$ **2. a.** No; the ball will only reach a height of 5 when $x = 5$, which is lower than the top of the wall at (5, 6). The ball will hit the wall on its way down. **b.** domain: $0 \leq x \leq 7$, range: $0 \leq y \leq 6\frac{1}{8}$ **3.** $y = -0.329x^2 + 9.798x + 15.571$; 88.5°F at 2:53 P.M. (although the meteorologist's prediction is 89° at 3 P.M.) **Lesson Check 1.** $y = -2x^2 + 3x - 1$ **2.** $y = 2x^2 + 6x + 7.5$ **3.** $y = -2x^2 + 10x - 13.5$ **4.** Answers may vary. Sample: A rough plot of the data will indicate whether the data are collinear (linear regression) or non-collinear where the data follows a curve (quadratic regression). **5.** Answers may vary. Sample: A rough plot will show that the four points do *not* lie on a single parabola. **6.** y is not a function of x since for one value of x, "3," there are 2 values of y, "4" and "0." **Exercises 7.** $y = -x^2 + 3x - 4$ **9.** $y = 2x^2 - x + 3$ **11.** $y = x^2 - 6x + 3$ **13.** $y = x^2 + 2x$ **15.** $y = -x^2 + x - 2$ **17. a.** $y = -16x^2 + 33x + 46$, where x is the number of seconds after release and y is the height in ft **b.** 28.5 ft **c.** about 63 ft **19.** yes; $y = -2x^2 + 3x + 5$ **21.** yes; $y = 0.625x^2 - 1.75x + 1$ **23.** $y = 0.005x^2 - 1.95x + 120$; 66 mm **25. a.** $y = -0.00357x^2 + 0.930x + 18.856$ **b.** Answers may vary. Sample: domain: integers from 0 to 27; range: whole numbers from 18 to 42 **c.** the year 2004 **d.** The year 2021; the year is outside the domain of the data pts. **27.** Answers may vary. Sample: $y = -\frac{1}{25}x^2$, $y = \frac{1}{25}x^2 - \frac{2}{5}x$, $y = \frac{1}{5}x^2 - \frac{6}{5}x$

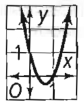
35.



36.



37.



38. (2, 5) **39.** (5, 8) **40.** (−1, −1) **41.** $\frac{4}{5}$ **42.** $-\frac{7}{2}$ **43.** $x^2 + 5x - 1$ **44.** $6x^2 - 10x - 3$ **45.** $4x^2 - x - 10$

Lesson 4-4 pp. 216–223

Got It? 1. a. $(x + 10)(x + 4)$ **b.** $(x - 5)(x - 6)$ **c.** $-(x + 2)(x - 16)$ **2. a.** $7(n^2 - 3)$ **b.** $9(x + 2)(x - 1)$ **c.** $4(x^2 + 2x + 3)$ **3. a.** $(x + 1)(4x + 3)$ **b.** $(x - 2)(2x - 3)$ **c.** No; $2x^2 + 2x + 2 = 2(x^2 + x + 1)$, there are no real factors of a and c whose product is 1 and whose sum is 1. **4.** $(8x - 1)^2$ **5.** $(4x - 9)(4x + 9)$

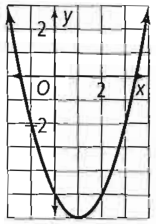
Lesson Check 1. $(x + 4)(x + 2)$ **2.** $(x - 12)(x - 1)$ **3.** $(x - 9)(x + 9)$ **4.** $(5y - 6)(5y + 6)$ **5.** $(y - 3)^2$ **6.** $(2x - 1)^2$ **7.** $5x$ **8.** $4a^2$ **9.** 6 **10.** $7h$ **11.** No; the middle term is not twice the product of the square root of the end terms. **12.** For $a \neq 1$, look for two factors whose sum is b and whose product is ac. For $a = 1$, look for two factors whose sum is b and whose product is c.

13. $a^2 - 2ab + b^2 - 25$ Group the first 3 terms.

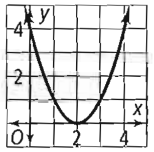
$$= \left(a^2 - 2ab + b^2\right) - 25$$
$$= (a - b)^2 - 5^2$$
$$= (a - b - 5)(a - b + 5)$$

Exercises 15. $(x + 2)(x + 3)$ **17.** $(x + 2)(x + 8)$ **19.** $(x + 2)(x + 20)$ **21.** $-(x - 1)(x - 12)$ **23.** $(x - 4)(x - 6)$ **25.** $(x - 4)(x - 9)$ **27.** $-(x - 4)(x + 5)$ **29.** $(c - 7)(c + 9)$ **31.** $-(t - 11)(t + 4)$ **33.** $5b$; $5b(5b - 4)$ **35.** 5; $5(t + 1)(t - 2)$ **37.** 9; $9(3p^2 - p + 2)$ **39.** $(x - 8)(2x - 3)$ **41.** $(m - 3)(2m - 5)$ **43.** $(x - 12)(2x - 3)$ **45.** $(y + 4)(5y - 8)$ **47.** $(z + 4)(2z - 7)$ **49.** $(4k + 3)(7k - 2)$ **51.** $(t - 7)^2$ **53.** $(2z - 5)^2$ **55.** $(9z + 2)^2$ **57.** $(c - 8)(c + 8)$ **59.** The rectangle would have one side equal to x and one side equal to $x + 2y$. **61.** $9(x + 2)(x - 2)$ **63.** $3(2y + 5)(2y - 5)$ **65.** $3(2x + 3)^2$ **67.** $2(a - 4)^2$ **69.** $2(3b + 5)(3b - 1)$ **71.** $3(y + 3)(y + 5)$ **73.** $2(x - 5)(2x - 1)$ **75.** $-6(z^2 + 100)$ **77.** $(x - y)(x + y)$ **79.** The third line should be $x(2x - 5) - (2x - 5)$, and the final line should be $(x - 1)(2x - 5)$. **81.** y; $y(y - 1)$ **83.** 10; $10(x - 3)(x + 3)$ **85.** 2; $2(x^2 - 37x + 6)$ **87.** D **106.** $y = -0.149x^2 + 5.171x + 16.971$ **107.** penny: 2.5g, nickel: 5g, dime: 2.3g

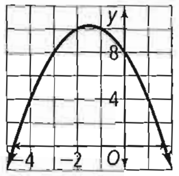
108.



109.



110.



Algebra Review p. 225 1. $3\sqrt{2}$ **3.** $-4\sqrt{2}$ **5.** $\frac{-\sqrt{91}}{13}$ **7.** $-10\sqrt{2}$ **9.** 108 **11.** $|xy|$ **13.** $-\frac{|x|\sqrt{35}}{5}$ **15.** $\frac{5\sqrt{14}}{7}$

Lesson 4-5 pp. 226–231

Got It? 1. 3, 4 **2.** 3, $\frac{1}{4}$ **3.** -6, 4 **4. a.** $53\frac{1}{3}$ m; $21\frac{1}{3}$ m; answers may vary. Sample: domain: $0 \leq x \leq 60$, range: $0 \leq y \leq 30$ **b.** No; domains and ranges are constrained by real-world limits.

Lesson Check 1. 3, -3 **2.** -4, -9 **3.** $-\frac{2}{3}$, 1 **4.** 4.372, -1.372 **5.** 2.608, -2.108 **6.** -1; since $y = 0$ when $x = 5$, substitute these values into the equation to find b. **7.** when the coefficients are integers and a recognizable pattern of factoring is evident **8.** One solution: when the table's range consists of zero and all positive numbers or zero and all negative numbers. No solution: when the table does not include zero and the y-values are either all positive or all negative numbers

Exercises 9. -4, -2 **11.** -1, $\frac{3}{2}$ **13.** -2, -1 **15.** 0, 4 **17.** 0, 4 **19.** 3, 8 **21.** -0.78, 1.28 **23.** 4, -10 **25.** 0.8, -1 **27.** -1.67, -1.5 **29.** -0.94, 2.34 **31.** -1, 0.25 **33.** -1.46, 5.46 **35.** -1.16, 2.16 **37.** 3 in. **39.** about 3.6 ft **41.** Answers may vary. Samples are given: **a.** $x^2 - 8x + 15 = 0$ **b.** $x^2 + x - 6 = 0$ **c.** $x^2 + 7x + 6 = 0$ **43.** $-\frac{3}{2}$, $-\frac{2}{3}$ **45.** -4, $\frac{5}{2}$ **47.** -1, 4 **49.** -4, 0 **51.** 4.37, -1.37 **53.** -1, $\frac{10}{3}$ **55.** $(0, -2)$, $(2, 2)$ **64.** $(4x - 1)(4x + 1)$ **65.** $(5x - 1)(x - 5)$ **66.** $(2x - 1)(x + 7)$ **67.** $(2, 0, -2)$ **68.** $(-2, 1, 5)$ **69.** $(7, 1, -1)$ **70.** vertex: $(-9, 4)$; axis of symmetry: $x = -9$; translation 9 units to the left and 4 units up **71.** vertex: $\left(\frac{7}{2}, 0\right)$; axis of symmetry: $x = \frac{7}{2}$; stretch by a factor of 2, translation $3\frac{1}{2}$ units to the right **72.** vertex: $(0, -1)$; axis of symmetry: $x = 0$; vert. compression by a factor of $\frac{3}{4}$, translation 1 unit down **73.** $x^2 + 8x + 13$ **74.** $4x^2 - 4x + 1$ **75.** $x^2 - 6x + 9$

Lesson 4-6 pp. 233–239

Got It? 1. a. $\sqrt{5}$, $-\sqrt{5}$ **b.** $\sqrt{2}$, $-\sqrt{2}$ **2.** 42 in. × 67.2 in. **3.** 2, 12 **4. a.** 9 **b.** No; $\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$, which is a function of b. **5.** $\frac{1}{2} \pm \frac{\sqrt{13}}{2}$ **6.** $y = \left(x + \frac{3}{2}\right)^2 - \frac{33}{4}$; vertex: $\left(-\frac{3}{2}, -\frac{33}{4}\right)$; y-intercept: $(0, -6)$

Lesson Check 1. 6, -6 **2.** 3, -3 **3.** 1 **4.** 25 **5.** 4 **6.** 36 **7.** 2500 **8.** 256 **9.** First, you rewrite the eq. to get all terms with x on one side. Then, you find $\left(\frac{b}{2}\right)^2$ and add it to both sides of the eq. Then, you factor the resulting trinomial.

10.

$x^2 + 12x + 5 = 3$	
$x^2 + 12x = -2$	Rewrite to get all terms with x on one side of the eq.
$\left(\frac{12}{2}\right)^2 = 6^2 = 36$	Find $\left(\frac{b}{2}\right)^2 = 36$.
$x^2 + 12x + 36 = -2 + 36$	Add 36 to each side.
$(x + 6)^2 = 34$	Factor the trinomial.

11. Your friend should also have subtracted 49;
$(x^2 - 14x + 49) + 36 - 49$
$= (x - 7)^2 - 13$

Exercises 13. 2, -2 **15.** $\frac{5}{3}$, $-\frac{5}{3}$ **17.** $2\sqrt{2}$, $-2\sqrt{2}$ **19.** -4, -2 **21.** -1, 3 **23.** -4, 3 **25.** $-\frac{4}{5}$, $\frac{2}{5}$ **27.** $-\frac{10}{3}$, $\frac{2}{3}$ **29.** $\frac{1}{4}$ **31.** 100 **33.** 4 **35.** $6 \pm \sqrt{29}$ **37.** $1 \pm \sqrt{6}$ **39.** $5 \pm \sqrt{13}$ **41.** $3 \pm \sqrt{11}$ **43.** $-\frac{5}{4} \pm \frac{1}{4}\sqrt{37}$ **45.** $-\frac{3}{5} \pm \frac{\sqrt{21}}{5}$ **47.** $y = 2(x - 2)^2 - 7$ **49.** $y = (x + 2)^2 - 11$ **51.** $y = -(x - 2)^2 + 3$ **53.** 10, -10 **55.** 22, -22 **57.** 18, -18 **59.** 1, -1 **61.** 84, -84 **63.** $\frac{-5 \pm \sqrt{37}}{2}$ **65.** $\frac{1 \pm \sqrt{21}}{2}$ **67.** $\frac{2 \pm \sqrt{10}}{3}$ **69.** $\frac{-3 \pm \sqrt{41}}{8}$ **71.** $\frac{1}{3}$, $-\frac{2}{3}$ **73.** $-3 \pm \sqrt{7}$ **75. a.** $-.01(x - 59)^2 + 36.81$; 36.81 ft **b.** 7.65 ft **c.** about 120 ft **90.** $\frac{1}{2}$, 1 **91.** -4, 1 **92.** 8, $-\frac{2}{3}$ **93.** yes; $y = \frac{1}{2}x^2 + \frac{7}{2}x + 9$ **94.** yes; $y = -\frac{1}{2}x^2 + x + 2$ **95.** yes; $y = 3x^2 - 5x + 2$ **96.** $(2, 0)$ **97.** $(3, 1)$ **98.** $(3, 1)$ **99.** 24 **100.** 84

Lesson 4-7 pp. 240–247

Got It? 1. a. -2 **b.** $-2 \pm \sqrt{7}$ **2. a.** \$10.74 **b.** Yes; a neg. profit means more money was spent than earned. **3. a.** no real solutions **b.** two real solutions **4.** Yes;
$b^2 - 4ac = (85)^2 - 4(-16)\left(-109\frac{11}{12}\right) = 190\frac{1}{3}$.
The discriminant is pos. So the eq. has two real solutions.

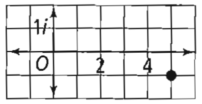
Lesson Check 1. $\frac{5 \pm \sqrt{53}}{2}$ **2.** $\frac{-3 \pm \sqrt{61}}{2}$ **3.** 3, $-\frac{1}{2}$ **4.** no real solutions **5.** -32; no real solutions **6.** 273; two real solutions **7.** 0; one real solution **8.** $k = \pm 6$ for one real solution; $k > 6$ or $k < -6$ for two real solutions **9.** Answers may vary. Sample: The discriminants of eqs. with one real solution are all zero and thus equal, but the solutions may or may not be equal. An example is $x^2 - 8x + 16$ and $x^2 - 4x + 4$. Each has a discriminant of zero, but the solutions are 4 and 2. **10.** Yes; the eqs. can share common factors such as for $x^2 + 2x - 8$ where the discriminant is 36 and the solutions are 2 and -4, and $x^2 - 4x + 4$ where the discriminant is zero and the solution is 2.

Exercises 11. 1, 3 **13.** $-\frac{7}{2}$, 1 **15.** -5 **17.** $\frac{3 \pm \sqrt{5}}{2}$ **19.** $\frac{2 \pm \sqrt{10}}{3}$ **21.** 1, 4 **23.** \$16.34 **25.** -4; no real solutions **27.** 0; one **29.** 169; two **31.** 1; two **33.** 0; one **35.** -23; no real solutions **37.** no

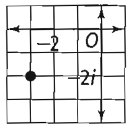
39. 2.29 in. × 15.71 in. **41.** $-\frac{1}{6}$, 1 **43.** −2.49, 0.89 **45.** −0.19, 2.69 **47.** 1, 10 **49.** $-\frac{3}{2}, \frac{1}{2}$ **51.** −1.70, 4.70 **53.** −8.47, 0.47 **55.** 1.47, −7.47 **57.** about 1.89 s **59.** one **61.** two **63.** two **65.** two **67. a.** k such that $|k| < 12$ **b.** 12 or −12 **c.** k such that $|k| > 12$ **69. a.** $x^2 = 100\pi$ **b.** 17.72 cm **82.** −2, 10 **83.** $\frac{2 \pm \sqrt{2}}{2}$ **84.** $\frac{3 \pm \sqrt{41}}{2}$ **85.** $9z^2 + 3z$ **86.** $4x + k$ **87.** $2y - 8x$ **88.** $2\sqrt{17}$ **89.** 5 **90.** 13

Lesson 4-8 pp. 248–255

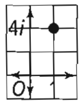
Got It? 1. a. $2i\sqrt{3}$ **b.** $5i$ **c.** $i\sqrt{7}$ **d.** $8i \neq -8$

2. a. 

; $\sqrt{26}$

b. 

; $\sqrt{13}$

c. 

; $\sqrt{17}$

3. a. $4 - i$ **b.** $-2 + 7i$ **c.** $12i$ **d.** $18i$ **4. a.** −21 **b.** $23 - 2i$ **c.** 41 **5. a.** $\frac{7}{25} - \frac{26}{25}i$ **b.** $-\frac{1}{6} - \frac{2}{3}i$ **c.** $\frac{15}{113} - \frac{112}{113}i$ **6. a.** $\pm 2i$ **b.** $\pm i\sqrt{15}$ **7. a.** $\frac{1 \pm i\sqrt{23}}{6}$ **b.** $2 \pm i$

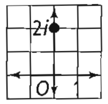
Lesson Check 1. $5i\sqrt{3}$ **2.** 5 **3.** $\pm 4i$ **4.** $7 - 3i$ **5.** $13 - 6i$ **6.** The add. inv. of a complex number, $a + bi$, is the opposite of the complex number, or $-a - bi$. The complex conjugate of a complex number, $a + bi$, is the real part plus the opposite of the imaginary part of the complex number, or $a - bi$. **7.** error in the sign of the last term of the first line, which carries through to the end of the calculation; the line should be: "...

$= 16 + 28i - 28i - 49i^2$

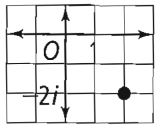
$= 16 + 49$

$= 65.$"

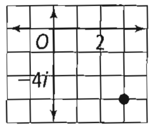
Exercises 9. $i\sqrt{7}$ **11.** $9i$

13. 

; 2

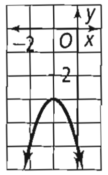
15. 

; $2\sqrt{2}$

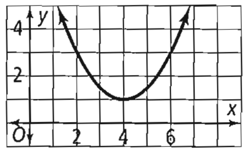
17. 

; $3\sqrt{5}$ **19.** $1 - 7i$ **21.** $10 + 6i$

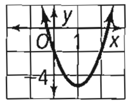
23. $9 + 58i$ **25.** −36 **27.** $-\frac{2}{5} - \frac{3}{5}i$ **29.** $\frac{8}{17} + \frac{19}{17}i$ **31.** $\frac{8}{13} + \frac{12}{13}i$ **33.** $\pm 5i$ **35.** $\pm 8i\frac{\sqrt{3}}{3}$ **37.** $\pm 6i$ **39.** $-1 \pm i\sqrt{2}$ **41.** $1 \pm i\frac{\sqrt{10}}{2}$ **43.** $2 \pm i$ **45. a.** A: −5; B: $3 + 2i$; C: $2 - i$; D: $3i$; E: $-6 - 4i$; F: $-1 + 5i$ **b.** A: 5; B: $-3 - 2i$; C: $-2 + i$; D: $-3i$; E: $6 + 4i$; F: $1 - 5i$ **c.** A: −5; B: $3 - 2i$; C: $2 + i$; D: $-3i$; E: $-6 + 4i$; F: $-1 - 5i$ **d.** A: 5; B: $\sqrt{13}$; C: $\sqrt{5}$; D: 3; E: $2\sqrt{13}$; F: $\sqrt{26}$ **47.** −5, 5 **49.** $-1 + 5i$ **51.** $8 - 2i$ **53.** $6 + 10i$ **55.** $10 + 11i$ **57.** trapezoid **59.** $\frac{1}{26} + \frac{3}{52}i$ **61.** sum: 2, product: 3 **63.** sum: $\frac{3}{2}$, product: $\frac{3}{2}$ **65.** Answers may vary. Sample: $x^2 - 4x + 29 = 0$ **67.** $x = -7, y = 3$ **69.** $x = -7, y = -3$ **77.** $\frac{-3 \pm \sqrt{41}}{4}$ **78.** $\frac{-1 \pm \sqrt{17}}{8}$ **79.** $\frac{-7 \pm \sqrt{17}}{2}$

80. 

; axis of symmetry: $x = -1$

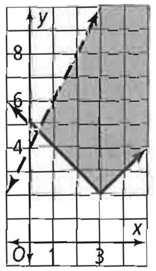
81. 

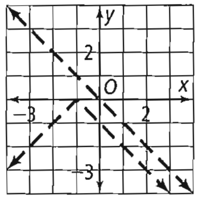
; axis of symmetry: $x = 4$

82. 

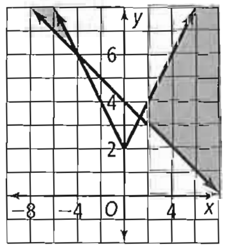
; axis of symmetry: $x = 1$

83. $y = 3x - 4$ **84.** $y = -0.5x - 2$ **85.** $y = -7x + 10$ **86.** $y = 2x + 8$

87. 

88. 

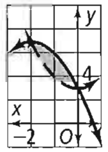
no solution

89. 

Lesson 4-9 pp. 258–264

Got It? 1. (−3, 0), (−2, 1) **2.** (−5, 0), (1, 6) **3. a.** (0, 5), (2, 1) **b.** no solution

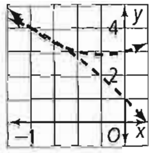
4. a.



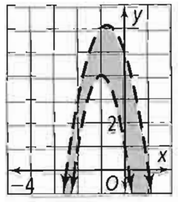
b. infinite, one, or none

Lesson Check **1.** (1, 2), (2, 3) **2.** (1, −1), (2, 0) **3.** (2, −5), $\left(-\frac{4}{3}, \frac{25}{9}\right)$

4.

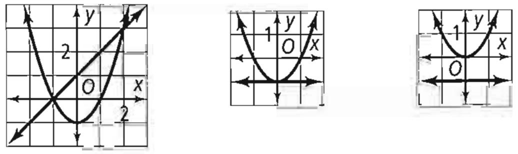


5.

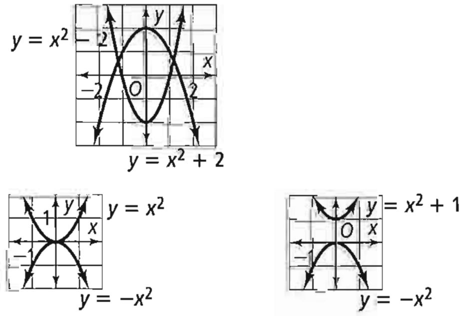


6. For each system of eqs., linear or quadratic, to solve the system you need to find the pt. (or pts.) of intersection or, in the case of inequalities, the regions of intersection. A linear system of eqs. can have one, infinite, or no solutions, whereas a quadratic systems of eqs. can have one, two, infinite, or no solutions.

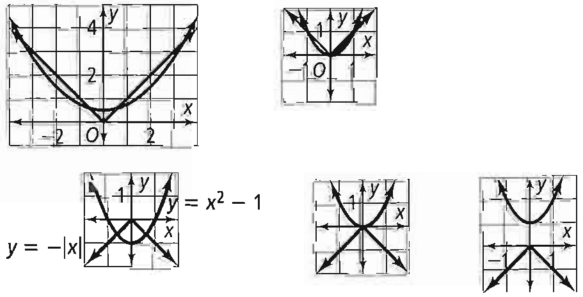
7. a. two, one, or zero



b. two, one, or zero

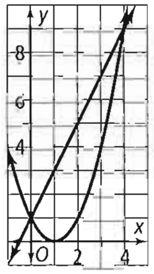


c. four, three, two, one, or zero

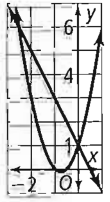


Exercises

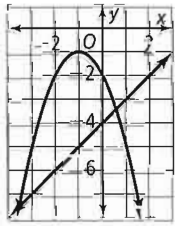
9. (0, 1), (4, 9);



11. (0, 1), $\left(-\frac{5}{2}, 6\right)$;

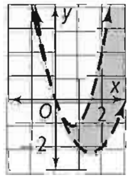


13. $\left(\frac{-3+\sqrt{17}}{2}, \frac{-11+\sqrt{17}}{2}\right), \left(\frac{-3-\sqrt{17}}{2}, \frac{-11-\sqrt{17}}{2}\right)$
$\approx$ (0.56, −3.44), (−3.56, −7.56)



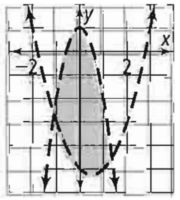
15. (3, 7), (−2, 2) **17.** (1, −2) **19.** $(-3+\sqrt{6}, -1+\sqrt{6}), (-3-\sqrt{6}, -1-\sqrt{6})$ **21.** (0, −1)
23. (0, −3), $\left(\frac{1}{3}, -\frac{31}{9}\right)$ **25.** no solution

27.

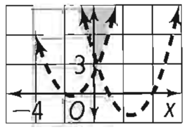


29. width = 20.45 in. and length = 21.45 in.; or width = 6.55 in. and length = 7.55 in.
31. (8.42, −5.42), (−1.42, 4.42) **33.** $\left(-1, -\frac{3}{2}\right)$
35. (0, −4), (2, −2)

37.

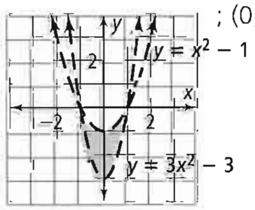


39.



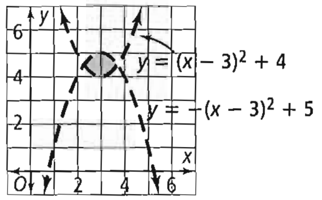
41. (0, −1), (1, 0) **43.** (10, 68), (−3, 16)
45. (−8, −16), $\left(-\frac{4}{3}, 4\right)$ **47. a.** $0 < p < 25$
b. 13 **49.** no solution **51.** $\left(\frac{1}{2}, \frac{7}{2}\right)$

53.



; (0, −2)

55.



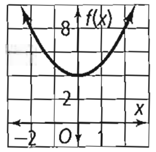
; $\left(3, \frac{9}{2}\right)$

66. $-4 + 3i$ **67.** $7 + 7i$ **68.** $3 + 3i$ **69.** $-1, -\frac{3}{2}$
70. 1, 3 **71.** $\frac{3}{5}$ **72.** $y = -(k - 2)^2 + 10$
73. $y = (x + 3)^2 - 8$ **74.** $y = 2(n - 2)^2 - 11$
75. $11q$ **76.** $2a^2b + ab^2$ **77.** $-y^2 + 2y$

Chapter Review pp. 267–272

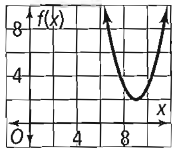
1. standard **2.** discriminant **3.** complex **4.** vertex: (−2, −6); axis of symmetry: $x = -2$; minimum: −6; domain: all real numbers; range: $y \geq -6$ **5.** vertex: (3, 2); axis of symmetry: $x = 3$; maximum: 2; domain: all real numbers; range: $y \leq 2$ **6.** vertex: (1, 5); axis of symmetry: $x = 1$; minimum: 5; domain: all real numbers; range: $y \geq 5$ **7.** vertex: (−9, −4); axis of symmetry: $x = -9$; minimum: −4 domain: all real numbers; range: $y \geq -4$

8.



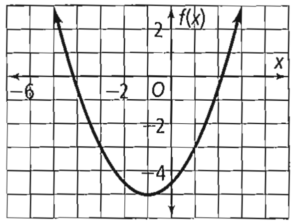
translation 4 units up

9.



translation 9 units to the right and 2 units up

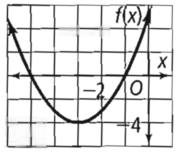
10.



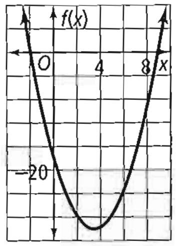
vert. compression by a factor of $\frac{1}{2}$, translation 1 unit to the left and 5 units down

11. $y = 2(x - 2)^2 + 1$ **12.** $y = \left(-\frac{1}{3}\right)(x + 5)^2 + 4$

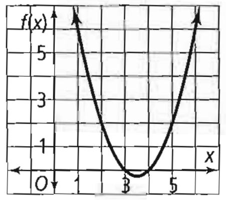
13.



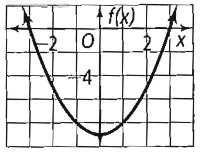
14.



15.

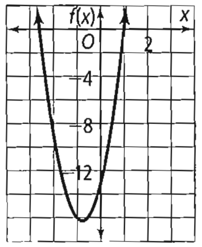


16.

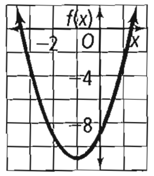


17. $f(x) = 4(x - 1)^2 - 2$ **18.** $f(x) = (x - 4)^2 - 4$
19. $f(x) = 8\left(x + \frac{1}{2}\right)^2 - 14$ **20.** $f(x) = -2\left(x + \frac{3}{2}\right)^2 + \frac{29}{2}$
21. 1s; 25 ft **22.** $y = x^2 - 6x + 5$
23. $y = -2x^2 + 8x - 8$ **24.** $y = x^2 + 3x - 18$
25. $y = -0.5x^2 + 2.5x - 7$
26. $y = -0.0043x^2 + 0.3521x + 0.3691$
27. $(x - 6)(x - 2)$ **28.** $(3x - 4)(x + 5)$
29. $-2(2x - 1)(x - 3)$ **30.** $(x + 10)(x + 4)$
31. $(x - 7)^2$ **32.** $(3x + 5)^2$ **33.** $4(3x - 2)(3x + 2)$
34. $(5x - 2)(5x + 2)$ **35.** $6x$; $6x(x - 4)$
36. 7; $-7(2x^2 + 7)$ **37.** −2, 6 **38.** $-2, \frac{7}{2}$
39. −4, 2 **40.** −9, 2

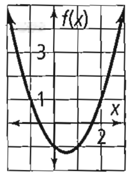
41. 1, −2.6;



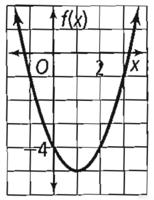
42. 1.345, −3.345;



43. 1.618, −0.618;

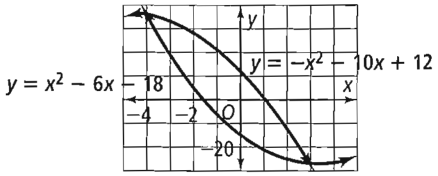


44. 3.236, −1.236;

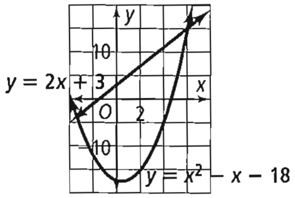


45. 2, 4 **46.** no real solutions **47.** 4.56, 0.44 **48.** 2, 4
49. ± 2 **50.** $\pm\sqrt{5}$ **51.** ± 3 **52.** $\pm 2\sqrt{3}$ **53.** 9 **54.** $\frac{9}{4}$
55. $-4 \pm \sqrt{10}$ **56.** $5 \pm \sqrt{38}$ **57.** $-1, \frac{1}{3}$ **58.** $1 \pm i\sqrt{3}$
59. $\frac{-3 \pm i\sqrt{91}}{2}$ **60.** $1, -\frac{3}{4}$ **61.** $1, -\frac{8}{3}$ **62.** 3 **63.** 4, −1
64. 1.744, −0.344 **65.** 164; two **66.** −3; none
67. 0; one **68.** 233; two **69.** 10.2736 ft × 16.5472 ft
70. $2i\sqrt{6}$ **71.** $-3 + i\sqrt{2}$ **72.** $-50 + 40i$
73. $6 + 4i\sqrt{6}$ **74.** $3 + 9i$ **75.** $13 + 20i$ **76.** $21 - 25i$
77. $-12 - 15i$ **78.** $-3 - 2i$ **79.** $-\frac{1}{2} - \frac{1}{2}i$ **80.** $\pm 3i$
81. $\frac{1}{5} \pm \frac{2}{5}i$ **82.** $2 \pm i\sqrt{6}$ **83.** $\frac{-4 \pm i\sqrt{26}}{7}$
84. (7, −6), (−2, 12) **85.** (5, −27), (−2, 8)

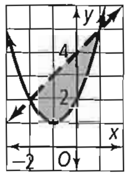
86. (−5, 37), (3, −27);



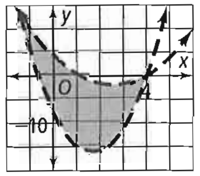
87. (6.32, 15.64), (−3.32, −3.64);



88.



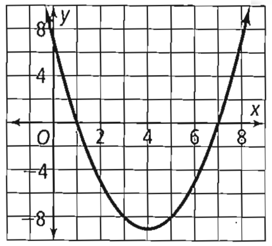
89.



Chapter 5

Get Ready! p. 277

1.



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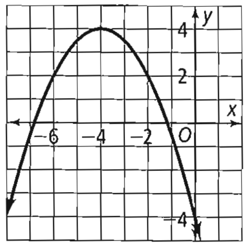
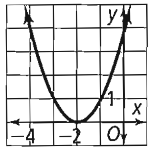


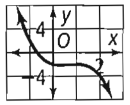
3.



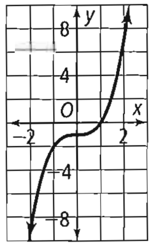
4. $y = x^2 + 3x - 4$ **5.** $y = x^2 - 2x + 1$ **6.** $y = -x^2 - x + 12$ **7.** 0.25, −1 **8.** −6.39, 4.39 **9.** −6.38, 0.38 **10.** −4, 5 **11.** −9, 3 **12.** 1, 2 **13.** 24; two real solutions **14.** 0; one real solution **15.** 0; one real solution **16.** south **17.** The highest pt. in Maine may be lower than the highest pt. in the United States. The relative maximum of a graph for a given region is the maximum for that region only, whereas the maximum of the graph may be greater than or equal to the relative maximum for the region. **18.** $4x^2 + 4x + 1$

Lesson 5-1 pp. 280–287

Got It? 1. a. $5x^4 + 3x^3 - x$; quartic trinomial **b.** $-4x^5 + 2x^2 + 13$; quintic trinomial **2.** up and down **3. a.** end behavior: up and down; two turning points



b. end behavior: down and up; no turning points



4. a. degree: 4 **b.** Answers may vary. Sample: $y = x^5$
Lesson Check 1. cubic monomial **2.** quadratic trinomial **3.** $5x^2 + 7x + 3$ **4.** $9x - 3$ **5.** up and down **6.** Yes; the graph of a linear binomial or linear monomial is a straight line; $f(x) = 2x$ **7.** The graph of $y = 4x^3 + 4$ has no turning points, *not* one turning point.
Exercises 9. $-3x + 5$; linear binomial **11.** $x^4 - x^3 + x$; quartic trinomial **13.** $3a^3 + 5a^2 + 1$; cubic trinomial **15.** $12x^4 + 3$; quartic binomial **17.** $-2x^3$; cubic monomial **19.** $-x^4 + 3x^2$; quartic binomial **21.** up and up **23.** down and up **25.** down and down **27.** up and down **29.** up and down **31.** up and up **33.** end behavior: up and down; two turning pts. **35.** end behavior: down and up; no turning pts. **37.** end behavior: down and up; no turning pts. **39.** 3 **41.** $-4a^4 + a^3 + a^2$; quartic trinomial **43.** $6x^2$; quadratic monomial **45.** $-9d^3 - 13$; cubic binomial **47.** negative; 3 **49.** positive; 4
51. For $f(x) = x^3 - 3x^2 - 2x - 6$,

x	f(x)	1st diff	2nd diff	3rd diff
0	−6	−4	0	6
1	−10	−4	6	6
2	−14	2	12	6
3	−12	14	18	
4	2	32		
5	34			

For $f(x) = ax^3 + bx^2 + cx + d$,

x	f(x)	1st diff	2nd diff	3rd diff
0	d	$a + b + c$	$6a + 2b$	$6a$
1	$a + b + c + d$	$7a + 3b + c$	$12a + 2b$	$6a$
2	$8a + 4b + 2c + d$	$19a + 5b + c$	$18a + 2b$	$6a$
3	$27a + 9b + 3c + d$	$37a + 7b + c$	$24a + 2b$	
4	$64a + 16b + 4c + d$	$61a + 9b + c$		
5	$125a + 25b + 5c + d$			

53. a.

x	y	1st diff	2nd diff
−2	8		
		−6	
−1	2		4
		−2	
0	0		4
		2	
1	2		4
		6	
2	8		

b.

x	y	1st diff	2nd diff
−2	20		
		−15	
−1	5		10
		−5	
0	0		10
		5	
1	5		10
		15	
2	20		

c.

x	y	1st diff	2nd diff
−2	18		
		−15	
−1	3		10
		−5	
0	−2		10
		5	
1	3		10
		15	
2	18		

d.

x	y	1st diff	2nd diff
−2	28		
		−21	
−1	7		14
		−7	
0	0		14
		7	
1	7		14
		21	
2	28		

e.

x	y	1st diff	2nd diff
−2	29		
		−21	
−1	8		14
		−7	
0	1		14
		7	
1	8		14
		21	
2	29		

f.

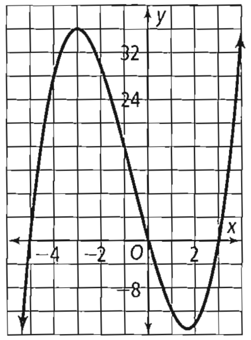
x	y	1st diff	2nd diff
−2	23		
		−18	
−1	5		14
		−4	
0	1		14
		10	
1	11		14
		24	
2	35		

Second differences of quadratic functions are constant.

61. (3.7, −4.4), (−2.7, 8.4) **62.** (−20, 360) **63.** (−3, 0), (−1, −8) **64.** $35x - 5y = -2$ **65.** $6x + 2y = -5$ **66.** $2x + 7y = 28$ **67.** $12x - 10y = 5$ **68.** $(x + 4)(x + 3)$ **69.** $(x + 10)(x - 2)$ **70.** $(x - 12)(x - 2)$

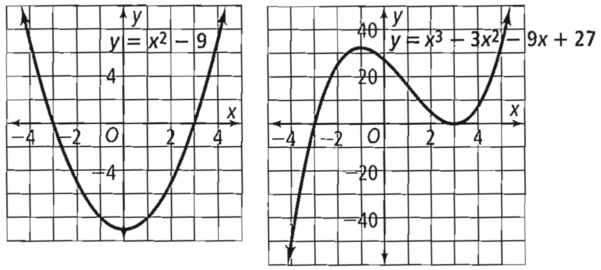
Lesson 5-2 pp. 288–295

Got It? 1. $x(x - 4)(x + 3)$
2. 0, 3, −5;



3. a. $f(x) = x^2 - 9$ **b.** $P(x) = x^3 - 3x^2 - 9x + 27$

c.



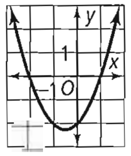
Both graphs have *x*-intercepts of 3 and −3. The quadratic has up and up end behavior and one turning pt., and the cubic has down and up end behavior and two turning pts.

4. 0 is a zero of multiplicity 1, the graph looks close to linear at $x = 0$; 2 is a zero of multiplicity 2, the graph looks close to quadratic at $x = 2$. **5.** relative maximum: (−0.86, 3.13); relative minimum: (0.64, −2) **6.** 2.28 in.^{3}

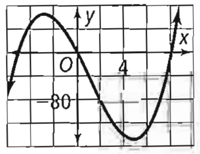
Lesson Check 1. 0, 6 **2.** −4, 5 **3.** −12, 7, 9 **4.** $f(x) = x^3 - x$ **5.** $h(x) = x^4 + 4x^3 - 26x^2 - 60x + 225$ **6.** Error in writing the factors: a function that has zeros at 3 and −1 has factors of $x - 3$ and $x + 1$ *not* $x + 3$ and $x - 1$, so $f(x) = x^2 - 2x - 3$ *not* $x^2 + 2x - 3$.

Exercises 7. $x(x + 5)(x + 2)$ **9.** $x(x - 7)(x + 3)$ **11.** $x(x + 4)^2$

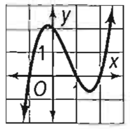
13. 1, −2;



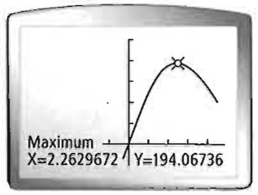
15. 0, −5, 8;



17. −1, 1, 2;



19. $y = x^3 - 18x^2 + 107x - 210$ **21.** $y = x^3 + 9x^2 + 15x - 25$ **23.** $y = x^3 + 2x^2 - x - 2$ **25.** $y = x^4 - 5x^3 + 6x^2$ **27.** −3 (multiplicity 3) **29.** $-1, 0, \frac{1}{2}$ **31.** 4 (multiplicity 2) **33.** $-\frac{3}{2}$, 1 (multiplicity 2) **35.** relative maximum: (−3.19, 24.19); relative minimum: (0.52, −1.38) **37.** relative maximum: (2.15, 12.32); relative minimum: (−0.15, −12.32) **39. a.** $\ell = 16 - 2x$; $w = 12 - 2x$; $h = x$ **b.** $V = x(16 - 2x)(12 - 2x)$ **c.**

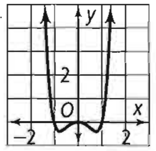


194 in.^{3}, 2.26 in.

41. $y = -2x(x + 5)(x - 4)$ **43.** 1 ft increase in each dimension **45.** $V = 12x^3 - 27x$ **47.** relative maximum: (2.53, 10.51); relative minimum: (5.14, −7.14); $\frac{3}{2}$, 4, 6 **49.** no relative maximum; relative minimum: (−1, −1); −2, 0 **51.** Answers may vary. Sample: The linear factors can be determined by examining the x-intercepts of the graph. **53.** $-1, 4, \frac{3}{2}$ **61.** $-3x^5 + 3x^2 - 1$; quintic trinomial **62.** $-7x^4 - x^3$; quartic binomial **63.** $x^3 - 2$; cubic binomial **64.** $(x + 4)(x + 1)$ **65.** $(x - 5)(x + 3)$ **66.** $(x - 6)^2$ **67.** −3, 2 **68.** $\frac{1}{2}$, 3 **69.** $\pm\frac{5}{2}$

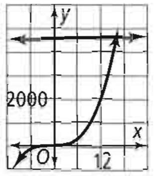
Lesson 5-3 pp. 296–302

Got It? 1. a. ±1, ±2i **b.** 0, 2, 3 **2. a.** ±2, ±2i **b.** 0, −4, 2 **c.** 2, $-1 \pm i\sqrt{3}$ **3. a.** −1.84 **b.** The second method seems to be a more reliable way to find the solutions because you do not risk missing a pt. of intersection. **4.** 7, 8, 9
Lesson Check 1. $(x - 6)(x + 3)$ **2.** $(x - 3)(x^2 + 3x + 9)$ **3.** $(x^2 + 4)(x + 3)$ **4.** $(x - 2)(x + 2)(x^2 + 2)$ **5.** $-4, \frac{1}{2}$ **6.** −2, 0, 1 **7. a.** difference of squares **b.** sum of cubes **c.** difference of cubes **d.** difference of squares **8.** Graphing; imaginary numbers don't exist on the x-axis. **9.** Method 1: Graph $y = x^6 - x^2$. Find the zeros for the real solutions.

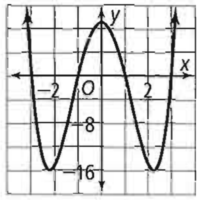


Method 2: Factor and solve x for $x^6 - x^2 = 0$.
$x^2(x^4 - 1) = x^2(x^2 - 1)(x^2 + 1) =$
$x^2(x - 1)(x + 1)(x^2 + 1) = 0$
$x = 0, \pm 1$

Exercises 11. $10, -5 \pm 5i\sqrt{3}$ **13.** $\frac{1}{4}, \frac{-1 \pm i\sqrt{3}}{8}$ **15.** $\frac{1}{3}, -\frac{5}{2}$ **17.** $4, -2 \pm 2i\sqrt{3}$ **19.** $-\frac{1}{2}, \frac{1 \pm i\sqrt{3}}{4}$ **21.** ±2 **23.** $\pm\sqrt{2}, \pm 3i$ **25.** −2, 1, 5 **27.** 0, 1 **29.** 0, −1, −2 **31.** 0, −0.5, 1.5 **33.** 1, 7 **35.** −2, 5 **37.** 16 yrs old; $x^2(x + 2) = 3x + 4560$;



39. $0, 5 \pm 2\sqrt{3}$ **41.** $0, \frac{5 \pm 5\sqrt{2}}{2}$ **43.** $\frac{4}{3}, \frac{-2 \pm 2i\sqrt{3}}{3}$ **45.** $1.9(1 \pm i), -1.9(1 \pm i)$ **47.** 0, ±1, ±2 **49.** −1, ±i **51.** 2 ft × 3 ft × 6 ft **53.** 5 m **55.** ±3, ±1; $y = (x - 1)(x + 1)(x - 3)(x + 3)$;



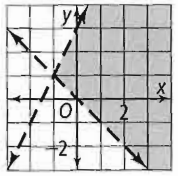
65. $3(x - 4)(x - 2)$ **66.** $2x(x + 3)^2(x - 3)$ **67.** $x^2(x - 5)(x + 1)$ **68.** −2, 6 **69.** ±6 **70.** $-\frac{1}{2}$, 3 **71.** 12 **72.** $-\frac{1}{5}$

Lesson 5-4 pp. 303–310

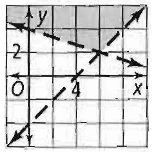
Got It? 1. $3x - 8$,R 0 **2. a.** yes; $P(x) = (x + 5)(x^4 - 1)$ **b.** $(x + 2)(3x + 1)$ **3.** $x^2 + 7x - 8$, R 0 **4.** width: $(x + 1)$ in.; height: $(x + 2)$ in.; length: $(x + 3)$ in. **5.** 0
Lesson Check 1. $2x + 3$, R 5 **2.** $x^2 + 2x + 5$ **3.** $x^2 + x - 2$ **4.** $4x^2 + x - 6$, R 6 **5.** $9x^2 + 12x + 40$, R 120 **6.** $x - a$ is a factor of $P(x)$. **7.** The polynomials need to be written in standard form since the leading coefficient of both polynomials determines the leading term of the quotient. **8.** Answers may vary. Check student's work.
Exercises 9. $x - 8$ **11.** $x^2 + 4x + 3$, R 5 **13.** $3x^2 + 3x + 2$ **15.** $x - 10$, R 40 **17.** no **19.** yes **21.** $x^2 + 4x + 3$ **23.** $x^2 - 11x + 37$, R −128 **25.** $x + 1$, R 4 **27.** $x^2 - 3x + 9$ **29.** $y = (x + 1)(x + 3)(x - 2)$ **31.** width = x; length = $x + 3$; height = $x - 2$ **33.** 0 **35.** 12 **37.** 10 **39.** 0 **41.** The constant term of the dividend is missing and the divisor is −1 *not* 1: $x^3 - x^2 - 2x = (x + 1)(x^2 - 2x) = x(x + 1)(x - 2)$

43. $x + 2$ **45.** $x^3 - 3x^2 + 12x - 35$, R 109
47. $x + 4$ **49.** no **51.** yes **53.** yes **55.** no
57. $x^3 - x^2 + 1$ **59.** $x^3 + 7x + 5$
61. $x^3 - 2x^2 - x + 6$ **71.** 0, −1 **72.** 0, 1
73. −5, 0, 5 **74.** $\frac{-3 \pm \sqrt{17}}{2}$ **75.** $-1 \pm \sqrt{3}$
76. 1, $-\frac{5}{7}$ **77.** $\frac{5 \pm \sqrt{5}}{2}$ **78.** $3 \pm \sqrt{2}$ **79.** $\frac{-7 \pm \sqrt{5}}{2}$

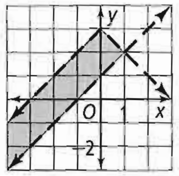
80.



81.



82.



83. 24 **84.** 5 **85.** $23 - 11i$

Lesson 5-5 pp. 312–317

Got It? 1. $\frac{2}{3}$ **2.** 2, −1, $-\frac{3}{2}$ **3.** $3 + 2i$
4. $P(x) = x^4 - 14x^3 + 69x^2 - 194x + 208$ **5. a.** There are three or one positive real roots and one negative real root. The graph confirms one negative and one positive real root. **b.** Real roots can be confirmed graphically because they are *x*-intercepts. Complex roots cannot be confirmed graphically because they have an imaginary component.
Lesson Check 1. ±1, ±2 **2.** ±1, ±2, ±3, ±6, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$ **3.** ±1, ±2, ±3, ±4, ±6, ±12, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{4}{3}$
4. $P(x) = x^2 - 14x + 45$ **5.** $P(x) = x^3 + 4x^2 + 4x + 16$
6. Answers may vary. Samples: $1 + 2i$, $1 - 2i$; $1 + \sqrt{2}$, $1 - \sqrt{2}$ **7. a.** never; 5 does not divide 8 evenly **b.** always; −2 divides 8 evenly **8.** Complex number roots come in pairs; if $-4i$ is a root, so is $4i$.
Exercises 9. ±1; no rational roots **11.** ±1, ±2, ±2, ±4, $\pm\frac{1}{2}$; no rational roots **13.** ±1, ±2, ±3, ±4, ±6,±12, $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, $\pm\frac{3}{4}$, $\pm\frac{3}{2}$; no rational roots **15.** ±1, ±2, ±5, ±10, $\pm\frac{1}{7}$, $\pm\frac{2}{7}$, $\pm\frac{5}{7}$, $\pm\frac{10}{7}$; no rational roots **17.** ±1, ±2, ±5, ±10, $\pm\frac{1}{2}$, $\pm\frac{5}{2}$, $\pm\frac{1}{5}$, $\pm\frac{2}{5}$, $\pm\frac{1}{10}$; no rational roots
19. $14 + \sqrt{2}$, $6i$ **21.** $\sqrt{3}$, $5 + \sqrt{11}$
23. $P(x) = x^2 + 24x + 135$ **25.** $P(x) = x^2 - 18x + 90$
27. $P(x) = x^4 - 10x^3 + 294x^2 - 1690x + 21{,}125$
29. $P(x) = x^4 - 58x^3 + 1290x^2 - 13{,}066x + 51{,}545$
31. two or no positive real roots; one negative real root
33. no rational roots **35.** no rational roots **37.** no rational roots **39.** $P(x) = x^4 + 3x^3 + 207x^2 + 675x - 4050$
41. $P(x) = x^4 + 6x^3 + 27x^2 - 366x - 518$ **43.** Error in second line, sign of second term; the line should be: $P(-1) = -x^3 + x^2 - x + 1$. Since there are three sign changes in $P(-x)$, there are three or one negative real roots.
45. height: 5 ft; bases: 10 ft, 14 ft **55.** $x^2 + 6x + 6$, R 3
56. $8x^2 - 36x + 216$, R −1289 **57.** $7x - 3$, R 2
58. $\pm 3i$ **59.** $\pm 9i$ **60.** $\pm 12i$ **61.** $-5x^4 + 6x^2 + 9x + 11$; quartic polynomial of four terms
62. $-4x^5 + 7x^3 + 13x + 2$; quintic polynomial of four terms

Lesson 5-6 pp. 319–324

Got It? 1. 0, 1, −5, 2 **2. a.** −1, 2, $\frac{1 \pm i\sqrt{23}}{4}$
b. i. A 5th degree polynomial function has four, two, or no turning pts. Three turning pts. are visible, so there must be a fourth one. This will turn the graph back across the *x*-axis. **ii.** The Fundamental Thm. of Algebra states there will be five roots, and the Conjugate Root Thm. requires pairs of irrational or complex roots. Only two zeros appear in the graph, so there are three zeros remaining. Of the remaining roots, either there are three real roots, or one real and two complex roots. Either way, there is at least one real root that does not appear in the graph.
Lesson Check 1. four roots **2.** fourteen roots
3. 5, $\pm 4i$ **4.** 0, 2, $\pm i$ **5.** By the Fundamental Thm. of Algebra, polynomial equation of degree *n* has exactly *n* roots. **6.** Answers may vary. Sample: $y = x^4 + 8x^2 + 16$ **7.** Use synthetic division to test for and factor out linear factors until a quadratic factor is obtained. Then use the Quadratic Formula if the quadratic factor cannot be factored further.
Exercises 9. −1, $\pm 2i$ **11.** −1, 2, 4 **13.** 2, $\pm\sqrt{3}$
15. 0, ±3, $\pm i$ **17.** 3, $\pm i$ **19.** 2, $\pm\sqrt{3}$ **21.** ±2, $\pm i$
23. −6, $\pm i$ **25.** ±4, $\frac{-1 \pm i\sqrt{3}}{2}$ **27.** five complex roots; one, three, or five real roots; possible rational roots: ±1, ±2, ±3, ±6, ±9, ±18 **29.** six complex roots; zero, two, four, or six real roots; possible rational roots: $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, $\pm\frac{3}{4}$, ±1, $\pm\frac{3}{2}$, ±2, ±3, ±4, ±6, ±8, ±12, ±24 **31.** −2, $\pm\sqrt{5}$ **33.** −2, $\frac{4}{3}$, 3
35. 3, $-1 \pm i\sqrt{2}$ **37.** $-\frac{1}{2}$, −1, ±3 **39.** 3 bridges
41. sometimes **43.** always **45.** Answers may vary. Sample: $y = x^4 + 3x^2 + 2$ **54.** $x^4 + 6x^3 + 14x^2 + 24x + 40 = 0$ **55.** $3 \pm 2\sqrt{2}$ **56.** $\frac{-5 \pm i\sqrt{47}}{4}$
57. $\frac{3 \pm i\sqrt{23}}{4}$ **58.** $f(x) = -x^2 + 2x + 3$
59. $f(x) = 2x^2 + 24x + 75$ **60.** $x^3 + 3x^2 + 3x + 1$
61. $x^3 - 9x^2 + 27x - 27$ **62.** $x^4 - 8x^3 + 24x^2 - 32x + 16$ **63.** $x^2 - 2x + 1$ **64.** $x^3 + 15x^2 + 75x + 125$
65. $-x^3 + 12x^2 + 48x + 64$

Lesson 5-7 pp. 326–330

Got It? 1. $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$ **2. a.** $16x^4 - 96x^3 + 216x^2 - 216x + 81$ **b.** If you express 11 as $(10 + 1)$ and calculate the powers using Pascal's triangle, it will be the coefficients.

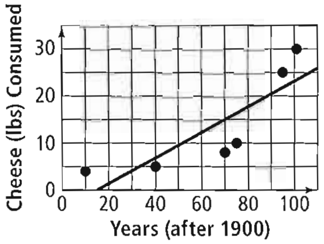
Lesson Check 1. $x^3 + 3x^2a + 3xa^2 + a^3$ **2.** $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$ **3.** $4x^2 + 16x + 16$ **4.** $27a^3 - 54a^2 + 36a - 8$ **5. a.** yes **b.** yes **c.** no **6.** The coefficients for the expansion of $(a + b)^n$ are equal to the numbers in the *n*th row of Pascal's Triangle, respectively. **7.** 13; $n + 1$

Exercises 9. $a^4 + 8a^3 + 24a^2 + 32a + 16$ **11.** $x^3 - 15x^2 + 75x - 125$ **13.** $x^{10} + 20x^9 + 180x^8 + 960x^7 + 3360x^6 + 8064x^5 + 13{,}440x^4 + 15{,}360x^3 + 11{,}520x^2 + 5120x + 1024$ **15.** $b^9 + 27b^8 + 324b^7 + 2268b^6 + 10{,}206b^5 + 30{,}618b^4 + 61{,}236b^3 + 78{,}732b^2 + 59{,}049b + 19{,}683$ **17.** $a^4 + 12a^3b + 54a^2b^2 + 108ab^3 + 81b^4$ **19.** $65{,}536 - 131{,}072x + 114{,}688x^2 - 57{,}344x^3 + 17{,}920x^4 - 3584x^5 + 448x^6 - 32x^7 + x^8$ **21.** $27a^3 - 189a^2 + 441a - 343$ **23.** $81y^4 - 1188y^3 + 6534y^2 - 15{,}972y + 14{,}641$ **25. a.** 6 **b.** 489,888 **27.** $135x^4$ **29.** $625b^8$ **31.** The challenge of the Binomial Theorem occurs when there is a coefficient with the *x*. However, it is much more efficient to use the Binomial Theorem than FOIL when expanding a binomial that is raised to a high power. **33.** $x^{20} + 40x^{18} + 720x^{16} + 7680x^{14} + 53{,}760x^{12} + 258{,}048x^{10} + 860{,}160x^8 + 1{,}966{,}080x^6 + 2{,}949{,}120x^4 + 2{,}621{,}440x^2 + 1{,}048{,}576$ **35.** $a^5 - 5a^4b^2 + 10a^3b^4 - 10a^2b^6 + 5ab^8 - b^{10}$ **37.** $256x^4 - 1792x^3y + 4704x^2y^2 - 5488xy^3 + 2401y^4$ **39.** $4096x^{18} + 12{,}288x^{15}y^2 + 15{,}360x^{12}y^4 + 10{,}240x^9y^6 + 3840x^6y^8 + 768x^3y^{10} + 64y^{12}$ **41.** $125a^3 + 150a^2b + 60ab^2 + 8b^3$ **43.** $-32y^{10} + 80y^8x - 80y^6x^2 + 40y^4x^3 - 10y^2x^4 + x^5$ **45.** Answers may vary. Sample: Since one of the terms is negative $(-y)$ and it is alternately raised to odd and even powers; the term is negative when raised to an odd power and positive when raised to an even power. **58.** $-3, -1, \frac{-3 \pm i\sqrt{11}}{2}$ **59.** $1, \pm i, \pm 3i$ **60.** $-4, \frac{-3 \pm i\sqrt{7}}{4}$ **61.** $-1, 1, 2, 7$ **62.** $-18 + 43i$ **63.** $600i$ **64.** -2 **65.** $\frac{7}{5} + \frac{31}{5}i$ **66.** $2x^3 + 5x^2 - x + 9$; cubic polynomial of 4 terms **67.** $-7x^2 + 4x + 1$; quadratic trinomial **68.** $12x^4 - 3x^3 - 9x^2 + x - 8$; quartic polynomial of 5 terms

Lesson 5-8 pp. 331–338

Got It? 1. $y = 1.667x^3 + 1.3 \times 10^{-12}x^2 - 4.667x + 5$ **2.** 15.58 billion lbs **3.** Answers may vary. Sample: The cubic model would fit the data better than the linear model because of the $(n + 1)$ Pt. Principle. Both models have down and up end behavior and increasing growth. The cubic shows slowing growth followed by rapidly increasing growth.

4. a. $y = 0.269867411x - 3.919692952$

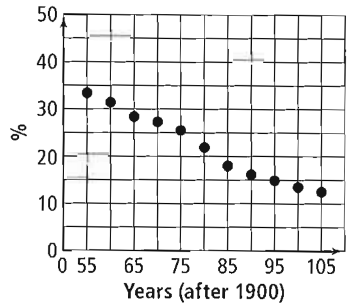


b. 1980: 17.7 lbs; 2000: 23.07 lbs; 2012: 26.31 lbs; most confident for the yrs 1980 and 2000, since they are within the domain of the data set; least confident for the yr 2012, since it is outside the domain of the data set

Lesson Check 1. linear **2.** quadratic **3.** cubic **4.** quartic **5.** interpolation since the data point is within the domain of the data set **6.** yes, since the four pts. pass the vertical-line test, a cubic function will fit the pts.; $y = -x^3 - 3x^2 - 3x$ **7.** cubic model; the closer R^2 is to 1, the better the fit

Exercises 9. $y = \frac{1}{2}x - 3$ **11.** $y = -0.929x^2 + 7.786x + 4$ **13.** $y = x^2 - 6x + 1$ **15.** $y = 2x^3 + x^2 - 4x + 6$ **17.** (where $x =$ yrs after 1900) quadratic: $y = -1.25 \times 10^{-4}x^2 - 0.003x + 2.804$, cubic: $y = -8.33 \times 10^{-6}x^3 + 0.002x^2 - 0.190x + 8.142$; cubic; cubic **19.** linear: $y = -0.057x + 19.93$, quadratic: $y = -0.025x^2 + 0.14x + 19.595$; quadratic; quadratic **21.** 1950: 2.60%; 1988: 1.23%; 2010: 0.35% **23.** January: 19.714 millions of barrels/day; March: 19.8 millions of barrels/day; October: 18.535 millions of barrels/day **25.** cubic: $y = 10.25x^3 + 5x^2 - 2.25x - 8.2$; quartic: $y = 2.042x^4 + 10.25x^3 - 4.042x^2 - 2.25x - 4$; quartic, $(R^2 = 1)$ **27.** $y = -0.275x^4 + 0.85x^3 - 4.025x^2 - 8.15x + 7$ **29.** $y = 0.0611511911x^3 - 0.9276466231x^2 + 6.184642324x + 1.750778723$; $R^2 = 0.9994739763$; good fit **31.** $y = 0.111x^4 - 45.618x^3 + 6997.73x^2 - 476{,}931.355x + 12{,}185{,}696.59$ **33.** A quadratic model would be more appropriate, given the real world context. According to the cubic model, there would be a negative number of students enrolled in the course in the year 2024.

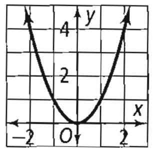
35. a.



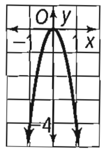
A cubic model seems to be most appropriate.

b. The model is $y = 0.00022688423x^3 - 0.0525547786x^2 + 3.505260295x - 38.64568765.$ **c.** 17.3% **d.** yes; $R^2 = 0.99288$ **43.** $32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$ **44.** $1331x^3 - 363x^2 + 33x - 1$ **45.** $4096 - 6144x + 3456x^2 - 864x^3 + 81x^4$ **46.** $64 + 432x + 972x^2 + 729x^3$ **47.** $|x - 8| < 1$ **48.** $|x - \frac{3}{8}| \leq \frac{1}{8}$ **49.** $|y - 2.8| < 1.1$ **50.** $|t - 750| < 250$ **51.** $s = \sqrt{A}$ **52.** $\ell = \frac{P}{2} - w$ **53.** $r = \frac{C}{2\pi}$ **54.** $b = \frac{A}{h}$

55.



56.



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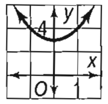

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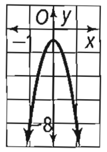


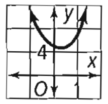
Lesson 5-9 pp. 339–345

Got It? 1. $y = 2(x + 3)^3 - 4$ **2.** $1 - \sqrt[3]{2}$ **3. a.** Answers may vary. Sample: $y = x^4 - 6x^3 + x^2 - 6x$ **b.** Yes; $-f(x)$ is the function reflected across the x-axis, so the zeros will stay the same. **4.** 972 kW
Lesson Check 1. −2 **2.** 3 **3.** $\frac{1}{3}$ **4.** No; a power function is of the form $y = ax^b$, where y varies directly with the bth power of x. **5.** $y = x^3$ has end behavior of down and up with no turning pt. Thus, at most, there is one real root. **6.** Both $y = x^3$ and $y = 4x^3$ pass through the origin, have the same end behavior of down and up, and no turning pts. $y = 4x^3$ is $y = x^3$ stretched vertically by a factor of 4.
Exercises 7. $y = -3(x - 1)^3 + 2$ **9.** $y = -(x + 5)^3 - 1$ **11.** $y = -3\left(x + \frac{1}{2}\right)^3 + \frac{3}{4}$ **13.** $\frac{8}{3}$ **15.** $-\frac{2}{15}$ **17.** $1 - \frac{1}{2}\sqrt[3]{20}$ **For Exercises 19–24, answers may vary. Samples: 19.** $x^4 - x^3 - x^2 - x - 2$ **21.** $x^4 - 2x^3 - 2x^2 - 2x - 3$ **23.** $x^4 + 5x^3 + 5x^2 + 5x + 4$ **25.** 38.4 ≈ 38 slices **27.** 90 lb·ft²/s² **29.** Yes; using parent function $y = x^2$, stretch vertically by a factor of 2, then translate 5 units up and 3 units to the right. **31.** Yes; using parent function $y = x^2$, translate 9 units down and 4 units to the right. **33.** Yes; using parent function $y = x^3$, stretch vertically by a factor of 4 and reflect across the x-axis. **35.** reflection across the x-axis, vert. stretch by a factor of 2, translation 1 unit up and 1 unit to the right **37.** vert. stretch by a factor of 3, translation 2 units down and 1 unit to the right **39.** reflection across the x-axis, translation 2 units up and 4 units to the right **41.** The parent function, $y = x^5$, has only one x-intercept. **43.** Error in (2) "a transformation of $y = x^2$." Some polynomials do not contain an x^2 term. **50.** $y = -2x^3 + 3x^2 - x - 2$ **51.** $y = 3x^3 - 5x - 3$ **52.** $y = -\frac{4}{5}x + \frac{16}{5}$ **53.** $y = -3x + 5$ **54.** yes **55.** no **56.** $x^2(x^8 + 1)$ **57.** $(x - y)(x + y)(x^2 + y^2)$ **58.** $13x^3y^6(13x^3y^6 - 1)$

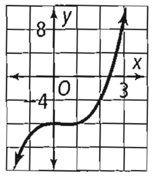
Chapter Review pp. 347–352

1. D **2.** B **3.** C **4.** A **5.** $y = -x^4 + 12$; quartic binomial; down and down **6.** $y = x^2 - x + 7$; quadratic trinomial; up and up **7.** $y = -x^4 + 2x^3 + 3x^2 - 6x + 12$; quartic polynomial of five terms; down and down **8.** $y = x^3 + 2x^2 - 4x + 8$; cubic polynomial of four terms; down and up **9.** $y = x^4 - 3x^3 + 3x^2 + 10$; quartic polynomial of four terms; up and up **10.** 3 **11.** If n is even there are an odd number of turning points; if n is odd there are an even number of turning points. **12.** $f(x) = x^3 - 4x^2 - 11x - 6$ **13.** $f(x) = x^3 - x^2 - 2x$ **14.** $f(x) = x^3 - 6x^2 + 11x - 6$ **15.** $f(x) = x^3 - 3x^2 - 6x + 8$ **16.** 0, −2 (multiplicity 3) **17.** 2 (multiplicity 2), −2 (multiplicity 2) **18.** 0, $-\frac{1}{2}$, 1 **19.** 5, −2 (multiplicity 2) **20.** relative maximum: (0.8672, −1.9351); relative minimums: (0, −3), (2.8828, −12.1704); zeros: $x = -0.5992$, $x = 3.7115$ **21.** relative maximum: (−0.8441, 9.3023); relative minimum: (0.7108, −0.0964); zeros: $x = -1.6180$, $x = 0.6180$, $x = 0.8$ **22.** relative minimum: (1, −4); zeros: $x = -0.2491$, $x = 1.6633$ **23.** relative maximum: (−0.4142, −3.3432); relative minimum: (2.4142, −14.6569); zero: $x = 4$ **24.** 3, 8 **25.** $-\frac{1}{2}$ **26.** 0, $\frac{-1 \pm \sqrt{37}}{2}$ **27.** $\frac{2 \pm i\sqrt{2}}{2}$

28. no real roots;



29. 2.3949;



30. 4.87 in. × 2.87 in. × 2.87 in. **31.** $x^2 + 6x + 9$ **32.** $2x^2 + x - 3$, R 1 **33.** yes **34.** no **35.** $x^2 - 1$ **36.** $2x^2 - 6x + 2$, R −20 **37.** $5x^2 + 18x + 36$, R 12 **38.** −14 **39.** 2 **40.** ±1, ±2, ±3, ±6 **41.** ±1, ±2, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$ **42.** ±1, ±2, ±3, ±4, ±6, ±12, $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, $\pm\frac{3}{4}$, $\pm\frac{3}{2}$ **43.** ±1, ±7, $\pm\frac{1}{3}$, $\pm\frac{7}{3}$ **44.** −3 **45.** −5 **46.** 1, −4, $-\frac{1}{2}$ **47.** 1, −2, $-\frac{2}{3}$ **48.** $1 + i$ **49.** $5 - \sqrt{3}$, $\sqrt{2}$ **50.** $3i$, $-7i$ **51.** $-2 - \sqrt{11}$, $-4 + 6i$ **52.** $y = x^2 - 17x + 70$ **53.** $y = x^3 + 3x^2 + 25x + 75$ **54.** $y = x^2 - 12x + 37$ **55.** $y = x^4 - 4x^3 - 10x^2 + 68x - 80$ **56.** one positive real zero; two or no negative real zeros **57.** two or no positive real zeros; one negative real zero **58.** four, two, or no positive real zeros; no negative real zeros **59.** two or no positive real zeros; two or no negative real zeros **60.** 3 **61.** 4 **62.** 5 **63.** 6 **64.** 1, $-3 \pm \sqrt{7}$ **65.** 2, $\pm\sqrt{5}$ **66.** 1, $\frac{1 \pm i\sqrt{15}}{4}$ **67.** −3, 6, $\frac{1 \pm \sqrt{5}}{2}$ **68.** 9 **69.** 1, 8, 28, 56, 70, 56, 28, 8, 1 **70.** 16 **71.** 105 **72.** $x^3 + 27x^2 + 243x + 729$ **73.** $b^4 + 8b^3 + 24b^2 + 32b + 16$ **74.** $27a^3 + 27a^2 + 9a + 1$ **75.** $x^3 - 15x^2 + 75x - 125$ **76.** $x^3 - 6x^2y + 12xy^2 - 8y^3$ **77.** $243a^5 + 1620a^4b + 4320a^3b^2 + 5760a^2b^3 + 3840ab^4 + 1024b^5$ **78.** $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$ **79.** $64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$ **80.** 108 **81.** $6a^2c^2$

82. $y = 3.5x^2 - 4.5x + 5$

83. $y = 2.082999x - 2.475234$; $y = 0.086232x^2 + 0.008929x + 5.963724$; $y = -0.002554x^3 + 0.178307x^2 - 0.913645x + 8.205128$; cubic is best fit since $R^2 = 1$.

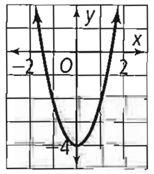
84. $y = -5.8667x^3 + 120.5333x^2 - 629.2667x + 1421$; 1097.2 ≈ 1097 **85.** $y = -(x - 2)^3 + 1$

86. $y = 6(x + 3)^3$ **87.** Answers may vary. Sample: $y = x^4 - 10x^3 + 25x^2 - 10x + 24$

Chapter 6

Get Ready! p. 357 **1.** domain: {1, 2, 3, 4}, range: {2, 3, 4, 5} **2.** domain: {1, 2, 3, 4}, range: {2} **3.** domain: all real numbers, range: $y \geq -8$ **4.** domain: all real numbers, range: $y \geq 3$

5.



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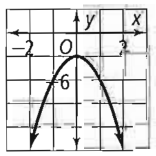
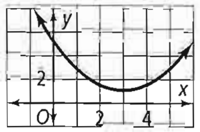


7.



8. $3y^2 - 14y + 8$ **9.** $49a^2 - 100$ **10.** $x^3 + 4x^2 - 15x - 18$ **11.** −2, 7 **12.** $\frac{5}{2}$, 3 **13.** −4, $\frac{2}{3}$ **14.** $\frac{1}{2}$ **15.** $\pm\frac{7}{2}$ **16.** $\pm\sqrt{7}$ **17.** Yes; it is a better deal to first take 50% off the shirt and then use the $10 coupon. **18.** A "one-to-one function" is a function where there is exact correspondence of every element of the domain with exactly one element of the range.

19. the non-negative root

Lesson 6-1 pp. 361–367

Got It? **1. a.** 0; −1; 2 **b.** ±0.1; no real square root; $\pm\frac{6}{11}$ **c.** Any negative number multiplied by itself an even number of times will always be positive. Therefore, there can be no real nth roots (where n is even) for a negative number b. **2. a.** −3 **b.** no real root **c.** 7 **d.** no real root **3. a.** $9x^2$ **b.** a^4b^5 **c.** $|x^3|y^4$ **4.** 0; 100

Lesson Check **1.** ±5 **2.** ±0.4 **3.** no real root **4.** $3|b|$ **5.** $a^4|b^9|$ **6.** $-5a$ **7.** 16 has two real fourth roots, ±2. **8.** The real roots of a number are the positive and negative (but not imaginary) roots of the number; the principal root of a number is the nonnegative root of the number. **9.** n is odd.

Exercises **11.** ±0.07 **13.** $\pm\frac{8}{13}$ **15.** 0.5 **17.** 0.07 **19.** none **21.** $\pm\frac{10}{3}$ **23.** 0.5 **25.** −3 **27.** $3y^2$ **29.** $2y^2$ **31.** ±10 **33.** ±0.5 **35.** about 0.8 in. **37. a.** about 79.01 ft **b.** about 44.44 ft **39.** $\frac{1}{3}$ **41.** $\frac{1}{4}$ **43.** Answers may vary. Sample: $\sqrt[3]{-8x^6}$, $-\sqrt[4]{16x^8}$, $\sqrt[5]{-32x^{10}}$ **45.** all; x^2 is always nonnegative **47.** some; they are equal for $x = -1, 0, 1$ **60.** $y = (x + 2)^3 + 3$ **61.** $y = \frac{1}{2}x^3 - 2$ **62.** 1, $\frac{3}{4}$ **63.** $\frac{5 \pm i\sqrt{11}}{6}$ **64.** $\frac{11}{6}$ **65.** $2x^3y^3$ **66.** $\frac{ac}{3}$ **67.** $\frac{4}{x^2}$

Lesson 6-2 pp. 367–373

Got It? **1. a.** No; the indexes are different. **b.** Yes; $\sqrt[5]{10}$ **2.** $4x^2\sqrt[3]{2x}$ **3.** $15x^3y^3\sqrt{7y}$ **4. a.** $5|x|$ **b.** yes; $\frac{3x^2\sqrt{2x}}{x\sqrt{2x}} = 3x$ **5. a.** $\frac{\sqrt[3]{175xy}}{5y}$ **b.** D; there is no y in the expression.

Lesson Check **1.** $\sqrt{10}$ **2.** $-3\sqrt[3]{4}$ **3.** Not possible, the indexes are different. **4.** not possible; $\sqrt{-4}$ is not a real number. **5.** $\sqrt[3]{3x}$ **6.** $x^2\sqrt{3x}$ **7.** $2x\sqrt[3]{4x}$ **8.** $x \leq 0$; for $x \leq 0$, $-4x^3 \geq 0$ and $\sqrt{-4x^3}$ is real. **9.** error in line 1: $\frac{\sqrt[7]{x^5}}{\sqrt[4]{x^2}} \neq \sqrt[7-4]{\frac{x^5}{x^2}}$

Exercises **11.** 4 **13.** not possible **15.** 5 **17.** 6 **19.** $2x\sqrt{5x}$ **21.** $5x^2\sqrt{2x}$ **23.** $3y^3\sqrt[3]{2y}$ **25.** $-5x^2y\sqrt[3]{2y^2}$ **27.** $-2xy\sqrt[5]{xy^2}$ **29.** $8y^3\sqrt{5y}$ **31.** $40x|y|\sqrt{3}$ **33.** $-2x^2y\sqrt[3]{30x}$ **35.** $4xy^2\sqrt[3]{y}$ **37.** 10 **39.** $2x^2y^2\sqrt{2}$ **41.** $\frac{2\sqrt[3]{x^2y}}{x}$ **43.** $\frac{\sqrt{2x}}{2}$ **45.** $\frac{\sqrt[3]{4x}}{2}$ **47.** $\frac{\sqrt[4]{250}}{5}$ **49.** $\frac{\sqrt{15y}}{5y}$ **51.** $\frac{\sqrt[3]{150ab^2c}}{5a}$ **53.** 6 cm^2 **55.** about 212 mi/h **57.** $5\sqrt{10}$ **59.** $3x^6y^5\sqrt{2y}$ **61.** $10 + 7\sqrt{2}$ **63.** $\frac{|x|\sqrt{10y}}{2y^2}$ **65.** $\frac{\sqrt[3]{3x^2}}{3x}$ **67.** $\frac{\sqrt[3]{2xy^2}}{xy}$ **69.** 4 g/cm^3 **71.** Check students' work. **73.** always **75.** sometimes **85.** $11|a^{45}|$ **86.** $9c^{24}d^{32}$ **87.** $4a^{27}$ **88.** $2y^5$ **89.** $y^2 - 4y + 16$, $R\ -128$ **90.** $6a^2 - 5a + 4$ **91.** 25 **92.** 25 **93.** $\frac{121}{4}$ **94.** $\frac{121}{4}$ **95.** $\frac{3}{5} + \frac{1}{5}i$ **96.** $\frac{10}{13} - \frac{15}{13}i$ **97.** $\frac{16}{17} - \frac{4}{17}i$ **98.** $-\frac{7}{74} - \frac{5}{74}i$

Lesson 6-3 pp. 374–380

Got It? **1. a.** The indexes are different. You cannot combine the expressions. **b.** $7x\sqrt{xy}$ **c.** $2\sqrt[5]{3x^2}$ **2. a.** about 84.9 in. **b.** The length of the diagonal of a square of side 6 can be found using the Pythagorean Thm. to be $\sqrt{6^2 + 6^2} = \sqrt{72}$. Using this information you can calculate the perimeter of the window and simplify the expression at the end. **3.** $6\sqrt[3]{2}$ **4.** $46 + 16\sqrt{5}$ **5. a.** 24 **b.** 1 **6. a.** $-\sqrt{21} - \sqrt{35}$ **b.** $\frac{1}{3}(12x + 4x\sqrt{6})$ **c.** after rationalizing; When the numerator is multiplied by the conjugate of the denominator it is more convenient if $\sqrt{8}$ is not yet simplified.

Lesson Check **1.** $12\sqrt{6}$ **2.** cannot combine **3.** $3\sqrt{3x}$ **4.** $7\sqrt{3}$ **5.** 13 **6.** $75 + 34\sqrt{5}$ **7.** $-16 - 3\sqrt{2}$ **8. a.** not like radicals **b.** like radicals; $9\sqrt{3xy}$ **c.** not like radicals **9.** They are alike in that you can also use the FOIL method and Distr. Prop. to multiply binomial radical expressions; they are different in that you cannot multiply like radicands together if they do not have the same index.

Exercises **11.** $4\sqrt[3]{3}$ **13.** $-2\sqrt{x}$ **15.** $5\sqrt[3]{x^2}$ **17.** $33\sqrt{2}$ **19.** $7\sqrt{2}$ **21.** $9\sqrt[3]{3} - 6\sqrt[3]{2}$ **23.** $8 + 4\sqrt{5}$ **25.** $63 - 38\sqrt{2}$ **27.** $49 + 12\sqrt{13}$ **29.** 14 **31.** -40 **33.** $-2 + 2\sqrt{3}$ **35.** $13 + 7\sqrt{3}$ **37.** 140.3 in.^2 **39.** $8\sqrt{3}$ **41.** $5\sqrt{3} - 4\sqrt{2}$ **43.** $-2\sqrt[3]{2}$ **45.** $-11 + \sqrt{21}$ **47.** $84 + 24\sqrt{6}$ **49.** 2 **51.** $4x\sqrt{3}$ **53.** Answers may vary. Sample: Without simplifying first, you must estimate three square roots and then add the estimates. If they are first simplified, then they can be combined as $13\sqrt{2}$. Then only one square root need be estimated. **55.** Answers may vary. Sample: $(\sqrt{7} + 2)(\sqrt{7} - 2), (2\sqrt{2} + \sqrt{5})(2\sqrt{2} - \sqrt{5})$ **57.** $2\sqrt{3} - \sqrt{2}$ **59.** $11|x| - 3|x|\sqrt{11}$ **61.** $\frac{3\sqrt{5} + 2\sqrt{3}}{3}$ **63.** $\frac{x + 5\sqrt[4]{x^3}}{x}$ **74.** $3\sqrt[3]{2}$ **75.** $\frac{2\sqrt[3]{x^2}}{x}$ **76.** 4 **77.** 6 **78.** $2x$ **79.** $7x^2\sqrt{2}$ **80.** $x\sqrt{15}$ **81.** $15x^2$ **82.** $2, -1 \pm i\sqrt{3}$ **83.** $-10, 5 \pm 5i\sqrt{3}$ **84.** $\frac{1}{5}, \frac{-1 \pm i\sqrt{3}}{10}$ **85.** $\sqrt{7}$ (multiplicity 2), $-\sqrt{7}$ (multiplicity 2) **86.** $\frac{2\sqrt{5}}{5}$ (multiplicity 2), $-\frac{2\sqrt{5}}{5}$ (multiplicity 2) **87.** $\pm\frac{1}{3}, \pm\frac{1}{3}i$ **88.** x^6 **89.** p^5q^5 **90.** 2^9 or 512 **91.** 3^3 or 27

Lesson 6-4 pp. 381–388

Got It? **1. a.** 8 **b.** 11 **c.** 6 **2. a.** $\frac{\sqrt[8]{w^3}}{w}, \sqrt[5]{w}$ **b.** $x^{\frac{3}{4}}, y^{\frac{4}{5}}$ **c.** If m is negative, a is in the denominator and $\frac{1}{a}$ is undefined when $a = 0$. **3. a.** The length of a Venusian year is about 0.61 Earth years. **b.** The length of a Jovian year is about 12.76 Earth years. **4. a.** $\sqrt[4]{27}$ **b.** $\sqrt[6]{x^5}$ **c.** $\sqrt[6]{16{,}807}$ **5. a.** $\frac{1}{8}$ **b.** 8 **c.** $\frac{1}{2187}$ **6. a.** $\frac{1}{2x^5}$ **b.** $27x\sqrt[8]{x^4y^3}$

Lesson Check **1.** 5 **2.** 5 **3.** $\frac{1}{125}$ **4.** $\frac{1}{128}$ **5.** $\sqrt[4]{11^3}$ **6.** $\frac{\sqrt{x}}{x}$ **7.** $(1 + \sqrt{2})$ or any nonzero number times $(1 + \sqrt{2})$ **8.** error in third line, second term; $5(5^{\frac{1}{2}}) = 5^{\frac{3}{2}}$. The third and fourth lines should be: $\begin{matrix} 20 - 5^{\frac{3}{2}} \\ 20 - 5\sqrt{5} \end{matrix}$ **9.** $(-64)^{\frac{1}{3}} = \sqrt[3]{-64} = -4$ and $-64^{\frac{1}{3}} = -\sqrt[3]{64} = -4$; $(-64)^{\frac{1}{2}} = \sqrt{-64}$, is not a real number, but $-64^{\frac{1}{2}} = -\sqrt{64} = -8$ is a real number.

Exercises **11.** 3 **13.** 10 **15.** $7\sqrt{3}$ **17.** 3 **19.** $\sqrt[6]{x}$ **21.** $\sqrt[7]{x^2}$ or $(\sqrt[7]{x})^2$ **23.** $\frac{1}{\sqrt[8]{y^9}}$ or $\frac{1}{(\sqrt[8]{y})^9}$ **25.** $\sqrt{x^3}$ or $(\sqrt{x})^3$ **27.** $(-10)^{\frac{1}{2}}$ **29.** $(7x)^{\frac{3}{2}}$ **31.** $a^{\frac{2}{3}}$ **33.** $c^{\frac{1}{2}}$ **35.** ≈72.8 m **37.** ≈7.9 m **39.** $\sqrt[12]{6^7}$ **41.** $\sqrt[10]{5^7}$ **43.** $\frac{\sqrt[6]{4^5}}{4}$ **45.** $\frac{\sqrt[6]{6^5}}{6}$ **47.** 4 **49.** 4 **51.** $\frac{1}{16}$ **53.** 64 **55.** $\frac{1}{x^2}$ **57.** $\frac{x^{\frac{1}{3}}}{3x}$ **59.** $-\frac{3}{x^3}$ **61.** $\frac{y^4}{x^3}$ **63.** $\frac{1}{x}$ **65.** x^3y^9 **67.** about 78%; 61%; 37% **69.** -7 **71.** 64 **73.** 2,097,152 **75.** $-\frac{1}{81}$ **77.** 125 **79.** $x^{\frac{1}{2}}$ **81.** $x^{\frac{3}{10}}$ **83.** $x^{\frac{1}{6}}y^{\frac{1}{4}}$ **85.** $\frac{4x^7}{9y^9}$ **87.** $\frac{2x^2}{3y^3}$ **89. a.** $(\sqrt{x})^4 = \sqrt{x} \cdot \sqrt{x} \cdot \sqrt{x} \cdot \sqrt{x} = x \cdot x = x^2$, so $\sqrt[4]{x^2} = \sqrt{x}$. **b.** $\sqrt[4]{x^2} = (x^2)^{\frac{1}{4}} = x^{\frac{2}{4}} = x^{\frac{1}{2}} = \sqrt{x}$ **102.** $4\sqrt[3]{3}$ **103.** $21\sqrt{2}$ **104.** $1 + 3\sqrt{5}$ **105.** -7 **106.** $-8\sqrt{3}$ **107.** $9\sqrt[4]{2}$ **108.** $4x(x^2 - 2x + 4)$ **109.** $(x + 2)^2$ **110.** $(x - 9)^2$ **111.** $(4a - 3b)(4a + 3b)$

112. $(5x - 4y)^2$ **113.** $(3x + 8)^2$ **114.** $-3, 2$ **115.** $7, -2$ **116.** $-\frac{3}{2}, 1$ **117.** $-\frac{1}{3}, 2$ **118.** $-\frac{5}{2}, \frac{1}{2}$ **119.** $-\frac{2}{3}, \frac{3}{2}$

Lesson 6-5 pp. 390–397

Got It? **1.** 6 **2.** 5, −11 **3.** 37,500,000 m^3 **4. a.** 10 **b.** when you raise each side of an equation to a power **5.** 9

Lesson Check **1.** 12 **2.** 27 **3.** $\frac{1}{25}$ **4.** 4 **5.** 1 **6.** 512 **7.** 3; The solution of 3 yields a negative value for $x - 6$, but the right side of the equation $(\sqrt{3(3)})$ cannot be negative. **8.** Solving square root equations is different from solving absolute value equations in that you use a different technique to isolate the variable. In square root equations, you square each side. In absolute value equations, you write two new equations and solve both. Solving square root equations is similar to solving absolute value equations in that both can introduce extraneous solutions.

Exercises **9.** 16 **11.** 22 **13.** 5 **15.** 4 **17.** $\frac{2}{3}$ **19.** −29, 25 **21.** 78 **23.** 0 **25.** about 4 in. **27.** 1 **29.** 3 **31.** −3, −4 **33.** 1 **35.** 3 **37.** 1 **39.** −2 **41.** 1 **43.** 5 **45.** $10\sqrt[4]{3}$ cm or about 13.16 cm **47.** 5 **49.** 8 **51.** 5 **53.** 1 **55.** 9, −7 **57.** 9 **59.** $x = 4$ is a solution, but $x = 1$ is an extraneous solution. **61.** Answers may vary. Sample: $\sqrt{x - 3} = \sqrt{3x + 5}$ **63.** C **65.** 0, 2 **67.** 0 **77.** 3 **78.** 2 **79.** 625 **80.** 512 **81.** $\frac{1}{1000}$ **82.** 16 **83.** 125 **84.** $6\sqrt{2}$ **85.** 3, 4 **86.** 3, 5 **87.** −5, −4 **88.** $-2, -\frac{2}{3}$ **89.** $-\frac{1}{3}, -\frac{4}{3}$ **90.** $-2, -\frac{3}{4}$
91. domain: {0, 2, 4}, range: {−5, −3, −1}; yes
92. domain: {−1, 0, 1}, range: {2, 0, 1}; yes
93. domain: {−2, 0, 1}, range: {−2, 0, 1}; yes
94. domain: {3, 4, 5}, range: {−1}; yes
95. domain: {0, 1, 2}, range: {0, 1, 2}; no
96. domain: {0}, range: {−2, 0, 2}; no

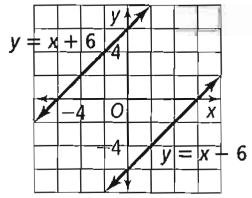
Lesson 6-6 pp. 398–404

Got It? **1.** $(f + g)(x) = 2x^2 + x + 5$, domain: all real numbers; $(f - g)(x) = 2x^2 - x + 11$, domain: all real numbers **2.** $(f \cdot g)(x) = 9x^3 - 30x^2 - 23x - 4$, domain: all real numbers; $\left(\frac{f}{g}\right)(x) = x - 4$, domain: all real numbers except $x = -\frac{1}{3}$ **3.** 4 **4.** Let $D(x)$ = cost after applying the 15% store discount, $E(x)$ = cost after applying the 20% employee discount, and x = cost of items. Then $D(x) = 0.85x$ and $E(x) = 0.80x$. **a.** $(E \circ D)(x) = 0.68x$ **b.** $(D \circ E)(x) = 0.68x$ **c.** The total discounts are the same.

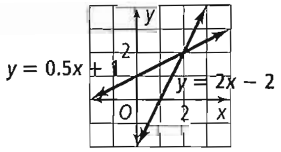
Lesson Check **1.** $3x^3 - 2x^2 + 3x - 2$ **2.** $-x^2 + 3x - 3$ **3.** $3x^2 + 1$ **4.** $x^2 + 3x - 1$ **5.** $x^2 - 3x + 3$ **6.** $-x^2 + 3x - 3$ **7.** Answers may vary. Sample: $f(x) = 3x^2 + 1$, $g(x) = 2x + 1$; $(f \circ g)(x) = 12x^2 + 12x + 4$; $(g \circ f)(x) = 6x^2 + 3$ **8.** Answers may vary. Sample: $f(x) = 2x$, $g(x) = 0.5x$; $f(g(x)) = x$

Exercises **9.** $x^2 + 7x + 5$; domain: all real numbers
11. $x^2 - 7x - 5$; domain: all real numbers
13. $\frac{7x + 5}{x^2}$; domain: all real numbers except $x = 0$
15. $2 - x + \frac{1}{x}$; domain: all real numbers except $x = 0$
17. $\frac{1}{x} + x - 2$; domain: all real numbers except $x = 0$
19. $2x - x^2$; domain: all real numbers except $x = 0$
21. $2x^2 + 2x - 4$; domain: all real numbers
23. $-2x^2 + 2$; domain: all real numbers
25. $2x + 3$; domain: all real numbers except $x = 1$
27. 8 **29.** 20 **31.** 8 **33.** $4a$ **35.** $4a^2 + 4$ **37.** 25 **39.** 9 **41.** 0.25 **43.** $a^2 - 3$ **45. a.** $f(x) = 0.95x$ **b.** $g(x) = x - 200$ **c.** $1225 **d.** $1235
47. $x^2 - x + 7$; domain: all real numbers
49. $x^2 - 5x - 3$; domain: all real numbers
51. $-x^2 + 5x + 13$; domain: all real numbers
53. $4x^2 - 14x + 3$; domain: all real numbers
55. $2x^3 - x^2 - 11x + 10$; domain: all real numbers
57. $\frac{2x + 5}{x^2 - 3x + 2}$; domain: all real numbers except $x = 1$ and 2 **59.** Substitute $5995x$ for y; $79,850 **61. a.** $g(x)$ is the bonus earned when x is the amount of sales over $5000. $h(x)$ is the excess sales over $5000. **b.** $(g \circ h)(x)$; you first need to find the excess sales over $5000 to calculate the bonus. **63.** 1 **65.** 0 **67.** 8 **69.** −2 **71. a.** ≈1963; The area after 2 seconds is about 1963 in.^2 **b.** ≈7854 in.^2 **73.** $x - 2$; $x - 2$ **75.** $x - 3$; $x - 6$ **77.** $\frac{x^2 + 5}{2}$; $\frac{x^2 + 10x + 25}{4}$ **90.** 1 **91.** −3 **92.** 4 **93.** 3 **94.** 2 **95.** 3 **96.** $x^8 + 32x^7 + 448x^6 + 3584x^5 + 17{,}920x^4 + 57{,}344x^3 + 114{,}688x^2 + 131{,}072x + 65{,}536$ **97.** $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$ **98.** $16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$ **99.** $128x^7 - 1344x^6y + 6048x^5y^2 - 15{,}120x^4y^3 + 22{,}680x^3y^4 - 20{,}412x^2y^5 + 10{,}206xy^6 - 2187y^7$ **100.** $59{,}049 - 65{,}610x + 29{,}160x^2 - 6480x^3 + 720x^4 - 32x^5$ **101.** $1024x^5 - 1280x^4y + 640x^3y^2 - 160x^2y^3 + 20xy^4 - y^5$ **102.** $x^8 + 4x^7 + 6x^6 + 4x^5 + x^4$ **103.** $x^{12} + 12x^{10}y^3 + 60x^8y^6 + 160x^6y^9 + 240x^4y^{12} + 192x^2y^{15} + 64y^{18}$

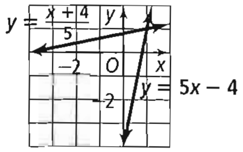
104. no solution



105. (2, 2)



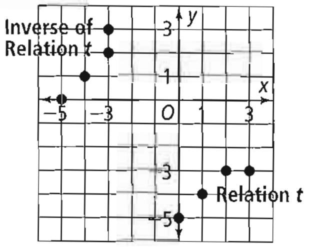
106. (1, 1)



Lesson 6-7 pp. 405–412

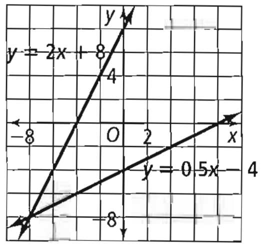
Got It?

1. a.



b. *t* is a function; the inverse of *t* is not a function; there are 2 *y*-values for one *x*-value **2.** $y = \frac{x}{2} - 4$

3.



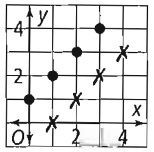
4. a. domain: all real numbers; range: all real numbers **b.** $g^{-1}(x) = -\frac{1}{4}x + \frac{3}{2}$ **c.** domain: all real numbers; range: all real numbers **d.** Yes; for each *x* in the domain of g^{-1}, there is only one value of *y* in the range. **5.** $v = \sqrt{19.6d}$; 21.7 m/s **6. a.** $g^{-1}(x) = \frac{4 - 2x}{x}$ **b.** 0 is not in the domain of g^{-1} so $(g \circ g^{-1})(0)$ does not exist. **c.** 0

Lesson Check **1.** $f^{-1}(x) = \frac{x - 3}{4}$; yes **2.** $f^{-1}(x) = \pm\sqrt{x + 1}$; no **3.** $f^{-1}(x) = -1 \pm \sqrt{x}$; no **4. a.** $h^{-1}(x) = -\frac{1}{x} - 2$ **b.** −2.25 **c.** 0 **5.** no; yes **6.** 2, 5 **7.** Answers may vary. Samples: $f(x) = 2x + 1$ and $g(x) = x - 2$; $f(x) = x^2$ and $g(x) = x + 1$

Exercises

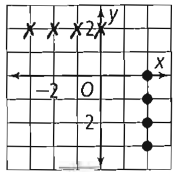
9.

x	0	1	2	3
y	1	2	3	4



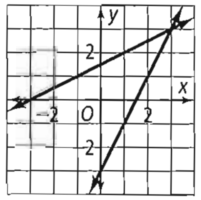
11.

x	2	2	2	2
y	−3	−2	−1	0

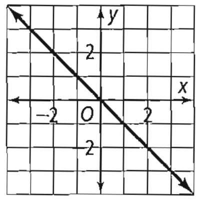


13. $y = \frac{1}{2}x + \frac{1}{2}$; yes **15.** $y = \pm\sqrt{\frac{5 - x}{2}}$; no **17.** $y = \pm\sqrt{\frac{x + 5}{3}}$; no **19.** $y = \frac{4 \pm \sqrt{x}}{3}$; no

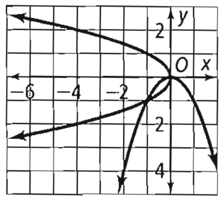
21.



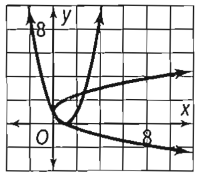
23.



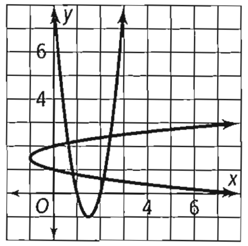
25.



27.



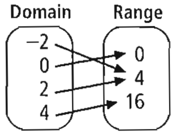
29.

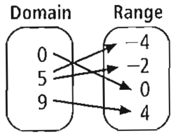


31. $f^{-1}(x) = x^2 + 5$, $x \geq 0$, domain of f: $x \geq 5$, range of f: $y \geq 0$, domain of f^{-1}: $x \geq 0$, range of f^{-1}: $y \geq 5$; f^{-1} is a function **33.** $f^{-1}(x) = \frac{3 - x^2}{2}$, $x \geq 0$, domain of *f*: $x \leq \frac{3}{2}$, range of *f*: $y \geq 0$, domain of f^{-1}: $x \geq 0$, range of f^{-1}: $y \leq \frac{3}{2}$; f^{-1} is a function **35.** $f^{-1}(x) = \pm\sqrt{1 - x}$, domain of *f*: all real numbers, range of *f*: $y \leq 1$, domain of f^{-1}: $x \leq 1$, range of f^{-1}: all real numbers; f^{-1} is not a function **37. a.** $r = \sqrt[3]{\frac{3V}{4\pi}}$; yes **b.** 20.29 ft **39.** −10 **41.** *d* **43.** $f^{-1}(x) = \pm\sqrt[4]{x}$; no **45.** $f^{-1}(x) = \pm\sqrt{\frac{2x + 8}{3}}$; no **47.** $f^{-1}(x) = \frac{x^2 - 6x + 10}{2}$, $x \geq 3$; yes **49.** −1 **51.** $f^{-1}(x) = x^2$, $x \leq 0$, domain of *f*: $x \geq 0$, range of *f*: $y \leq 0$, domain of f^{-1}: $x \leq 0$, range of f^{-1}: $y \geq 0$; f^{-1} is a function **53.** $f^{-1}(x) = 3 - x^2$, $x \geq 0$, domain of *f*: $x \leq 3$, range of *f*: $y \geq 0$, domain of f^{-1}: $x \geq 0$, range of f^{-1}: $y \leq 3$; f^{-1} is a function **55.** $f^{-1}(x) = \pm\sqrt{2x}$, domain of *f*: all real numbers, range of *f*: $y \geq 0$, domain of f^{-1}: $x \geq 0$, range of f^{-1}: all real numbers; f^{-1} is not a function **57.** $f^{-1}(x) = \pm\sqrt{x} + 4$, domain of *f*: all real numbers, range of *f*: $y \geq 0$, domain of f^{-1}: $x \geq 0$, range of f^{-1}: all real numbers; f^{-1} is not a function **59.** $f^{-1}(x) = \pm\frac{1}{\sqrt{x}} - 1$, domain of *f*: $x \neq -1$, range of *f*: $y > 0$, domain of f^{-1}: $x > 0$, range of f^{-1}: $y \neq -1$; f^{-1} is not a function **61.** $f^{-1}(x) = \left(\frac{3}{x}\right)^2$, $x \geq 0$, domain of

f: $x > 0$, range of f: $y > 0$, domain of f^{-1}: $x > 0$, range of f^{-1}: $y > 0$; f^{-1} is a function

63. a–b. Answers may vary. Samples are given.

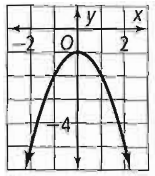
a.


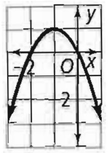
b.


65. $h = s\sqrt{2}$; $s = \frac{h\sqrt{2}}{2} = 3\sqrt{2} \approx 4.2$ in. **67. a.** The horizontal line test tells you if there is more than one x-value for every y-value. Since the graph of f^{-1} interchanges the x and y values of f, if f passes the horizontal line test, f^{-1}will pass the vertical line test and it will be a function. **b.** no **79.** $2x + 7$ **80.** $-x - 10$

81. $-\frac{3}{2}x + 11$ **82.** $2x^2 + 28x$ **83.** 32 **84.** $2x + 28$

85. −2 **86.** no real root **87.** 3 **88.** −3 **89.** −3
90. 0.4 **91.** 30 **92.** 0.05

93.


94.


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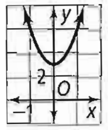
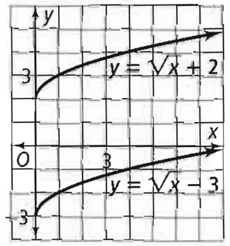


Lesson 6-8 pp. 414–420

Got It?

1.


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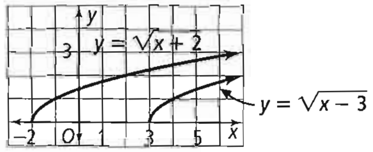

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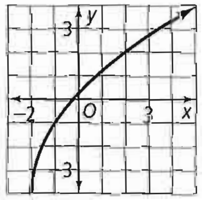
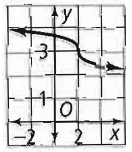
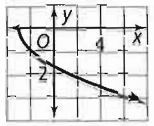


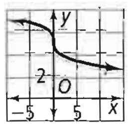
4. 1999

5.


6. a. $y = \sqrt[3]{8x + 32} - 2$ is the graph of $y = 2\sqrt[3]{x}$ translated 4 units to the left and 2 units down.
b. $y = 9|x + 2|$; the graph of $y = 9|x + 2|$ is the graph of $y = 9|x|$ translated 2 units to the left; You are rewriting the function so that x has a coefficient of 1.

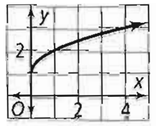
Lesson Check

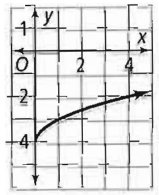
1.


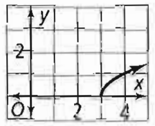
2.


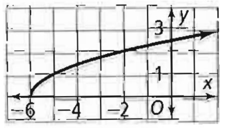
3. $y = 2\sqrt{x - 1}$; the graph of $y = 2\sqrt{x}$ translated 1 unit to the right **4.** $y = 2\sqrt[3]{x + 2}$; the graph of $y = 2\sqrt[3]{x}$ translated 2 units to the left **5.** When $|a| < 1$, a will vertically compress $y = a\sqrt{x}$ and when $|a| > 1$, a will vertically stretch $y = a\sqrt{x}$; this is similar to its effect on other functions. **6.** $g(x)$ is the reflection of $f(x)$ across the x-axis and again across $x = -1$.

Exercises

7.


9.


11.


13.


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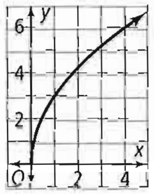

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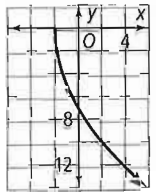

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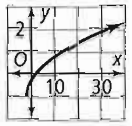

21. 147 **23.** −1

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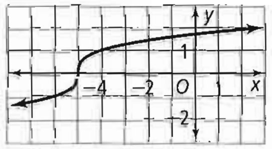

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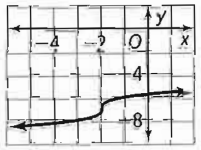

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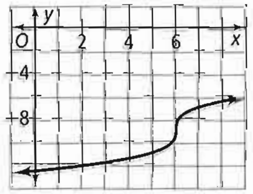
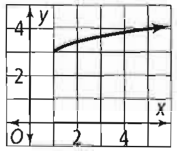


31. $y = 3\sqrt{x - 1}$; the graph of $y = 3\sqrt{x}$ translated 1 unit to the right **33.** $y = -4\sqrt{x + 4}$; the graph of $y = -4\sqrt{x}$ translated 4 units to the left
35. $y = 5\sqrt{x + 5} - 3$; the graph of $y = 5\sqrt{x}$ translated 5 units to the left and 3 units down
37. ≈16.44 ft; ≈29.22 ft

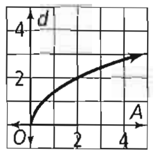
39. 

domain: $x \geq 1$, range: $y \geq 3$

41. a. $y = \sqrt{x-2} - 2$ **b.** domain: $x \geq 2$, range: $y \geq -2$ **c.** No; the function pairs the number 3 with the number -1, which is not a non-negative real number. **43. a.** $y = 5\sqrt{x-4} - 1$; the graph of $y = 5\sqrt{x}$, translated 4 units to the right and 1 unit down

45. $y = -2\sqrt[3]{x - \frac{1}{4}}$; the graph of $y = -2\sqrt[3]{x}$, translated $\frac{1}{4}$ unit to the right **47.** $y = 10 - \frac{1}{3}\sqrt[3]{x+3}$; the graph of $y = -\frac{1}{3}\sqrt[3]{x}$, translated 3 units to the left and 10 units up

49. $\frac{1}{3}$ **51.** 0, 1, 9

53. a. 

b. $15\sqrt{2}$ in. ≈ 21.2 in.

64. $f^{-1}(x) = \frac{3(x+3)}{2}$; yes

65. $f^{-1}(x) = (x+4)^2 - 3, x \geq -4$; yes

66. $f^{-1}(x) = \frac{-1 \pm \sqrt{x}}{2}$; no **67.** $\frac{x\sqrt{3xy}}{y}$ **68.** $\frac{\sqrt{6xy}}{2y}$

69. $\frac{\sqrt[3]{9xy^2}}{3y}$ **70.** $\frac{\sqrt[5]{48x^3y^4}}{2y}$ **71.** $\frac{9 \pm \sqrt{21}}{2}$

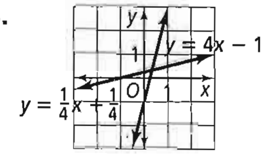
72. $\frac{-3 \pm 3\sqrt{5}}{2}$ **73.** $\frac{-1 \pm \sqrt{61}}{10}$ **74.** 8 **75.** 16 **76.** 2

Chapter Review pp. 422–426

1. radicand **2.** radical functions **3.** rational exponent **4.** composite function **5.** 5 **6.** 0.7 **7.** −2 **8.** −2 **9.** $9|x|$ **10.** $4x^2$ **11.** $2|x^3|$ **12.** $0.2x$ **13.** $\frac{x^2}{2}$ **14.** $5x^2y^3$ **15.** 3 **16.** −7 **17.** 4 **18.** $4x^2$ **19.** $30y$ **20.** 4 **21.** $3xy$ **22.** $\frac{3|x|}{y^2}$ **23.** $\frac{2\sqrt{3}}{3}$ **24.** $\frac{\sqrt{3x}}{8}$ **25.** $\frac{y\sqrt[3]{150x}}{10x^2}$ **26.** $22\sqrt{3}$ **27.** $26\sqrt{5x}$ **28.** $x\sqrt[3]{2}$ **29.** $14 + 7\sqrt{2}$ **30.** −6 **31.** $100 + 10\sqrt{6} - 10\sqrt{3} - 3\sqrt{2}$ **32.** $\frac{5 + 2\sqrt{5}}{5}$ **33.** $\frac{9 + 3\sqrt{2}}{7}$ **34.** 5 **35.** 3 **36.** 4 **37.** 25 **38.** x **39.** $-2y^3$ **40.** $81x^2y^4$ **41.** $\frac{1}{x^3y^6}$ **42.** $\frac{1}{x}$ **43.** x^3y^6 **44.** −1 **45.** 15 **46.** 5 **47.** 10, −8 **48.** 2, −1 **49.** −2 **50.** 0, 16 **51.** 0, 36 **52.** 9.05 W **53.** $x^2 + x - 20$; domain: all real numbers **54.** $x^2 - x - 12$; domain: all real numbers **55.** $x^3 - 4x^2 - 16x + 64$; domain: all real numbers **56.** $x + 4$; domain: all real numbers except $x = 4$ **57.** 50 **58.** 5 **59.** 23 **60.** $5a^2 + 3$ **61.** $D(C(x)) = 0.5x - 0.5$, $C(D(x)) = 0.5x - 1$; use the coupon after the store discount.

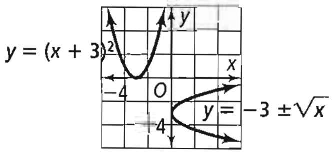
62. $f^{-1}(x) = \pm\sqrt{\frac{x+8}{2}}$; no **63.** $f^{-1}(x) = 5 - \frac{1}{3}x$; yes

64. $f^{-1}(x) = x^2 - 6, x \geq 0$; yes **65.** $f^{-1}(x) = \frac{3 \pm \sqrt{x}}{2}$; no

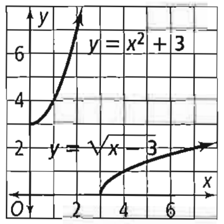
66. 

domain of f: all real numbers, range of f: all real numbers, domain of f^{-1}: all real numbers, range of f^{-1}: all real numbers

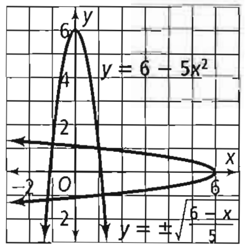
67. domain of f: all real numbers, range of f: $y \geq 0$; domain of f^{-1}: $x \geq 0$, range of f^{-1}: all real numbers



68. domain of f: $x \geq 3$, range of f: $y \geq 0$, domain of f^{-1}: $x \geq 0$, range of f^{-1}: $y \geq 3$

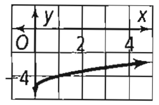


69. domain of f: all real numbers, range of f: $y \leq 6$, domain of f^{-1}: $x \leq 6$, range of f^{-1}: all real numbers

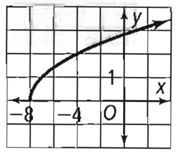


70. $s = \sqrt[3]{V}$; 4 ft

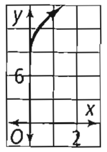
71. domain: $x \geq 0$, range: $y \geq -5$



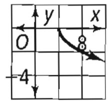
72. domain: $x \geq -8$, range: $y \geq 0$



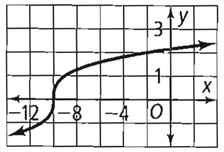
73. domain: $x \geq 0$, range: $y \geq 9$



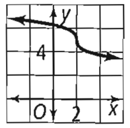
74. domain: $x \geq 4$, range: $y \leq 0$



75. domain: all real numbers, range: all real numbers

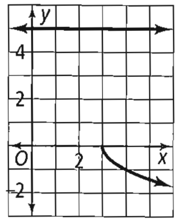


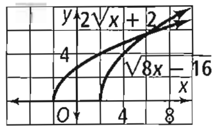
76. domain: all real numbers, range: all real numbers



77. $y = 3\sqrt{x - 3} + 4$; the graph of $y = 3\sqrt{x}$ translated 3 units to the right and 4 units up **78.** $y = -6\sqrt{x - 4}$; the graph of $y = -6\sqrt{x}$ translated 4 units to the right **79.** $y = 2\sqrt[3]{x + 3}$; the graph of $y = 2\sqrt[3]{x}$ translated 3 units to the left **80.** $y = \frac{1}{2}\sqrt{x - 4} + 6$; the graph of $y = \frac{1}{2}\sqrt{x}$ translated 4 units to the right and 6 units up

81. no solution **82.** 6

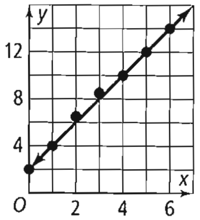




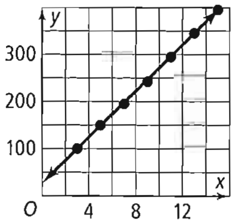
Chapter 7

Get Ready! p. 431 **1.** 0.1; 10; 1000 **2.** $\frac{4}{9}$; 1; $\frac{9}{4}$ **3.** $-\frac{1}{625}$; $-\frac{1}{25}$; −1 **4.** $-\frac{1}{3}$; −1; −3

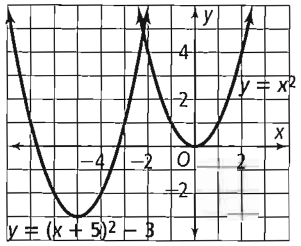
5. ; $y = 2x + 2$



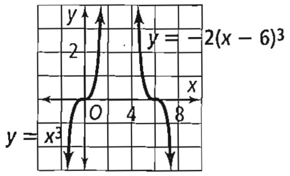
6. ; $y = 25x + 25$



7. $y = x^2$



8. $y = x^3$

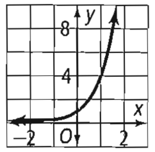


9. x^2 **10.** $16x^4$ **11.** $y = \pm\sqrt{\frac{10 - x}{2}}$; no **12.** $y = -4 + \sqrt[3]{x + 1}$; yes **13.** decrease **14.** exponential **15.** no

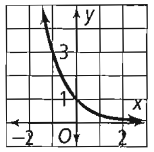
Lesson 7-1 pp. 434–441

Got It?

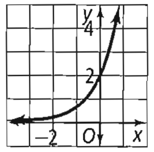
1. a.



b.



c.

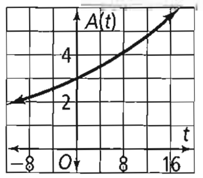


d. domain: all real numbers, range: $y > 0$; y-intercept: $(0, a)$ where $y = ab^x$

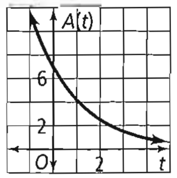
2. a. exponential growth; 3 **b.** exponential decay; 11 **c.** exponential growth; 2000 **3.** \$593.84 **4. a.** after 8 yrs **b.** after 11 yrs; then the account contains \$1710.34. **5. a.** ≈3 **b.** No; the function is asymptotic to the x-axis.

Lesson Check **1.** decay; 10 **2.** growth; 0.75 **3.** growth; 1 **4.** decay; 1

5.



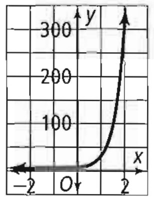
6.



7. If $a > 0$ and $b > 1$, then the function represents exponential growth; if $a > 0$ and $0 < b < 1$, then the function represents exponential decay. **8. a.** quadratic; degree 2 with $3x^2$ as the leading term **b.** exponential; the equation is of the form $y = ab^x$ **c.** linear; degree 1 with x as the leading term **d.** exponential; the equation is of the form $y = ab^x$ **9.** $0.3 < 1$, so 0.3 is the decay factor

Exercises

11.



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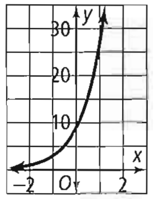

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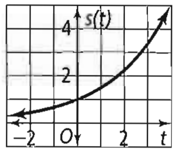

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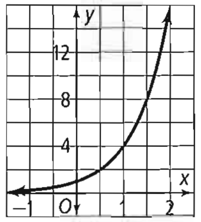
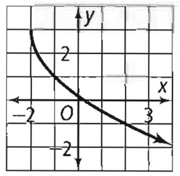


19. exponential decay; 2 **21.** exponential decay; 0.8 **23.** exponential growth; 0.45 **25.** exponential decay; 1 **27.** $y = 120{,}000(1.012)^x$; 143,512 **29. a.** $y = 72(\frac{1}{2})^x$ **b.** 2.25 in. **31. a.** about 5.6% **b.** about 0.0017% **33.** Answers may vary. Sample: $y = 59.5(0.6)^x$ **35.** 6 **37.** 0.45 **39.** 0.999 **41.** 2

51.



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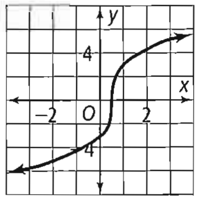

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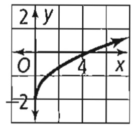
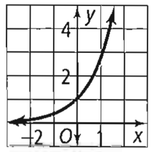


54. $(2 + 3x)(4 - 6x + 9x^2)$ **55.** $(3x - 1)(x + 4)$ **56.** $(4x - 5)(4x - 5)$ **57.** $(1, -1)$ **58.** $(0, 0)$ **59.** $(3, -3, 9)$

60.



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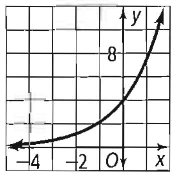

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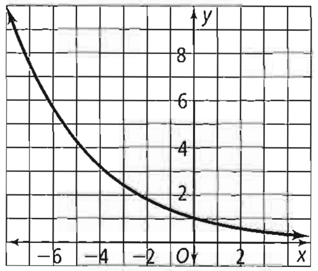
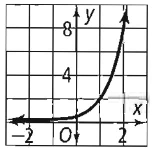


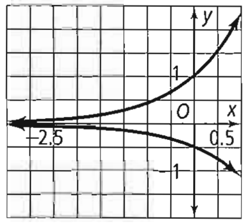
63.



Lesson 7-2 pp. 442–450

Got It?

1. reflects across the x-axis; compresses by a factor of 0.5

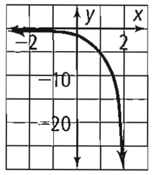


2. a. translate 2 units to the left; the y-intercept becomes 16 **b.** Stretch the graph of $y = (0.25)^x$ by a factor of 5 and translate the graph of $y = 5 \cdot 0.25^x$ 5 units up
3. a. about 31.9 min **b.** No; a hot coffee cannot cool below room temperature. So, to use exponential data, it is important to translate the data by 68 units.
4. $e^8 \approx 2980.957987$; three methods: use the e^x key, $x = 8$; graph $y = e^x$ and find y for $x = 8$; or use the table of values for $y = e^x$ and find y for $x = 8$
5. about $4475

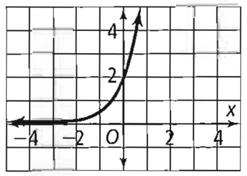
Lesson Check **1.** stretch by a factor of 2 and reflection across the x-axis **2.** compress by a factor of $\frac{1}{2}$ **3.** translate 5 units to the right **4.** translate 3 units up **5.** yes **6.** no; $2000e^{0.05t} \neq 1000(e^{0.04t} + e^{0.06t})$

Exercises

7.



9.



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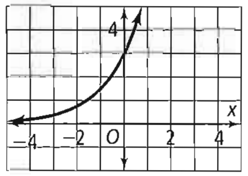

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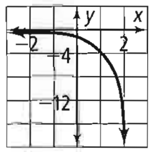

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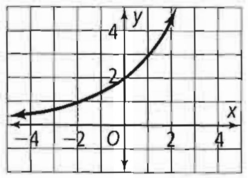

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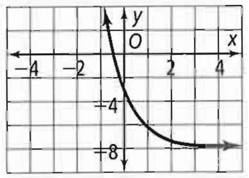
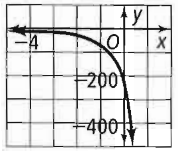
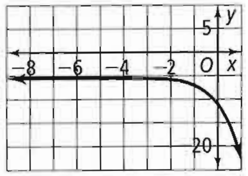


19.

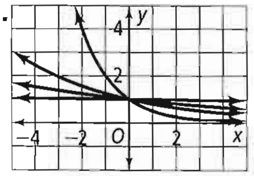


21.



23. 403.4288 **25.** 1 **27.** 15.1543 **29.** $448.30 **31.** $6168.41 **33.** graph is a shift of the parent function 2 units to the left and 1 unit up

35.

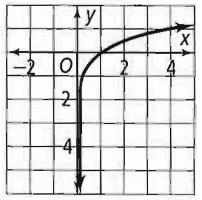


As the value of *b* approaches 1, the graph comes closer to being a straight line.

37. $y = 24\left(\frac{1}{2}\right)^{\frac{1}{5730}x}$; 0.64 mg **39.** $y = -3^x$; $y = -3^{x-8} + 2$ **41.** $y = -3\left(\frac{1}{3}\right)^x$; $y = -3\left(\frac{1}{3}\right)^{x+15} - 1$ **51.** exponential growth; 23 **52.** exponential growth; 3 **53.** exponential decay; 2 **54.** exponential growth; 5 **55.** $6\sqrt{5}$ **56.** $-\sqrt[3]{4}$ **57.** $5(\sqrt{3} + \sqrt{5})$ **58.** $2(\sqrt[4]{2} + \sqrt[4]{8})$ **59.** $\sqrt{3}$ **60.** $11\sqrt{7}$ **61.** $f^{-1}(x) = \frac{x+1}{4}$; yes **62.** $f^{-1}(x) = x^{\frac{1}{7}}$; yes **63.** $f^{-1}(x) = \left(\frac{x-1}{5}\right)^{\frac{1}{3}}$; yes

Lesson 7-3 pp. 451–458

Got It? 1. a. $\log_6 36 = 2$ **b.** $\log_{\frac{2}{3}} \frac{8}{27} = 3$ **c.** $\log_3 1 = 0$ **2. a.** 3 **b.** $\frac{5}{2}$ **c.** $-\frac{5}{6}$ **3.** ≈16 times **4. a.** domain: $x > 0$; range: all real numbers; no *y*-intercept; vertical asymptote: $x = 0$



b.

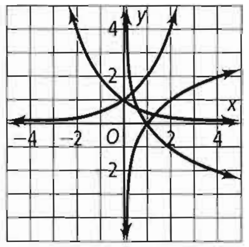
x	$2^y = x$	y
−1	$2^y = -1$	undefined
0	$2^y = 0$	undefined
1	$2^y = 1$	0
2	$2^y = 2$	1

5. a. translates the graph of the parent function 3 units to the right and 4 units up; The asymptote changes from $x = 0$ to $x = 3$. The domain changes from $x > 0$ to $x > 3$. The range remains all real numbers. **b.** stretch the graph of the parent function by a factor of 5; The asymptote, domain, and the range remain the same.

Lesson Check 1. $\log_5 25 = 2$ **2.** $\log_4 64 = 3$ **3.** $\log_3 243 = 5$ **4.** $\log_5 25 = 2$ **5.** 3 **6.** 1 **7.** 2 **8.** −2 **9. a.** no **b.** yes **c.** yes **d.** no **10.** Choose a few points on the graph of $y = 6^x$, reverse their coordinates, and plot them. **11.** $y = \log_2 (x + 4)$ translates the graph of $y = \log_2 x$ 4 units to the left. Asymptote changes from $x = 0$ to $x = -4$. Domain changes from $x > 0$ to $x > -4$. Range remains the same.

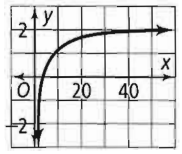
Exercises 13. $\log 1000 = 3$ **15.** $\log \frac{1}{10} = -1$ **17.** $\log_{\frac{1}{2}} 4 = -2$ **19.** $\log 0.01 = -2$ **21.** $\frac{1}{2}$ **23.** $\frac{3}{2}$ **25.** $\frac{1}{2}$ **27.** 2 **29.** 1 **31.** 3 **33.** The earthquake in Chile was about 39.81 times more intense. **35.** The earthquake in Missouri was about 10 times more intense.

36–39.

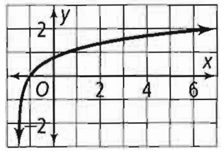


41. translate the graph 2 units to the right **43.** translate the graph 2 units to the left and 1 unit down **45.** $\approx 3.16 \times 10^{-9}$ **47.** $10^{-4} = 0.0001$ **49.** $4^0 = 1$ **51.** $2^{-1} = \frac{1}{2}$ **53.** $10^1 = 10$ **55.** −2 **57.** 7 **59.** First rewrite $y = \log_1 x$ as $1^y = x$. For any real number y, $x = 1$. **61.** $y = 4^x$ **63.** $y = 10^x$ **65.** $y = 10^x - 1$ **67.** $y = 2^{x-2}$

69.

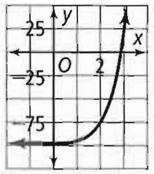


71.

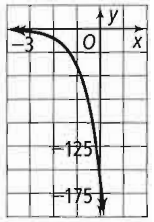


73. domain $x > 0$, range: all real numbers **75.** domain $x > 3$, range: all real numbers **77.** $4 = \log_3 (81)$ **79.** $8 = \log_6 (a + 1)$

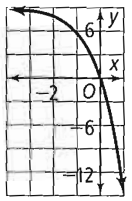
89.



90.



91.



92. $(2x - 3)(2x - 1)$ **93.** $4(b - 5)(b + 5)$ **94.** $(5x - 2)(x + 3)$ **95.** 2 **96.** 256 **97.** $\frac{1}{4}$ **98.** 12

Lesson 7-4 pp. 462–468

Got It? 1. a. $\log_4 15x^2$ **b.** 1 **2. a.** $\log_3 2 + 3\log_3 5 - \log_3 37$ **b.** $2 + 5\log_3 x$ **3. a.** $\frac{5}{3}$ **b.** ≈2.085 **4.** Substance B; log 2; $-\log[H^+{}_B] + \log[H^+{}_B] = \log 2$
Lesson Check 1. $\log_4 16$ **2.** $\log_6 6$ **3.** $\log_3 x - \log_3 y$ **4.** $2\log m + 5\log n$ **5.** $\frac{1}{2}\log_2 x$ **6. a.** Product Prop. and Power Prop. **b.** Quotient Prop. **7.** 0.00001 **8.** Answers may vary. Samples: $\log 150 = \log 25 + \log 6$
Exercises 9. log 14 **11.** log 972 **13.** $\log \frac{m^4}{n}$ **15.** $\log_6 5x$ **17.** $\log_3 32xy$ **19.** $2 + \log_7 x + \log_7 y + \log_7 z$ **21.** $2\log a$ **23.** $2\log_3 2 + 2\log_3 x$ **25.** $2\log a + 3\log b - 4\log c$ **27.** $1 + \frac{1}{2}\log_8 3 + \frac{5}{2}\log_8 a$ **29.** $1 + 4\log m - 2\log n$ **31.** ≈1.2 **33.** ≈1.43 **35.** ≈3.631 **37.** ≈3.183 **39.** −2 **41.** 1 **43.** 2 **45.** Yes, because the loudness of the sound is 102 dB. **47.** The coefficient $\frac{1}{2}$ is missing in $\log_4 s$;

$$\begin{aligned}\log_4 \sqrt{\tfrac{t}{s}} &= \tfrac{1}{2}\log_4 \tfrac{t}{s}\\ &= \tfrac{1}{2}(\log_4 t - \log_4 s)\\ &= \tfrac{1}{2}\log_4 t - \tfrac{1}{2}\log_4 s\end{aligned}$$

49. The log of a product is equal to the sum of the logs. $\log(MN) = \log M + \log N$. **51.** false; $\frac{1}{2}\log_3 3 = \log_3 3^{\frac{1}{2}}$, not $\log_3 \frac{3}{2}$ **53.** false; $\log_b \frac{x}{y} = \log_b x - \log_b y$ **55.** false; $\log_4 7 - \log_4 3 = \log_4 \frac{7}{3}$ not $\log_4 4$. **57.** $\log_x \frac{2\sqrt{y}}{z^3}$ **59.** $\log_b \frac{\sqrt[3]{x^2}\,\sqrt[4]{y^3}}{z^5}$ **61.** $\log s + \frac{1}{2}\log 7 - 2\log t$ **63.** $3\log m - 4\log n + 2\log p$ **65.** $\frac{1}{2}\log_b x + \frac{2}{3}\log_b y - \frac{2}{5}\log_b z$ **67.** $\frac{1}{2}\log(x + 2) + \frac{1}{2}\log(x - 2) - 2\log(x + 3)$ **69.** $\frac{\log 8}{\log 3}$ **71.** $\frac{\log 3.3}{\log 9}$ **73.** A 1.0 magnitude star is about 2.5 times brighter than a 2.0 magnitude star. **84.** $\log_7 49 = 2$ **85.** $\log_8 \frac{1}{4} = -\frac{2}{3}$ **86.** $-3 = \log_5 \frac{1}{125}$ **87.** ±8 **88.** $\frac{64}{7}$ **89.** 2 **90.** $x^3 + 5x^2 - 3x - 15$ **91.** $x^4 + 17x^2 + 16$ **92.** $x^4 - 2x^3 - 2x^2 + 14x - 35$ **93.** 2 **94.** 3 **95.** $\frac{1}{3}$

Lesson 7-5 pp. 469–476

Got It? 1. $\frac{4}{9}$ **2. a.** ≈1.5122 **b.** because the terms cannot be written with a common base **3. a.** ≈0.8588 **b.** ≈1.2114 **4.** ≈13.51 yrs **5.** 1.45 **6.** 200
Lesson Check 1. 2 **2.** ≈3.6439 **3.** 25 **4.** 2000
5. The log bases are not equal.

$$\begin{aligned}\log_2 x &= 2\log_3 9\\ \log_2 x &= \log_3 9^2\\ \log_2 x &= 4\\ x &= 2^4\\ x &= 16\end{aligned}$$

6. Yes; $5^x = 0$ has no solution.
Exercises 7. 3 **9.** 1 **11.** $\frac{4}{5}$ **13.** 2 **15.** 1.5850 **17.** 3 **19.** 0.9534 **21.** 0.2720 **23.** 0.5690 **25.** 4.7027 **27.** 6 **29.** 0.64 **31.** about the yr 2012 **33.** $\frac{\sqrt{10}}{10}$ or about 0.3162 **35.** 10,000 **37.** $\sqrt{10}$ or ≈3.1623 **39.** 2 **41.** $100{,}000\sqrt{5}$ or ≈223,606.8 **43.** $\frac{1}{4}$ **45.** 7 **47. a.** 18.9658 **b.** 18.9658 **c.** Answers may vary. Sample: You don't have to use the Change of Base Formula with the base-10 method, but there are fewer steps with the base-2 method. **49.** ≈7.6 yrs **51.** 3 **53.** 3 **55.** −2 **57.** $-\frac{1}{2}$ **59.** Answers may vary. Sample: $\log x = 1.6$; $x \approx 39.81$ **61.** 143.6 **63. a.** top up: 10^{-5} W/m^2 top down: $10^{-2.5}$ W/m^2 **b.** 99.68% **65.** 625 **67.** 10 **69.** 1.5 **71.** 2.7944 **73.** 500 **75.** $114.\overline{3}$ **77.** $x = y = 2$ **89.** $\log 2 + 3\log x - 2\log y$ **90.** $\log_3 x - \log_3 y$ **91.** $1 + \frac{1}{2}\log_3 x$ **92.** $x^2 - 3x - 1$ **93.** $3x^2 - 3$ **94.** $9x^2 - 1$ **95.** $1, \pm i$ **96.** $\pm 2, \pm 2i$ **97.** $\pm\sqrt{3}, \pm\sqrt{2}$ **98.** $\log_2 3$ **99.** $\log 3x^4$ **100.** $\log_7 \frac{32}{y^2}$

Lesson 7-6 pp. 478–483

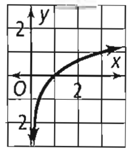
Got It? 1. a. ln 175 **b.** $\ln \frac{x}{4}$ **c.** $\ln 5x^3y^2$ **2. a.** e^2, or about 7.39 **b.** $\frac{-5 \pm e^2}{2}$, or about 0.8 or −4.13 **c.** $\frac{e^2}{6}$, or about 1.23 **3. a.** ln 2 + 2, or about 4.48 **b.** −ln 10, or about −2.3 **c.** $\frac{\ln 10}{3}$, or about 0.77 **4. a.** No; the maximum velocity of 5.4 km/s is less than the 7.7 km/s needed for a stable orbit. **b.** Yes; if, R could be changed so that $V > 7.7$.
Lesson Check 1. ln 81 **2.** ln 1.8 **3.** ln 12 **4.** −ln 4 **5.** ≈10.9 **6.** ≈14.4 **7.** ≈7.39 **8.** ≈−0.718
9. error in 3rd line: $4x = 5$
should be: $4x = e^5$
$x = \frac{e^5}{4}$; $x \approx 37.1$
10. No; ln 5 has base e and $\log_2 10$ has base 2.
Exercises 11. ln 125 **13.** ln 4 **15.** $\ln \frac{\sqrt[3]{xy}}{z^4}$ **17.** ln 40,960 **19.** ln 1 **21.** 0.135 **23.** ±11.588 **25.** ±2.241

27. 1488.979 **29.** ≈2.890 **31.** ≈1.242 **33.** ≈2.401 **35.** 0 **37.** ≈2.2 **39.** at least 25 s **41.** ≈11,552 yrs **43.** $\frac{1}{4}$ **45.** 83 **47.** 2 **49.** 10 **51.** $\frac{1}{2}$ **53.** ≈301 days **55.** never **57.** 10.8 **59.** ≈19.8 h **72.** 4 **73.** 2.846 **74.** 0.272 **75.** $3333.\overline{3}$ **76.** 1.002 **77.** 9.0×10^{-5} **78.** $y = \frac{x-7}{5}$; yes **79.** $y = \sqrt[3]{\frac{x-10}{2}}$; yes **80.** $y = \pm\sqrt{5-x}$; no **81.** $y = \frac{x-2}{3}$; yes **82.** 10 **83.** 15 **84.** $\frac{6}{5}$

Chapter Review pp. 487–490

1. exponential decay; exponential growth **2.** asymptote **3.** logarithm; natural logarithm function **4.** continuously compounded interest **5.** natural logarithmic function **6.** exponential growth; (0, 1) **7.** exponential growth; (0, 2) **8.** exponential growth; (0, 0.2) **9.** exponential decay; (0, 3) **10.** exponential growth; $\left(0, \frac{25}{7}\right)$ **11.** exponential growth; (0, 0.0015) **12.** exponential decay; (0, 2.25) **13.** exponential decay; (0, 0.5) **14.** $y = 12{,}500(0.91)^x$; \$7800 **15.** $y = 50(1.03)^x$; \$58 **16.** The parent graph $y = 2^x$ is stretched by a factor of 5, translated 1 unit to the left, and 3 units up. **17.** The parent graph $y = \left(\frac{1}{3}\right)^x$ is reflected across the x-axis, stretched by a factor of 2, and translated 2 units to the right. **18.** \$1100.76 **19.** \$291.91 **20.** 0.0498 **21.** 0.3679 **22.** 148.4132 **23.** 0.6065 **24.** $2 = \log_6 36$ **25.** $-3 = \log_2 0.125$ **26.** $3 = \log_3 27$ **27.** $-3 = \log 0.001$ **28.** 6 **29.** −2 **30.** −5 **31.** 0

32.



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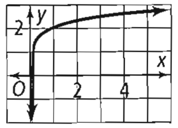

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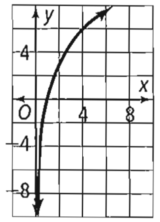

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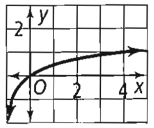
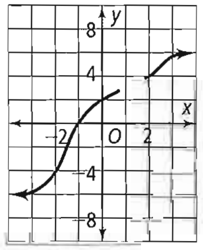


36. The parent graph $y = \log_4 x$ is stretched by a factor of 3 and translated 1 unit to the left. **37.** The parent graph $y = \ln x$ is reflected across the x-axis and translated 2 units up. **38.** log 24; Product Prop. **39.** $\log_2 \frac{5}{3}$; Quotient Prop. **40.** $\log_3 7x^4$; Power and Product Prop. **41.** $\log \frac{x}{y}$; Quotient Prop. **42.** $\log \frac{5}{x^2}$; Power and Quotient Prop. **43.** $\log_4 x^5$; Power and Product Prop. **44.** $2 \log_4 x + 3 \log_4 y$; Product and Power Prop. **45.** $\log 4 + 4 \log s + \log t$; Product and Power Prop. **46.** $\log_3 2 - \log_3 x$; Quotient Prop. **47.** $2 \log(x + 3)$; Power Prop. **48.** $3 \log_2 2 + 3 \log_2 (y - 2)$; Power and Product Prop. **49.** $2 \log z - \log 5$; Power and Quotient Prop. **50.** ≈2.8 **51.** ≈2.1 **52.** 0.75 **53.** 3.2619 **54.** 4.6542 **55.** 1.3652 **56.** 3.3333 **57.** 8 **58.** 50 **59.** 7.6256×10^{12} **60.** 0.9307 **61.** 0.6599 **62.** 0.6658 **63.** 3.0589 **64.** ≈18.2 h **65.** ≈0.83 **66.** ≈2.26 **67.** ≈4.31 **68.** ≈0.54 **69.** ≈3.77 **70.** ≈6.03 **71.** ≈3.4%

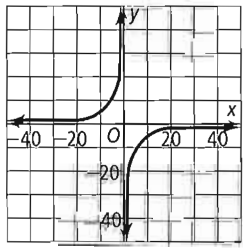
Chapter 8

Get Ready! p. 495 **1.** $\frac{4}{3}$; −4 **2.** $-\frac{2}{3}$; 2 **3.** $-\frac{10}{3}$; 10 **4.** $-\frac{16}{7}$; $\frac{48}{7}$ **5.** $(x + 3)(x - 2)$ **6.** $(4x + 5)(x + 3)$ **7.** $(3x - 5)(3x + 5)$ **8.** $(x - 6)^2$ **9.** $(3x + 4)(x + 2)$ **10.** $(x - 3)(x - 2)$ **11.** 1, −8 **12.** −6, −8 **13.** 4, 2 **14.** 0, $-\frac{2}{3}$ **15.** 8, $\frac{1}{2}$ **16.** 15, −2 **17.** Answers may vary. Sample: Inverse is used when one quantity increases as the other quantity decreases.
18. Answers may vary. Sample:



Lesson 8-1 pp. 498–505

Got It? **1. a.** direct; $y = 40x$ **b.** inverse; $y = \frac{8}{x}$ **c.** neither **2. a.** $y = -\frac{56}{x}$
b.



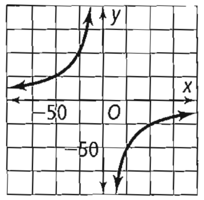
c. −28

3. a. $t = \frac{225}{n}$ **b.** 9 students **4.** 23 bags **5. a.** 4018 joules **b.** 12 m; No, you need not calculate PE to find the height. Substitute the mass and height of the first diver, and the mass of the second diver in $PE = mgh$ and set the two expressions equal. Solve the equation for h to calculate the height of the second diver.
Lesson Check **1.** inverse; $y = \frac{6}{x}$ **2.** direct; $y = 5x$ **3.** In direct variation, two positive quantities either increase together or decrease together. In an inverse variation, as one quantity increases, the other quantity decreases and vice versa. **4.** p varies directly with q, r, and

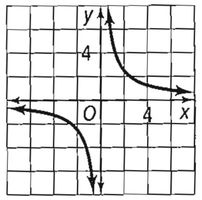
t and inversely with s. **5.** d varies directly with the cube root of r and inversely with the square of t.

Exercises **7.** neither **9.** inverse; $y = \frac{0.3}{x}$

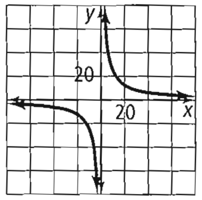
11. $y = -\frac{1300}{x}$; -130



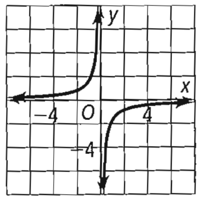
13. $y = \frac{5}{x}$, $\frac{1}{2}$



15. $y = \frac{250}{x}$; 25

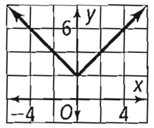


17. $y = -\frac{5}{3x}$, $-\frac{1}{6}$

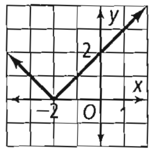


19. a. $s = 2.5m$ **b.** 100 muffins **21.** $PE = 2gh$ **23.** $F = \frac{km}{d^2}$ **25. a.** ≈76.58 L **b.** ≈20 moles **27.** $z = 10xy$; 360 **29.** 10 **31.** $18\frac{2}{3}$ **33.** 18 **35.** $\frac{1}{4}$ **37.** 2.5 **39.** 2.625 **49.** $\frac{e^5}{4} \approx 37.1$ **50.** $3e^4 \approx 163.79$ **51.** $\frac{e^2}{8} \approx 0.92$ **52.** $-90x^2$ **53.** $84x^2$ **54.** $10x^2y^3\sqrt{2y}$ **55.** $|x^5|y^{50}$ **56.** $-4ab^2$ **57.** $2m^2|n|\sqrt[4]{4}$ **58.** x, if n is odd; $|x|$ if n is even

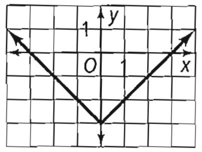
59. $y = |x|$ translated 2 units up



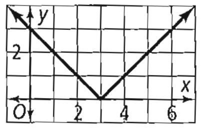
60. $y = |x|$ translated 2 units to the left



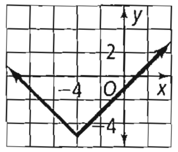
61. $y = |x|$ translated 3 units down



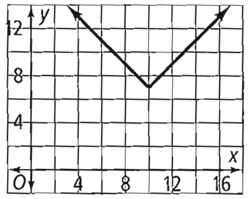
62. $y = |x|$ translated 3 units to the rt.



63. $y = |x|$ translated 4 units to the left and 5 units down



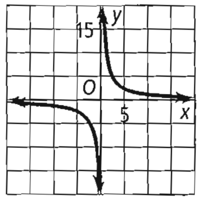
64. $y = |x|$ translated 10 units to the rt. and 7 units up



Lesson 8-2 pp. 507–514

Got It?

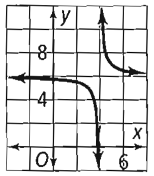
1. a.



no x- or y-intercept; horizontal asymptote: $y = 0$; vertical asymptote: $x = 0$; domain: all real numbers except $x = 0$, range: all real numbers except $y = 0$

b. Yes; because all the functions have similar graphs. **2. a.** $y = \frac{1}{2x}$ is a shrink of the graph of $y = \frac{1}{x}$ by a factor of $\frac{1}{2}$. **b.** $y = \frac{2}{x}$ is a stretch of the graph of $y = \frac{1}{x}$ by a factor of 2. **c.** $y = -\frac{1}{2x}$ is a reflection across the x-axis and a shrink of the graph of $y = \frac{1}{x}$ by a factor of $\frac{1}{2}$.

3.

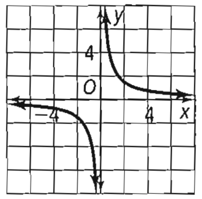


domain: all real numbers except $x = 4$, range: all real numbers except $y = 6$

4. $y = \frac{2}{x - 1} - 4$ **5. a.** $C = \frac{1200}{n}$; domain: whole numbers from 1 to 312; 160 students **b.** $C = \frac{1200}{n - 30}$; Domain: whole numbers from 1 to 282; 190 students

Lesson Check

1.



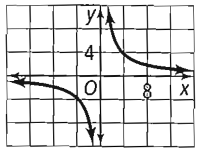
2. $y = \frac{1}{x}$ translated 5 units up **3.** $y = \frac{1}{x}$ reflected across the x-axis and stretched by a factor of 4 **4.** horizontal asymptote: $y = -7$, vertical asymptote: $x = -2$

5. shrink of the graph of $y = \frac{1}{x}$ by a factor of $\frac{1}{2}$

6. Answers may vary. Sample: $y = -\frac{2}{x}$ **7.** For $y = \frac{2}{x}$: stretch if $|a| > 1$ and shrink if $0 < |a| < 1$

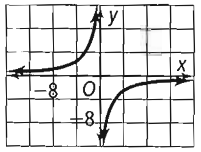
Exercises

9.



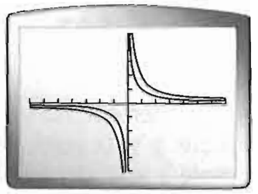
no x- or y-intercept; horizontal asymptote: $y = 0$, vertical asymptote: $x = 0$; domain: all real numbers except $x = 0$, range: all real numbers except $y = 0$

11.



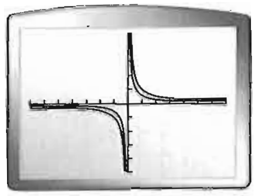
no x- or y-intercept; horizontal asymptote: $y = 0$, vertical asymptote: $x = 0$; domain: all real numbers except $x = 0$, range: all real numbers except $y = 0$

13.



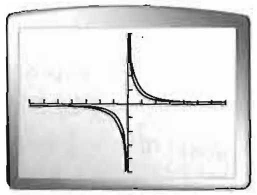
stretch by a factor of 2

15.



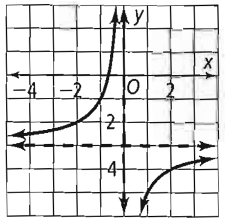
compression by a factor of 0.5

17.



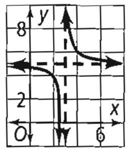
compression by a factor of 0.75

19.



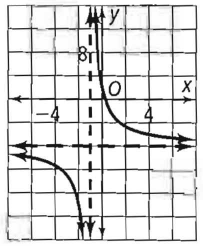
domain: all real numbers except $x = 0$, range: all real numbers except $y = -3$

21.



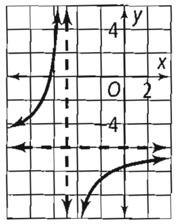
domain: all real numbers except $x = 3$, range: all real numbers except $y = 4$

23.



domain: all real numbers except $x = -1$, range: all real numbers except $y = -8$

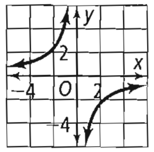
25.



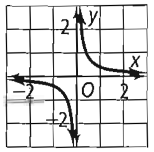
domain: all real numbers except $x = -5$, range: all real numbers except $y = -6$

27. $y = \frac{2}{x + 2} + 3$ **29.** 7.67 ft

33.

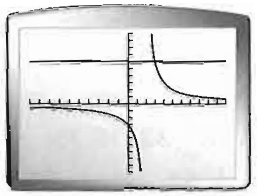


35.



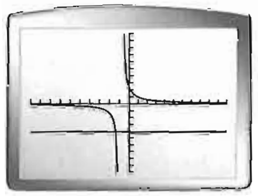
37. Answers may vary. Sample: The graph of the translation looks similar to the graph of $y = \frac{1}{x}$, so knowing the asymptotes helps to position the translation; check students' work.

39.



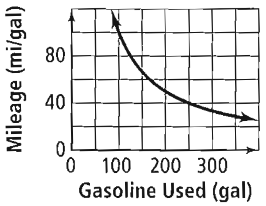
; (3, 6)

41.

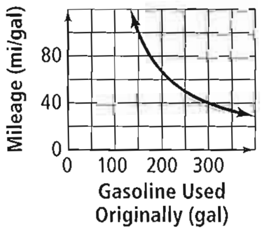


; (−1.75, −4)

43. a. $m = \frac{10,000}{g}$



b. $m = \frac{10,000}{g - 50}$ **c.** 25 mi/gal; 28.57 mi/gal



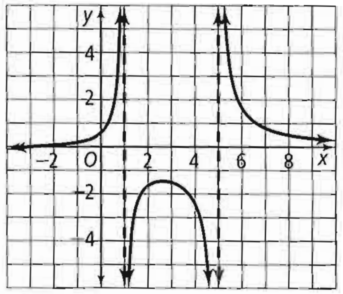
54. $y = \frac{24}{x}$; $-\frac{24}{5}$ **55.** $y = \frac{50}{x}$; -10 **56.** $y = \frac{48}{x}$; $-\frac{48}{5}$
57. exponential growth; 3 **58.** exponential growth; 0.1

59. exponential decay; 5 **60.** exponential decay; 3
61. $79 - 20\sqrt{3}$ **62.** 6 **63.** −2 **64.** $(x - 4)(x - 2)$
65. $(x + 9)(x - 3)$ **66.** $(2x - 7)(x + 4)$
67. $(2x - 3)(x - 8)$

Lesson 8-3 pp. 515–523

Got It? 1. a. domain: all real numbers except $x = 4$ and $x = -4$; points of discontinuity: non-removable at $x = 4$ and $x = -4$; no x-intercept, y-intercept: $\left(0, -\frac{1}{16}\right)$ **b.** domain: all real numbers; no points of discontinuity; x-intercepts: (1, 0) and (−1, 0), y-intercept: $\left(0, -\frac{1}{3}\right)$ **c.** domain: all real numbers except $x = -2$ and $x = -1$; points of discontinuity: non-removable at $x = -2$, removable at $x = -1$; no x-intercept, y-intercept: $\left(0, \frac{1}{2}\right)$ **2. a.** $x = 1$ and $x = -3$ **b.** $x = -3$ **c.** no vertical asymptotes **3. a.** $y = -2$ **b.** $y = 0$ **c.** no horizontal asymptote

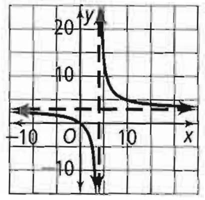
4.



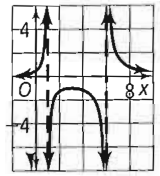
5. a. 4 gal **b.** No, because the graph changes when $y_1 = 0.8$ and intersects the graph of $y_2 = \frac{2 + (0.1)x}{2 + x}$ at $x \approx 0.6$. So, to have 80% orange juice, about 0.6 gal should be added.

Lesson Check 1. $x = -5$ and $x = -4$ **2.** $x = 9$ and $x = -2$ **3.** $x = -1$ **4.** $x = \frac{1}{3}$ and $x = 2$ **5.** $x = -5$ **6.** $x = -2$ and $x = -3$ **7.** $x = 1$ **8.** $x = 1$ and $x = -3$

9.

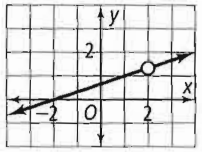


10.

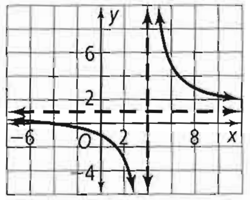


11. the function is undefined at $x = 1$ and $x = -3$
12. degree 2; function is discontinuous at 2 values of x
Exercises 13. domain: all real numbers except $x = 0$ and $x = 2$; points of discontinuity: non-removable at $x = 0$ and $x = 2$; no x- or y-intercept **15.** domain: all real numbers except $x = \pm 1$; pts. of discontinuity: non-removable at $x = -1$, removable at $x = 1$; no x-intercept; y-intercept: (0, 3) **17.** vertical asymptote at $x = -2$
19. vertical asymptotes at $x = -\frac{3}{2}$ and $x = 1$
21. hole at $x = -2$ **23.** $y = 0$ **25.** $y = 1$ **27.** $y = 0$

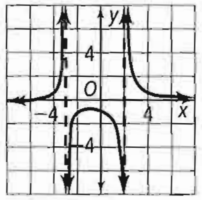
29.



31.

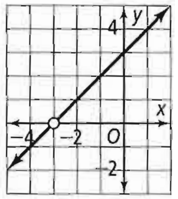


33.

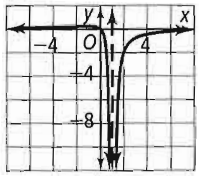


35. 900 ml **37.** vertical asymptote at $x = -2$
39. 6 free throws
41. correct answer: vertical asymptotes: $x = -5$ and $x = -1$, horizontal asymptote: $y = 1$

43.

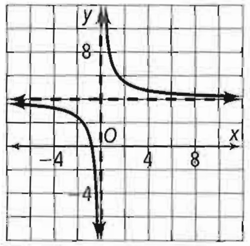


45.



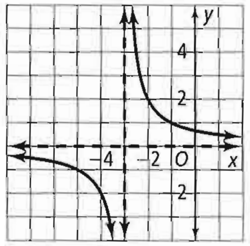
47. Answers may vary. Sample: There is no value of x for which the denominator equals 0.

55.



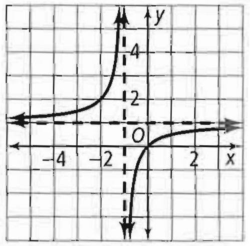
domain: all real numbers except $x = 0$, range: all real numbers except $y = 4$

56.



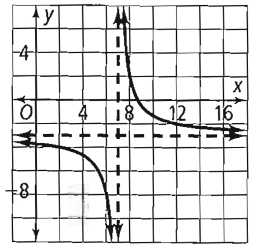
domain: all real numbers except $x = -3$, range: all real numbers except $y = 0$

57.



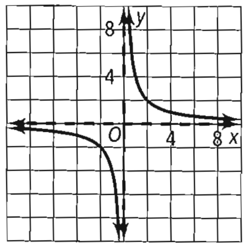
domain: all real numbers except $x = -1$, range: all real numbers except $y = 1$

58.



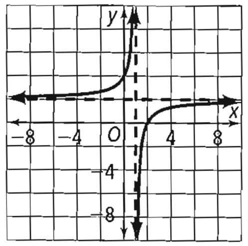
domain: all real numbers except $x = 7$, range: all real numbers except $y = -3$

59.



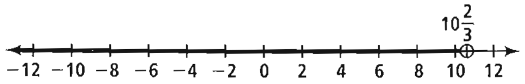
domain: all real numbers except $x = 0$, range: all real numbers except $y = 0$

60.

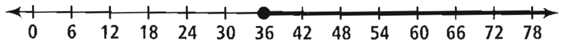


domain: all real numbers except $x = 1$, range: all real numbers except $y = 2$

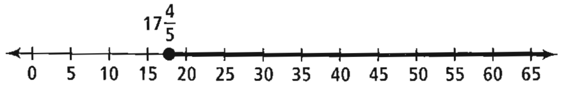
61. $y = \frac{x + 3}{2}$; yes **62.** $y = 6 - x$; yes **63.** $y = \pm\sqrt{\frac{x}{2}}$; no **64.** $y = \pm\sqrt{5x}$; no **65.** $y = \frac{1}{x} - 2$; yes
66. $y = (x - 1)^2 + 2$; yes
67. $a < 10\frac{2}{3}$



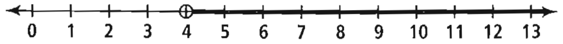
68. $x \geq 36$



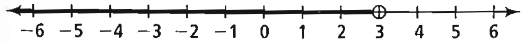
69. $x \geq 17\frac{4}{5}$



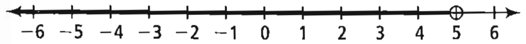
70. $y > 4$



71. $x < 3$



72. $b < 5$



73. $(2x - 1)(x - 1)$ **74.** $(2x - 3)(2x + 3)$
75. $(5x + 1)(x + 1)$ **76.** $10(x - 1)(x + 1)$

Lesson 8-4 pp. 527–533

Got It? 1. a. $-\frac{4x}{y}$; $x \neq 0, y \neq 0$ **b.** $\frac{x + 4}{x - 3}$; $x \neq 2$ or 3 **c.** $-\frac{4}{x + 3}$; $x \neq \pm 3$ **2.** $\frac{2(x + 1)}{(x + 4)^2}$; $x \neq \pm 4$ **3. a.** $\frac{2x}{x - 1}$; $x \neq 1, -1, -4, 0$, or 3 **b.** 6 restrictions; 2 in each of the original denominators, and 2 in the denominator of the reciprocal of the second rational expression. **4.** a square

Lesson Check 1. $\frac{z - 3}{2(z + 3)}$; $z \neq -3$ **2.** $\frac{3}{x}$; $x \neq 0$ or 1 **3.** $\frac{3(x + 5)}{x + 3}$; $x \neq -3, -6$, or 2 **4.** $-\frac{x + 6}{x + 2}$; $x \neq -6, -2, 2, 3$, or 5 **5.** Yes; the numerator and denominator are polynomials with no common factor. **6.** No; $x = 2$ will make the denominator of $\frac{x}{x - 2}$ 0, so $x = 2$ is not a solution. There is no solution to the eq.

7. Length $= \frac{2(a + 8)}{a + 5}$, $-10 < a < -5, a \neq -8$

Exercises 9. $\frac{1}{2x - 1}$; $x \neq 0$ or $\frac{1}{2}$ **11.** $7 - z$; $z \neq -7$ **13.** $-\frac{x + 4}{x - 5}$; $x \neq 5$ or 3 **15.** $\frac{xy^5}{4}$; $x \neq 0, y \neq 0$ **17.** $-\frac{4(x + 6)}{3(3x + 8)}$; $x \neq 3$ or $-\frac{8}{3}$ **19.** 1; $x \neq -2, -1, 2$, or 3 **21.** $\frac{y}{2x^2}$; $x \neq 0, y \neq 0$ **23.** 1; $y \neq -2$ or 4 **25.** $\frac{4(y - 3)}{y(y + 5)}$; $y \neq 2, -5$, or 0 **27.** $\frac{x - 8}{x - 10}$; $x \neq -3$ or 10 **29.** $\frac{y(y + 3)}{12(y + 4)}$; $x \neq 0, y \neq -4$ or 3 **31.** $R_{cylinder} = \frac{V_{cylinder}}{SA_{cylinder}} = \frac{rh}{2(r + h)}$; $R_{cube} = \frac{V_{cube}}{SA_{cube}} = \frac{s}{6}$; if $r = h = s$, then $R_{cylinder} = \frac{r}{4}$ and $R_{cube} = \frac{r}{6}$. $R_{cylinder} > R_{cube}$. The cylindrical shaped box is more efficient. If $s = h = 2r$ (diameter), then $R_{cylindrical} = R_{cube}$. The boxes are equally efficient. **33.** $\frac{18x}{(x + 9)(x + 3)}$; $x \neq -9, -3$, or 3 **35.** $\frac{x + 1}{x - 1}$; $x \neq -\frac{1}{2}, \frac{1}{2}$, 1, or −2 **37.** They are equally efficient. **39.** never **41.** never **43.** 2; $x \neq -3$ or 1 **54.** hole at $x = 3$ **55.** vertical asymptotes at $x = -\frac{2}{3}$ and $x = -1$ **56.** hole at $x = 4$, vertical asymptote at $x = -3$ **57.** 3 **58.** −5 **59.** $\frac{3}{2}$ **60.** $\frac{3}{4}$ **61.** 49 **62.** 168 **63.** 2 **64.** $\frac{17}{38}$ **65.** $\frac{19}{75}$ **66.** $\frac{11}{72}$ **67.** $\frac{137}{180}$

Lesson 8-5 pp. 534–541

Got It? 1. a. $2(x + 2)(x - 3)$ **b.** $(x - 1)(x - 2)^2(x + 4)$ **2. a.** $\frac{x + 2}{x}$, $x \neq 1$ or 0

b. $\frac{2(x-1)}{x^2-4}$; $x \neq \pm 2$ **c.** Yes, however the denominator could have to be factored more and there would be additional, incorrect limitations on x. **3. a.** $\frac{x-2}{x-1}$, $x \neq 1$ or 2 **b.** $\frac{x^2-x-4}{x^2+6x+5}$; $x \neq -5$ or -1 **4. a.** $\frac{x^2y}{x+y}$ **b.** $\frac{(x-1)^2}{2x}$; $x \neq 0, -2,$ or ± 1 **5.** Option 1 still gives the better combined mpg since Option 3 gives 18.46 mpg.

Lesson Check **1.** $\frac{2a-10}{3a-5}$; $a \neq \frac{5}{3}$ **2.** $\frac{6x-11}{x^2-4}$; $x \neq \pm 2$ **3.** $\frac{-11m}{3m+6}$; $m \neq -2$ **4.** $\frac{-4(2b-5)}{(b-4)(b+4)(b-2)}$; $b \neq 2$ or ± 4

5. error in finding a common denominator:

$$\begin{aligned}\frac{1+\frac{1}{x}}{\frac{3}{x}} &= \frac{\frac{x+1}{x}}{\frac{3}{x}}\\ &= \frac{x+1}{x}\cdot\frac{x}{3}\\ &= \frac{x+1}{3}\end{aligned}$$

6. Answers may vary. Sample: $\frac{x^2-1}{x^2-6x+5}$, $\frac{x^2+6x+5}{x^2-25}$

Exercises **7.** $9(x+2)(2x-1)$ **9.** $5(y+4)(y-4)$ **11.** $\frac{1}{x}$; $x \neq 0$ **13.** $\frac{-3}{x}$; $x \neq 0$ **15.** $\frac{xy+8y+4}{2xy^2}$; $x \neq 0, y \neq 0$ **17.** $\frac{y-6}{2(y+2)}$; $y \neq -2$ **19.** $\frac{-x+6}{(x-3)(x+3)}$; $x \neq \pm 3$ **21.** $\frac{-2x(x+3)}{(x-2)(x-1)(x+1)}$; $x \neq \pm 1$ or 2 **23.** $\frac{15}{28}$ **25.** $\frac{b}{9}$ **27.** $\frac{3x}{2+xy}$ **29.** $\frac{3}{x-6}$ **31.** $\frac{3x-8}{4x^2}$; $x \neq 0$ **33.** $\frac{7x-17}{(x-3)(x+3)}$; $x \neq \pm 3$ **35.** $\frac{x(3x^2+x-1)}{x^2-2}$; $x \neq \pm\sqrt{2}$ **37.** 3.84 in. **41.** $\frac{3x+2y}{7x-5y}$ **43.** x **45. a.** $\frac{2}{3}$ **b.** $\frac{3}{5}$ **c.** $\frac{2}{3}$ **d.** $\frac{1}{3}$ **53.** $\frac{12x}{x+3}$; $x \neq 2$ or ± 3 **54.** $\frac{3(x+2)}{4(x-3)}$; $x \neq \pm 2$ or 3 **55.** $\frac{3(x+1)}{2(x+3)}$; $x \neq \pm 1$ or -3 **56.** $\log_3 yt^4$ **57.** $\log p^7q^2$ **58.** $\log_5 \frac{x}{\sqrt[5]{y}}$ **59.** 30 **60.** 82 **61.** $\frac{15}{4}$ **62.** 101 **63.** $-\frac{4}{5}$ **64.** 21 **65.** 18

Lesson 8-6 pp. 542–548

Got It? **1. a.** 1 **b.** 0 **2. a.** ≈ 4.47 m/h **b.** The direction of wind affects the speed (rate) of the bike. Since the speed is inversely related to time, change in speed will lead to change in time. Since there is no wind, the speed of the bike will remain same to and from the store, hence the time to and from the store will remain the same. **3.** $0.\overline{27}$

Lesson Check **1.** 5 **2.** −1 **3.** −2 **4.** 310 mi/h **5.** LCD was not found. The correct answer is

$$\begin{aligned}\frac{35+9x}{7x} &= \frac{28(7)}{7x}, x \neq 0\\ 9x &= 161\\ x &= \frac{161}{9} = 17.\overline{8}\end{aligned}$$

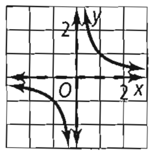
6. Answers may vary. Sample: $\frac{2}{x-3} + \frac{1}{x+3} = \frac{5x}{x^2-9}$ **7.** Answers may vary. Sample: (1) Substitute the solution into the original equation. (2) Check to see if the solution is in the domain of the graph of the original equation.

Exercises **9.** 10 **11.** 2 **13.** −1, 12 **15.** ≈−1.45, ≈1.65 **17.** −3, −2 **19.** 1 **21.** 0.6 **23.** 1.5 **25.** 1.75 **27.** ±2 **29.** ≈1.69, ≈−0.44 **31.** $E = mc^2$ **33.** $c = \pm\sqrt{a^2-b^2}$ **35.** $B = \pm\sqrt{\frac{2Vm}{r^2q}}$ **37.** $1\frac{5}{7}$ h **39.** 4 test scores **41. a.** $2250 **b.** $\frac{15{,}000}{24+x}(3.60)$ **c.** $2250 - \frac{15{,}000}{24+x}(3.60)$ **d.** ≈32.7 mpg **43.** 3 **45.** no solution **47.** no solution **49.** no solution **51.** $1, -\frac{2}{3}$ **61.** $\frac{-y-13}{4(y+1)}$ **62.** $\frac{5xy-12}{2y(y+2)}$ **63.** $\frac{x^2+3}{2(x-1)(x+3)}$ **64.** $x = -3$ **65.** $x = -1$ **66.** $x = -0.875$ **67.** $y = \frac{5-x}{2}$; yes **68.** $y = \pm\sqrt{x-1}$; no **69.** $y = \sqrt[3]{x+4}$; yes **70.** add 2; 9, 11, 13 **71.** subtract 2; −10, −12, −14 **72.** multiply by 5; 625, 3125, 15625 **73.** subtract 5; 30, 25, 20 **74.** multiply by 2; 128, 256, 512 **75.** subtract 4; −19, −23, −27

Chapter Review pp. 553–556

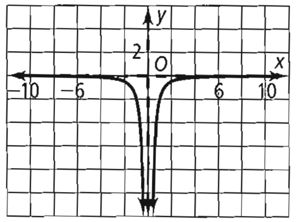
1. simplest form **2.** combined variation **3.** complex fraction **4.** point of discontinuity **5.** branch **6.** 12 **7.** $y = \frac{72}{x}$ **8.** $y = 6x$ **9.** $z = \frac{7}{4}xy$; 56 **10.** $z = \frac{4x}{y}$; 2

11.



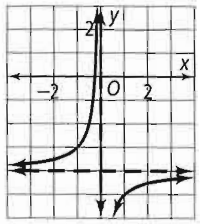
no x- or y-intercept; vert. asymptote: $x = 0$, horizontal asymptote: $y = 0$

12.



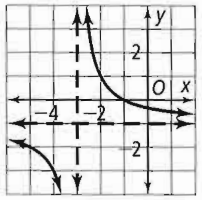
no x- or y-intercept; vert. asymptote: $x = 0$, horizontal asymptote: $y = 0$

13.



x-intercept: $(-0.25, 0)$, no y-intercept; vert. asymptote: $x = 0$, horizontal asymptote: $y = -4$

14.

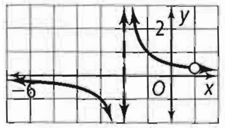


x-intercept: $(-1, 0)$, y-intercept: $(0, -\frac{1}{3})$; vert. asymptote: $x = -3$, horizontal asymptote: $y = -1$

15. $y = \frac{4}{x} + 3$ **16.** $y = \frac{4}{x - 2} + 2$ **17.** $y = \frac{4}{x + 3} - 4$

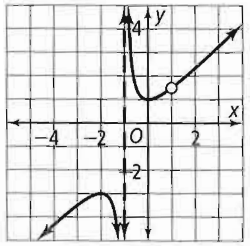
18. $y = \frac{4}{x - 4} - 3$

19. pts. of discontinuity: $x = -2, 1$;



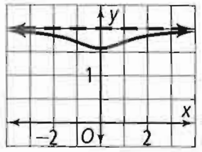
vert. asymptote: $x = -2$, horizontal asymptote: $y = 0$; hole at $x = 1$

20. 1, −1



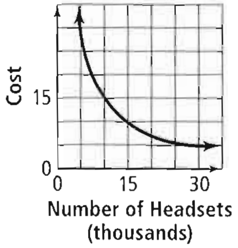
vert. asymptote: $x = -1$; hole at $x = 1$

21. no pts. of discontinuity



horizontal asymptote: $y = 2$

22.



≈31,056 headsets

23. $\frac{x + 5}{x + 4}$; $x \neq -4$ or -5 **24.** $\frac{(x - 1)(x + 1)}{x + 3}$; $x \neq -4, -3,$ or 6 **25.** $\frac{(2x - 1)(x + 1)}{x + 4}$; $x \neq -4, -1,$ or 0

26. $\frac{r}{3}$, where r is the radius **27.** $\frac{3(3x - 4)}{(x - 2)(x + 2)}$; $x \neq \pm 2$

28. $\frac{-x^2 + 3x + 2}{x(x + 1)(x - 1)(x + 3)}$; $x \neq \pm 1, 0,$ or -3

29. $\frac{2(x - 1)}{3x - 1}$ **30.** $\frac{1}{4(x + y)}$ **31.** −1 **32.** no solution

33. −12, 9 **34.** you: 10 mi/h friend: 8 mi/h

Chapter 9

Get Ready! p. 561 **1.** 9, 11, 13, 15 **2.** 1, 6, 11, 16 **3.** 0.9, 1.1, 1.3, 1.5 **4.** −2, −7, −12, −17 **5.** $3\frac{1}{3}, 7\frac{1}{3}, 11\frac{1}{3}, 15\frac{1}{3}$ **6.** −12, −15, −18, −21 **7.** subtract 5; −11, −16, −21 **8.** mult. by 2; 16, 32, 64 **9.** alternate subtract 9 and add 1; −7, −6, −15 **10.** add 3; 19, 22, 25 **11.** $\frac{4}{3}$ **12.** $\frac{3}{4}$ **13.** $\frac{5}{3}$ **14.** $\frac{5}{18}$ **15.** Answers may vary. Sample: $f(x) = 2x - 1$; 1, 3, 5, 7, 9 **16.** Answers may vary. Sample: $g(x) = 1 - 2x$; −1, −3, −5, −7, −9; yes; common difference: −2 **17.** Answers may vary. Sample: $h(x) = 5(2)^x$; 10, 20, 40, 80, 160; yes; common ratio: 2

Lesson 9-1 pp. 564–571

Got It? **1.** 147 **2. a.** $a_1 = 1$ and $a_n = na_{n-1}$ **b.** $a_1 = 1$ and $a_n = a_{n-1} + n^2$ **3. a.** $a_n = n^2 - 1$; 399 **b.** To find the nth term using an explicit formula, you simply substitute for n in the formula. To find the nth term using a recursive definition may require many iterations. **4.** 18 months

Lesson Check **1.** 2, 7, 12, 17, 22 **2.** −1, 0, 3, 8, 15 **3.** $a_1 = 3$ and $a_n = 2a_{n-1}$ **4.** $a_n = 2 + 3n$ **5.** A recursive formula defines the terms in a sequence by relating each term after the first term to the one before it and requires that the previous term be known to find a given term. An example of a recursive formula for the sequence 8, 4, 2, 1, . . . is $a_1 = 8$ and $a_n = \frac{1}{2}a_{n-1}$. An explicit formula describes the nth term of a sequence using the variable n and only requires the number of the term to be known. An example of an explicit formula for the sequence 1, 3, 5, 7, . . . is $a_n = 2n - 1$. **6.** The "+1" in $a_n = 3n + 1$ is incorrect for the sequence 1, 4, 7, 10, ... The correct explicit formula is $a_n = -2 + 3n$.

Exercises **7.** 5, 8, 11, 14, 17, 20 **9.** $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ **11.** 2, 10, 24, 44, 70, 102 **13.** $-\frac{1}{2}, 3, \frac{25}{2}, 31, \frac{123}{2}, 107$ **15.** $a_1 = 80$ and $a_n = a_{n-1} - 3$ **17.** $a_1 = 0$ and $a_n = a_{n-1} + (n + 1)$ **19.** $a_1 = 100$ and $a_n = \frac{1}{10}a_{n-1}$ **21.** $a_1 = 4$ and $a_n = -2a_{n-1}$ **23.** $a_1 = 1$ and $a_n = a_{n-1} + n^2$ **25.** $a_n = 3n + 1$; 31 **27.** $a_n = \frac{n - 6}{2}$; 2 **29.** $a_n = n^2 + 1$; 101 **31.** $a_n = 3^{n-1}$; 19,683 **33.** 5 **35.** $\frac{5}{16}$ **37.** $\frac{9}{1024}$ **39.** −47 **41.** $-\frac{9}{8}$ **43.** recursive; 3, 9, 21, 45, 93 **45.** explicit; −24, −21, −16, −9, 0

47. explicit; −6, −18, −38, −66, −102 **49.** 25, 36, 49, 64 **51.** $\frac{16}{5}, \frac{25}{6}, \frac{36}{7}, \frac{49}{8}$ **53.** \$140 **55.** 20, 23; $a_n = 3n + 2$, explicit OR $a_n = a_{n-1} + 3$; $a_1 = 5$, recursive **57.** 216, 343; $a_n = n^3$, explicit **59.** 144, 169; $a_n = (n + 6)^2$, explicit OR $a_n = a_{n-1} + 2n + 11$, $a_1 = 49$, recursive **61.** −1, $-\frac{1}{2}$; $a_n = \frac{-32}{2^n}$, explicit OR $a_n = \frac{a_{n-1}}{2}$, $a_1 = -16$, recursive **63.** −11, −19; $a_n = 29 - 8n$, explicit OR $a_n = a_{n-1} - 8$, $a_1 = 21$, recursive **65. a.** 25 boxes **b.** 110 boxes **c.** 9 levels **76.** 2 **77.** 4 **78.** −5 **79.** −1 **80.** 1 **81.** 2 **82.** subtract 2; −2, −4, −6 **83.** add 17; 185, 202, 219 **84.** add $\frac{3}{7}$; $\frac{17}{7}, \frac{20}{7}, \frac{23}{7}$

Lesson 9-2 pp. 572–577

Got It? 1. a. not arithmetic **b.** arithmetic **2. a.** 93 **b.** 95, 110 **3. a.** 115 **b.** yes, use the formula for arithmetic mean and solve for a_7, $a_7 = 2a_6 - a_5$ **4.** 65 seats

Lesson Check 1. 56 **2.** 87 **3.** 13 **4.** 39 **5.** In an arithmetic sequence, the diff. between any two consecutive terms is always the same number. **6.** Answers may vary. Sample: 2, 4, 8, 16, 32, . . .

Exercises 7. yes; 10 **9.** yes; 3 **11.** yes; 4 **13.** 127 **15.** 240 **17.** 12.5 **19.** −7 **21.** 13 **23.** 7.5 **25.** \$135 **27.** 18 **29.** 36 **31.** 2 **33.** The student multiplied the third term by 2 instead of adding 2. The correct answer is 6. **35.** 120 **37.** 1.1 **39.** 0
41. $a_n = 2 + 2(n - 1)$; $a_n = a_{n-1} + 2$, $a_1 = 2$
43. $a_n = -5 + 1(n - 1)$; $a_n = a_{n-1} + 1$, $a_1 = -5$
45. $a_n = -5 + 1.5(n - 1)$; $a_n = a_{n-1} + 1.5$, $a_1 = -5$
47. $a_n = 1 + \frac{1}{3}(n - 1)$; $a_n = a_{n-1} + \frac{1}{3}$, $a_1 = 1$
49. $a_n = 27 - 12(n - 1)$; $a_n = a_{n-1} - 12$, $a_1 = 27$
51. Answers may vary. Sample: An advantage of a recursive formula is that only the preceding term must be known to find the next term; a disadvantage is that many calculations may be required to find a term. An advantage of an explicit formula is that it is easy to find any term. Use the recursive formula when the previous term and common diff. are known. Use the explicit formula when the term number and common diff. are known. **53.** −4, −10, −16
55. −8, −17, −26 **57.** 17, 17, 17 **59.** −12.5, −8, −3.5
61. \$5055 **78.** recursive; −2, −7, −12, −17, −22
79. explicit; 6, 18, 36, 60, 90 **80.** explicit; 0, 3, 8, 15, 24
81. recursive; −121, −108, −95, −82, −69
82. $y - 3 = \frac{8}{3}x$ or $y - 11 = \frac{8}{3}(x - 3)$
83. $y - 6 = 4(x - 4)$ or $y - 30 = 4(x - 10)$
84. $y - 10 = 8(x - 1)$ or $y - 42 = 8(x - 5)$
85. $r = \frac{\sqrt[3]{6\pi^2 V}}{2\pi}$ **86.** 32 **87.** 625 **88.** −81

Lesson 9-3 pp. 580–586

Got It? 1. a. yes; $a_1 = 2$, $r = 2$ **b.** no **c.** yes; $a_1 = 2^3$, $r = 2^4$ **2.** 6 or −6 **3. a.** explicit; it is easier to use because only one calculation is needed. **b.** about 16.8 cm, about 4 cm **4.** ±60

Lesson Check 1. no **2.** yes; 2 **3.** 729 **4.** 0.0064
5. The third term would be the geometric mean of 5 and 80 which is 20. Since *a* is pos. and r^2 is always pos., the third term, ar^2, cannot be neg.
6. For both the arithmetic mean and the geometric mean, the middle term of any three consecutive terms can be determined using the first and last of the three terms. The arithmetic mean is the sum of the first and last terms divided by 2, whereas the geometric mean is the square root (or its opposite) of the product of the first and the last terms.

Exercises 7. yes; 2 **9.** yes; −2 **11.** yes; 0.4 **13.** yes; $-\frac{1}{3}$ **15.** yes; 1.5 **17.** yes; 6 **19.** 6561 **21.** 0.078125 **23.** $\frac{-3}{2048}$ **25.** about 656.1 g; about 182.5 g; about 96.2 g
27. ±1530 **29.** ±1.5 **31.** ±6 **33.** $a_n = 100(-20)^{n-1}$; (100, −2000, 40,000, −800,000, 16,000,000
35. $a_n = 1024(0.5)^{n-1}$; 1024, 512, 256, 128, 64
37. $a_n = 10(-1)^{n-1}$; 10, −10, 10, −10, 10 **39.** arithmetic; 125, 150 **41.** geometric; −80, 160 **43.** neither; 25, 36
45. 7.5, 22.5, 67.5 or −7.5, 22.5, −67.5
47. −6.64, −11.02, −18.30 or 6.64, −11.02, 18.30
49. about 74.3 mi **51.** 768 **53.** 3×4^{19} or 824,633,720,832 **55.** 4 **57.** 10 **59.** Both the common diff. and the common ratio are used to find the next term in a sequence, but a common diff. is added and a common ratio is multiplied.
68. $a_n = -3 + 3(n - 1)$; $a_n = a_{n-1} + 3$, $a_1 = -3$
69. $a_n = 17 - 9(n - 1)$; $a_n = a_{n-1} - 9$, $a_1 = 17$
70. $a_n = -2 - 11(n - 1)$; $a_n = a_{n-1} - 11$, $a_1 = -2$
71. $7\sqrt{14}$ **72.** $\frac{3\sqrt{3x}}{7x}$ **73.** $\frac{x}{y}$ **74.** $5\sqrt[3]{6}$ **75.** vert. asymptote: $x = -3$ **76.** vert. asymptote: $x = -1$
77. vert. asymptotes: $x = 0, 1$ **78.** vert. asymptote: $x = 3$; hole at $x = -3$ **79.** $a_n = a_{n-1} + n$, $a_1 = 1$
80. $a_n = a_{n-1} + (2n - 1)$, $a_1 = 1$
81. $a_n = a_{n-1} + n^2$, $a_1 = 1$

Lesson 9-4 pp. 587–593

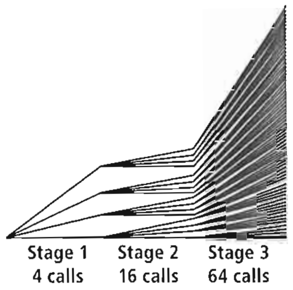
Got It? 1. a. 1030 **b.** Yes; no; the sum of any number of even numbers is always even. The sum of an odd number of odd numbers is odd, but the sum of an even number of odd numbers is even. **2.** 59 sales; 1725 sales **3. a.** $\sum_{n=1}^{40}(-12 + 7n)$ **b.** $\sum_{n=1}^{50}(510 - 10n)$ **4. a.** 2140 **b.** 100 **c.** 1 **5.** 41,650

Lesson Check **1.** 91 **2.** 780 **3.** $\sum_{n=1}^{7} 3n$
4. $\sum_{n=1}^{12}(-3 + 4n)$ **5.** An arithmetic sequence is a list of numbers for which successive numbers have a common difference. **6.** The lower limit should not be 3, it should be one and the expression $5n - 2$ should be in parentheses. The correct summation notation is $\sum_{n=0}^{8}(3 + 5n)$. **7.** Yes; $44 = 2(a_1 + a_4)$, so any combination of a_1 and a_4 with a sum of 22 is a possible series.

Exercises **9.** 92 **11.** 176 **13.** −165 **15.** $\sum_{n=1}^{5} 4n$
17. $\sum_{n=1}^{12}(2 + 3n)$ **19.** $\sum_{n=1}^{10}(-3n)$ **21.** 25 **23.** 20
25. −2 **27.** 2400 **29.** 682 **31.** −8556 **33.** 432 seats
35. sequence; finite **37.** series; infinite **39.** series; finite
41. −48 **43.** 35 **45.** −146 **47. a.** $a_n = n + 1$
b. $\sum_{n=1}^{9}(n + 1)$ **c.** 18 cans **d.** No; no; 13 rows have 104 cans, 14 rows have 119 cans, 15 rows have 135 cans, and 16 rows have 152 cans. The number of rows would not be an integer for 110 cans or 140 cans. **49.** −765
62. $a_n = 2^{n-1}$; 1, 2, 4 **63.** $a_n = -1(-1)^{n-1}$; −1, 1, −1
64. $a_n = 3\left(\frac{3}{2}\right)^{n-1}$; 3, $\frac{9}{2}$, $\frac{27}{4}$ **65.** $\frac{x+3}{x-4}$; $x \neq 4, x \neq -1$
66. $\frac{c-2}{c-5}$; $c \neq 5, c \neq 6$ **67.** $\frac{z^2 + 12z + 20}{z - 1}$; $z \neq 1$, $z \neq 0$ **68.** $-\frac{1}{3}$ **69.** $\frac{3}{4}$ **70.** $-\frac{1}{2}$

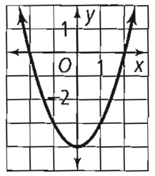
Lesson 9-5 pp. 595–601

Got It? **1. a.** 315 **b.** −1705 **2.** about $2138.43
3. a. diverges **b.** converges; $\frac{1}{4}$ **c.** converges; 2 **d.** Yes; if $|r| < 1$, the series converges. If $|r| \geq 1$, the series diverges.
Lesson Check **1.** $\frac{31}{80}$ **2.** $\frac{55}{9}$ **3.** converges **4.** diverges
5. Since $r = 1.1 > 1$, the series diverges and does not have a sum. **6.** An infinite geometric series has a sum only when the series converges, which is when $|r| < 1$.
7. The sum of a finite arithmetic series is $S_n = \frac{n}{2}(a_1 + a_n)$. The sum of finite geometric series is $S_n = \frac{a_1(1 - r^n)}{1 - r}$. The formulas are similar in that each sum requires the first term and the number of terms in the series. The formulas are different in that the sum of a finite arithmetic series needs the last term, while the sum of a finite geometric series needs the common ratio.
Exercises **9.** 1456 **11.** −5115 **13.** $\frac{15}{32}$ **15.** $\frac{121}{81}$
17. converges; $\frac{4}{3}$ **19.** converges; 8 **21.** diverges; no sum
23. diverges; no sum **25.** diverges; no sum **27.** 1 **29.** $\frac{9}{2}$
31. $\frac{9}{5}$ **33.** arithmetic; 420 **35.** geometric; about 96.47
37. geometric; about 121.5
39. a.



b. 4 + 16 + 64 + 256 + 1024 + 4096
c. 5460 employees **41.** $\frac{5}{4}$ **43.** $\frac{3}{4}$ **45.** $0.8\overline{3}$ **47. a.** $\frac{7}{8}$
b. 10 **49. a.** Answers may vary. Sample: The student used $r - 1$ instead of $1 - r$ in the formula for the sum of an infinite geometric series. **b.** $\frac{1}{2}$
51. a. $rS_n = r(a_1 + a_1r + \cdots + a_1r^{n-1}) = a_1r + a_1r^2 + \cdots + a_1r^n$
b.
$$\begin{aligned} S_n - rS_n &= a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} \\ &\quad - a_1r - a_1r^2 - \cdots - a_1r^{n-1} - a_1r^n \\ &= a_1 - a_1r^n \end{aligned}$$
c.
$$\begin{aligned} S_n - rS_n &= a_1 - a_1r^n \\ S_n(1 - r) &= a_1 - a_1r^n \\ S_n &= \frac{a_1 - a_1r^n}{1 - r} = \frac{a_1(1 - r^n)}{1 - r} \end{aligned}$$
59. 140 **60.** −825 **61.** $\frac{7c - 4}{2c^2}$ **62.** $\frac{10(2y + 3)}{(y + 3)(y - 3)}$
63. $\frac{x^2 + 6x + 4}{(x + 6)(x - 6)}$ **64.** 0 **65.** 2 **66.** 1

67.



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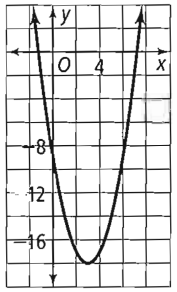

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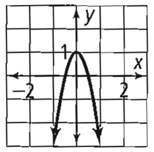


Chapter Review pp. 603–606

1. limits **2.** sequence **3.** converges **4.** common ratio
5. explicit formula **6.** 1, −1, −3, −5, −7
7. 1, 0, −3, −8, −15 **8.** 2, 3, 5, 9, 17 **9.** 20, 10, 5, 2.5, 1.25 **10.** $a_n = a_{n-1} + 17$, $a_1 = 5$ **11.** $a_n = a_{n-1} + 9$, $a_1 = -2$ **12.** $a_n = 3n - 2$ **13.** $a_n = 6.5 - 2.5n$ **14.** no
15. yes; $d = 15$, $a_{32} = 468$ **16.** yes; $d = 3$, $a_{32} = 100$
17. no **18.** 5 **19.** 101.5 **20.** 5 **21.** −4.9
22. −10.5, −8, −5.5 **23.** 1.4, 0.8, 0.2
24. $a_n = -2 + 9(n - 1)$ **25.** $a_n = 62 - 3(n - 1)$
26. yes; $r = \frac{1}{2}$; $\frac{1}{16}$, $\frac{1}{32}$ **27.** no **28.** yes; $r = 1.2$; 6.2208, 7.46496 **29.** ±6 **30.** ±0.04

31. ±10, −5, ±2.5 **32.** $a_n = 2^{n-1}$

33. $a_n = 25\left(\frac{1}{5}\right)^{n-1}$ **34.** 2560 **35.** 1536

36. $\sum_{n=1}^{5}(13 - 3n)$; 20 **37.** $\sum_{n=1}^{7}(45 + 5n)$; 455

38. $\sum_{n=1}^{11}(4.6 + 1.4n)$; 143 **39.** $\sum_{n=1}^{8}(23 - 2n)$; 112

40. 3; −8, 26; 27 **41.** 9; 4, 8; 54 **42.** 31 **43.** $53\frac{1}{8}$

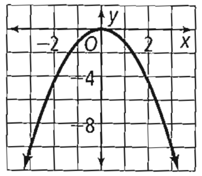
44. $14\frac{7}{18}$ **45.** converges; $S = 187.5$ **46.** diverges

47. diverges **48.** converges; $S = 2$

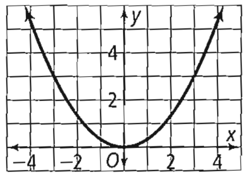
Chapter 10

Get Ready! p. 611

1.



2.



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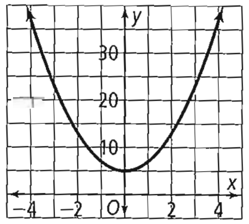

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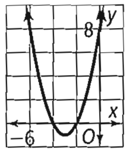


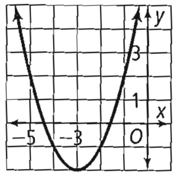
5. quadratic; $-x^2$, $6x$, 1 **6.** linear; none, $-12x$, −18

7. linear; none, x, $-\frac{13}{2}$ **8.** quadratic; $-8x^2$, $28x$, none

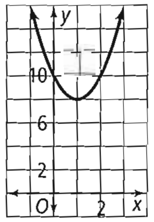
9. quadratic; $-2x^2$, $-3x$, 6 **10.** linear; none, $-x$, −10

11. 16 **12.** $\frac{25}{4}$ **13.** 49

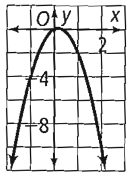
14. $y = (x + 3)^2 - 2$



15. $y = 2(x - 1)^2 + 8$



16. $y = -3\left(x - \frac{1}{6}\right)^2 + \frac{1}{12}$



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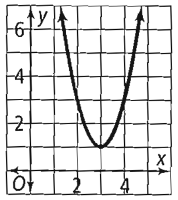

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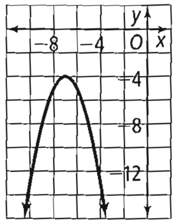

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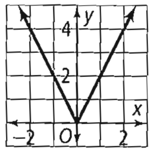

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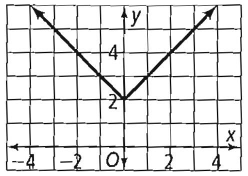
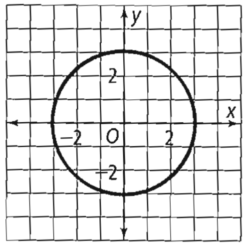


21. The radius of a circle is the distance from the center of the circle to any pt. on the circle. The radius extends in every direction from the center and ends on the circle. All radii of the same circle are equal. **22.** The vertex of a parabola is the lowest or highest pt. of a parabola; it is the pt. where the parabola changes direction.

Lesson 10-1 pp. 614–621

Got It?

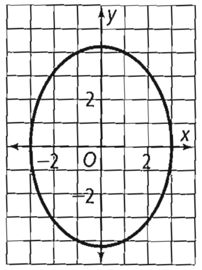
1. a.



circle: center (0,0); radius 3; lines of sym.: every line through the origin; domain: $-3 \leq x \leq 3$, range: $-3 \leq y \leq 3$

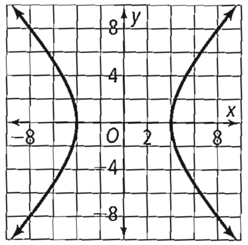
b. 6 is outside the domain of x.

2.



ellipse: center (0,0); lines of sym.: x-axis and y-axis; domain: $-3 \leq x \leq 3$, range: $-3\sqrt{2} \leq y \leq 3\sqrt{2}$

3.

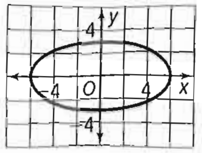


hyperbola: center (0,0); lines of sym.: x-axis and y-axis; domain: $x \leq -4$ or $x \geq 4$, range: all real numbers

4. center of hyperbola: (0, 0); no x-intercepts, y-intercepts: (0, 1), (0, −1); domain: all real numbers, range: $y \leq -1$ or $y \geq 1$ **5. a.** Ellipse; the eq. $9x^2 + 25y^2 = 225$ represents a conic section with two sets of intercepts, (±3, 0) and (0, ±5). Since the intercepts are not equidistant from the center, the eq. models an ellipse. **b.** Hyperbola; the eq. $x^2 - y^2 = 1$ represents a conic section with one set of intercepts, (±1, 0), so the eq. must model a hyperbola.

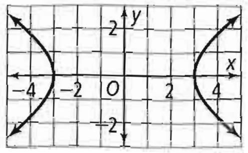
Lesson Check

1.



lines of sym.: x-axis and y-axis; domain: $-6 \leq x \leq 6$, range: $-3 \leq y \leq 3$

2.

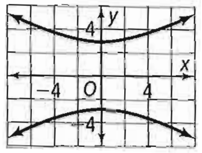


lines of sym.: x-axis and y-axis; domain: $x \leq -3$ or $x \geq 3$, range: all real numbers

3. domain: $x \leq -2.5$ or $x \geq 2.5$, range: all real numbers **4.** domain: $-6 \leq x \leq 6$, range: $-1.5 \leq y \leq 1.5$ **5. a.** hyperbola **b.** circle **6.** Answers may vary. Sample answer: The domain of an ellipse is an interval between two real numbers, such as $-a \leq x \leq a$. The domain of a hyperbola is two intervals, such as $x \leq -a$ or $x \geq a$, if there are x-intercepts, or all real numbers if there are no x-intercepts.

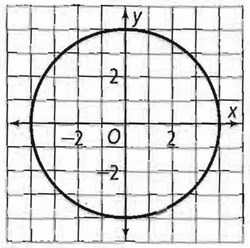
Exercises

7.



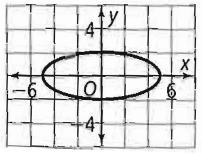
hyperbola; center: (0, 0); no x-intercepts, y-intercepts: $\left(0, \pm\frac{5\sqrt{3}}{3}\right)$; lines of sym.: x-axis and y-axis; domain: all real numbers, range: $y \leq -\frac{5\sqrt{3}}{3}$ or $y \geq \frac{5\sqrt{3}}{3}$

9.



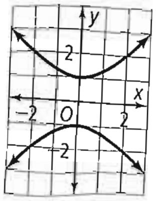
circle; center: (0, 0); radius: 4; x-intercepts: (±4, 0), y-intercepts: (0, ±4); infinitely many lines of sym.; domain: $-4 \leq x \leq 4$, range: $-4 \leq y \leq 4$

11.



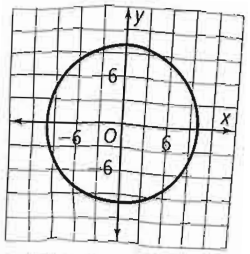
ellipse; center: (0, 0); x-intercepts: (±5, 0), y-intercepts: (0, ±2); lines of sym.: x-axis and y-axis; domain: $-5 \leq x \leq 5$, range: $-2 \leq y \leq 2$

13.



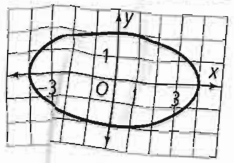
hyperbola; center: (0, 0); no x-intercepts, y-intercepts: (0, ±1); lines of sym.: x-axis and y-axis; domain: all real numbers, range: $y \leq -1$ or $y \geq 1$

15.



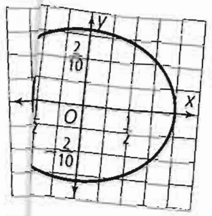
circle; center: (0, 0); radius: 10; x-intercepts: (±10, 0), y-intercepts: (0, ±10); infinitely many lines of sym.; domain: $-10 \leq x \leq 10$, range: $-10 \leq y \leq 10$

17.



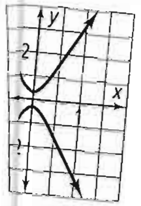
ellipse; center: (0, 0); x-intercepts: (±4, 0), y-intercepts: (0, ±2); lines of sym.: x-axis and y-axis; domain: $-4 \leq x \leq 4$, range: $-2 \leq y \leq 2$

19.



ellipse; center: (0, 0); x-intercepts: (±1, 0), y-intercepts: $\left(0, \pm\frac{1}{3}\right)$; lines of sym.: x-axis and y-axis; domain: $-1 \leq x \leq 1$, range: $-\frac{1}{3} \leq y \leq \frac{1}{3}$

21.

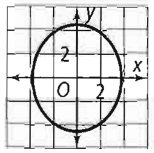


hyperbola; center: (0, 0); no x-intercepts, y-intercepts: $\left(0, \pm\frac{1}{2}\right)$; lines of sym.: x-axis and y-axis; domain: all real numbers, range: $y \leq -\frac{1}{2}$ or $y \geq \frac{1}{2}$

23. la; center: (0, 0); no x-intercepts, y-intercepts: (0, ±ain: all real numbers, range: $y \leq -2$ or $y \geq$ yperbola; center: (0, 0); x-intercepts: (±3, 0), no y-ts; domain: $x \leq -3$ or $x \geq 3$, range: all real numl hyperbola; center: (0, 0); no x-intercepts, y-int(0, ±3); domain: all real numbers, range: $y \leq \geq 3$ **29.** 22 **31.** 24 **33.** 27

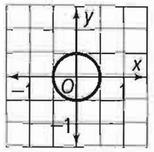
35.

circle; center: (0, 0); radius: 2; x-intercepts: (±2, 0), y-intercepts: (0, ±2); infinitely many lines of sym.; domain: $-2 \leq x \leq 2$, range: $-2 \leq y \leq 2$

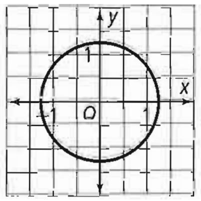
37. 

ellipse; center: (0, 0); x-intercepts: $\left(\pm\frac{8\sqrt{5}}{5}, 0\right)$, y-intercepts: $(0, \pm 2\sqrt{5})$; lines of sym.: x-axis and y-axis; domain: $-\frac{8\sqrt{5}}{5} \le x \le \frac{8\sqrt{5}}{5}$, range: $-2\sqrt{5} \le y \le 2\sqrt{5}$

39. a. All lines in the plane that pass through the center of a circle are axes of sym. of the circle. **b.** The axes of sym. of an ellipse intersect at the center of the ellipse. The same is true for a hyperbola. This can be confirmed using, for example, $4x^2 + 9y^2 = 36$ and $4x^2 - 9y^2 = 36$.

41. 

$x^2 + y^2 = \frac{1}{4}$

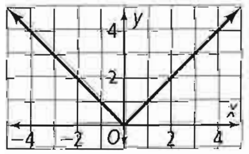
43. 

$x^2 + y^2 = 1.5625$

45. Sample: $(\sqrt{2}, 1)$ **47.** Sample: (2, 0) **49.** $(0, -\sqrt{7})$
59. diverges **60.** diverges **61.** converges
62. $x^3 - 3x^2y + 3xy^2 - y^3$
63. $p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$ **64.** $x^4 - 8x^3 + 24x^2 - 32x + 16$
65. $243 - 405x + 270x^2 - 90x^3 + 15x^4 - x^5$

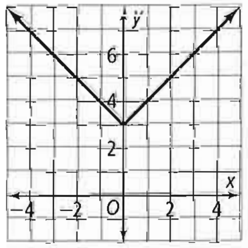
66.

x	−2	−1	0	1	2
y	2	1	0	1	2



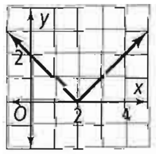
67.

x	−2	−1	0	1	2
y	5	4	3	4	5



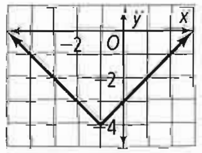
68.

x	0	1	2	3	4
y	2	1	0	1	2



69.

x	−3	−2	−1	0	1
y	−2	−3	−4	−3	−2



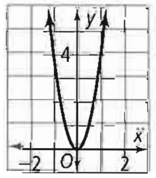
Lesson 10-2 pp. 622–629

Got It? 1. a. $y = -\frac{1}{6}x^2$ **b.** vertex: (0, 0); focus: (0, 1); directrix: $y = -1$ **c.** As the distance between the vertex and focus increases, the width of the parabola increases. **2. a.** $x = \frac{1}{10}y^2$ **b.** vertex: (0, 0); focus: $\left(-\frac{1}{16}, 0\right)$; directrix: $x = \frac{1}{16}$ **3.** 1 cm **4.** vertex: (−4, 2); focus: $\left(-4, 2\frac{1}{4}\right)$; directrix: $y = 1\frac{3}{4}$ **5.** $y = \frac{1}{8}(x - 1)^2 + 4$

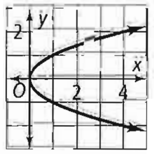
Lesson Check 1. $y = \frac{1}{2}x^2$ **2.** $x = \frac{1}{4}(y - 2)^2 + 3$
3. vertex: (0, 0); focus: (4, 0); directrix: $x = -4$
4. vertex: (−3, −4); focus: (−3, −3.75); directrix: $y = -4.25$ **5.** 6 units **6.** With the focus one unit away from the vertex of a parabola at the origin, $c = \pm 1$. Given this information, the student cannot tell whether the parabola opens in the vert. direction, with one of the eqs. $y = \frac{1}{4}x^2$ or $y = -\frac{1}{4}x^2$, or whether the parabola opens in the horizontal direction, with one of the eqs. of $x = \frac{1}{4}y^2$ or $x = -\frac{1}{4}y^2$.

Exercises 7. $x = \frac{1}{24}y^2$ **9.** $y = \frac{1}{28}x^2$ **11.** $x = \frac{1}{8}y^2$

13. vertex: (0, 0);
focus: $\left(0, \frac{1}{16}\right)$;
directrix: $y = -\frac{1}{16}$

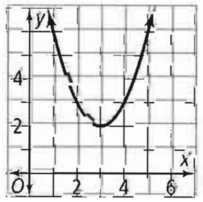


15. vertex: (0, 0);
focus: $\left(\frac{1}{4}, 0\right)$;
directrix: $x = -\frac{1}{4}$

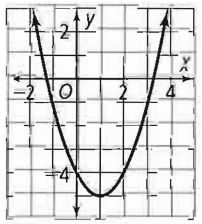


17. $x = \frac{1}{12}y^2$ **19.** $y = \frac{3}{4}x^2$ **21.** $y = -\frac{5}{56}x^2$ **23.** Answers may vary. Sample: $y = x^2$. The light produced by the bulb will reflect off the parabolic mirror in parallel rays.

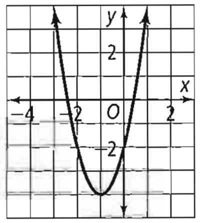
25. vertex: (3, 2);
focus: $\left(3, \frac{9}{4}\right)$;
directrix: $y = \frac{7}{4}$



27. vertex: (1, −5);
focus: $\left(1, -4\frac{3}{4}\right)$;
directrix: $y = -5\frac{1}{4}$

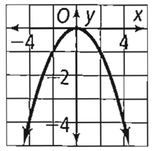


29. vertex: $(-1, -4)$;
focus: $\left(-1, -3\frac{7}{8}\right)$;
directrix: $y = -4\frac{1}{8}$

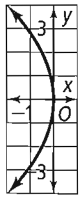


31. $x = -\frac{1}{32}(y - 3)^2$ **33.** $y = -\frac{1}{16}(x - 7)^2 + 2$

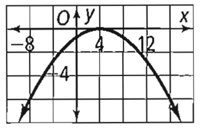
35. vertex: $(0, 0)$; focus: $(0, -1)$; directrix: $y = 1$



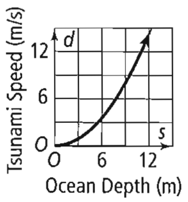
37. vertex: $(0, 0)$;
focus: $(-2, 0)$;
directrix: $x = 2$



39. vertex: $(4, 0)$;
focus: $(4, -6)$;
directrix: $y = 6$

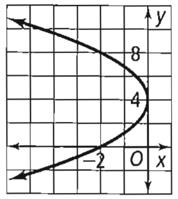


41.

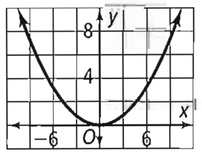


43. $y = \frac{1}{4}x^2$ **45.** 3.5 in.

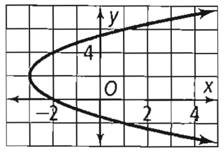
47.



49.



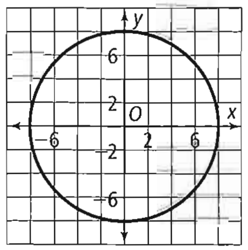
51.



53. $x = -\frac{1}{2}(y - 1)^2 + 1$

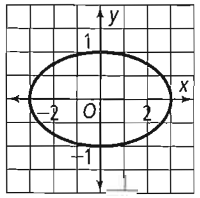
55. Answers may vary. Sample: Write the eq. in the form $x = \frac{1}{4\left(\frac{1}{8}\right)}y^2$. The distance from the focus to the directrix is $2\left(\frac{1}{8}\right)$, or $\frac{1}{4}$.

63.



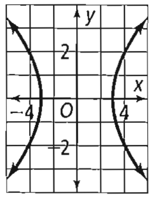
circle: center $(0, 0)$, radius 8; x-intercepts: $(\pm 8, 0)$, y-intercepts: $(0, \pm 8)$; infinitely many lines of sym.; domain: $-8 \le x \le 8$, range: $-8 \le y \le 8$

64.



ellipse: center $(0, 0)$; x-intercepts: $(\pm 3, 0)$, y-intercepts: $(0, \pm 1)$; lines of sym.: x-axis and y-axis; domain: $-3 \le x \le 3$, range: $-1 \le y \le 1$

65.



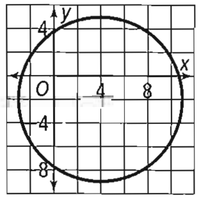
hyperbola: center $(0, 0)$; x-intercepts $(\pm 3, 0)$, no y-intercept; lines of sym.: x-axis and y-axis; domain: $x \le -3$ or $x \ge 3$, range: all real numbers

66. 1 **67.** 4 **68.** 25 **69.** 9

Lesson 10-3 pp. 630–636

Got It? 1. $(x - 5)^2 + (y + 2)^2 = 64$ **2. a.** $(x + 5)^2 + (y + 3)^2 = 1$ **b.** $(x - 2)^2 + (y - 3)^2 = 9$ **3. a.** $(x - 7)^2 + (y + 10)^2 = 144$ **b.** Yes; the values h and k determine the position of the circle and r determines the size. **4. a.** center $(-8, -3)$, radius 11 **b.** center $(3, -7)$, radius $\sqrt{66}$

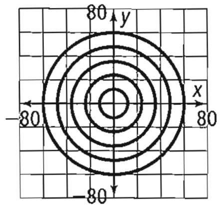
5.



Lesson Check 1. $(x + 1)^2 + (y + 5)^2 = 4$ **2.** $x^2 + y^2 = 36$ **3.** $x^2 + (y - 3)^2 = 121$ **4.** $(x + 5)^2 + (y + 3)^2 = 16$ **5.** The circle with equation $(x + 7)^2 + (y - 7)^2 = 8$ is a translation of the circle with equation $x^2 + y^2 = 8$ as 7 units left and 7 units up, not right and down. **6.** If $P(x, y)$ is one of the pts. $(r, 0)$, $(-r, 0)$, $(0, r)$, or $(0, -r)$, subst. shows that $x^2 + y^2 = r^2$. If $P(x, y)$ is any other pt. on the circle, drop a perpendicular $\overline{PK}$ from P to K on the x-axis. $\triangle OPK$ is a rt. triangle with legs of lengths $|x|$ and $|y|$ and with hypotenuse of length r. By the Pythagorean Thm., $|x|^2 + |y|^2 = r^2$, so $x^2 + y^2 = r^2$.

Exercises 7. $x^2 + y^2 = 100$ **9.** $(x - 2)^2 + (y - 3)^2 = 20.25$ **11.** $(x - 1)^2 + (y + 3)^2 = 100$ **13.** $x^2 + (y + 1)^2 = 9$ **15.** $(x - 2)^2 + (y + 4)^2 = 25$ **17.** $x^2 + (y + 5)^2 = 100$

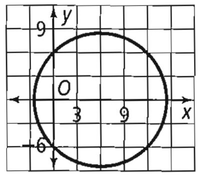
19.



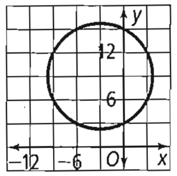
; $x^2 + y^2 = 144$,
$x^2 + y^2 = 576$,
$x^2 + y^2 = 1296$,
$x^2 + y^2 = 2304$,
$x^2 + y^2 = 3600$

21. $(x - 2)^2 + (y + 6)^2 = 16$ **23.** center (−2, 10), radius 2 **25.** center (−3, 5), radius 9 **27.** center (−6, 0), radius 11

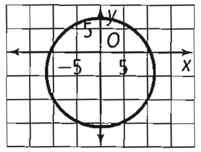
29.



31.



33.

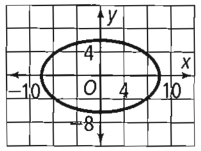


35. $x^2 + y^2 = 9$ **37.** $x^2 + y^2 = 3$ **39.** $x^2 + y^2 = 169$ **41.** $x^2 + y^2 = 26$ **43.** $x^2 + y^2 = 36$, $(x - 8)^2 + (y - 6)^2 = 16$, $(x - 8)^2 + (y)^2 = 4$ **45.** $(x + 6)^2 + (y - 13)^2 = 49$ **47.** $(x + 2)^2 + (y - 7.5)^2 = 2.25$ **49.** $(x - 2)^2 + (y - 1)^2 = 25$ **51.** $(x + 1)^2 + (y + 7)^2 = 36$ **53.** center (0, 0), radius $\sqrt{2}$ **55.** center (0, 0), radius $\sqrt{14}$ **57.** center (−5, 0), radius $3\sqrt{2}$ **59.** center (−3, 5), radius $\sqrt{38}$ **61.** center (3, 1), radius $\sqrt{6}$ **72.** $x = -\frac{1}{12}y^2$ **73.** at $x = -1$ **74.** at $x = 2$, and $x = 3$ **75.** no points of discontinuity **76.** 4 **77.** 2 **78.** −3 **79.** 4 **80.** $\frac{1}{2}$ **81.** 1 **82.** −2, −9 **83.** 0, ±4, $\pm 4i$ **84.** ±2, $\pm 2\sqrt{2}$

Lesson 10-4 pp. 638–644

Got It? 1. $\frac{x^2}{4} + \frac{y^2}{25} = 1$

2. a. (±8, 0).



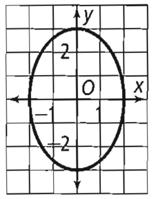
b. The vertex and co-vertex approach the same distance from the center of the ellipse; a circle

3. 24 ft **4.** $\frac{x^2}{36} + \frac{y^2}{53} = 1$

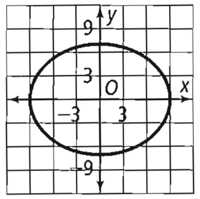
Lesson Check 1. $\frac{x^2}{64} + \frac{y^2}{36} = 1$ **2.** $(\pm\sqrt{21}, 0)$ **3.** $\frac{x^2}{169} + \frac{y^2}{25} = 1$ **4.** $6\sqrt{23}$ ft ≈ 28.77 ft **5.** The student used a and b instead of a^2 and b^2; $\frac{x^2}{1681} + \frac{y^2}{841} = 1$ **6.** The eq. of an ellipse with center at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. For a circle, the major axis and the minor axis are of equal length such that $a = b = r$. Thus, by subst., $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$ or $x^2 + y^2 = r^2$.

Exercises 7. $\frac{x^2}{16} + \frac{y^2}{9} = 1$ **9.** $\frac{x^2}{9} + y^2 = 1$ **11.** $\frac{x^2}{16} + \frac{y^2}{49} = 1$ **13.** $\frac{x^2}{81} + \frac{y^2}{4} = 1$

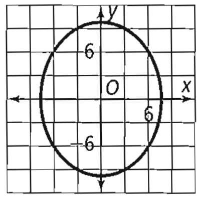
15. $(0, \pm\sqrt{5})$



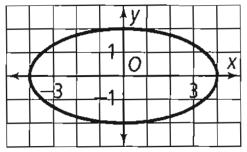
17. $(\pm 4\sqrt{2}, 0)$



19. (0, ±6)



21. $(\pm 2\sqrt{3}, 0)$

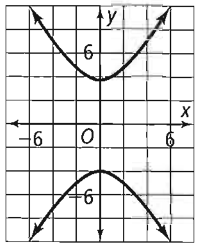


23. 32 **25.** 6 **27.** 12 **29.** $24\sqrt{2}$ **31.** $\frac{x^2}{100} + \frac{y^2}{64} = 1$ **33.** $\frac{x^2}{89} + \frac{y^2}{64} = 1$ **35. a.** about 22.25 ft **b.** Due to the reflective prop. of an ellipse, you can aim your putt at any part of the border. The ball will reflect off the border and go directly into the hole. **37.** $(0, \pm 2\sqrt{3})$ **39.** $(0, \pm\sqrt{21})$ **41.** (0, ±1) **43. a.** 0.9 **b.** 0.1 **c.** The shape is close to a circle. **d.** The shape is close to a line segment. **45.** $\frac{x^2}{16} + y^2 = 1$ **49.** $\frac{x^2}{25} + \frac{y^2}{4} = 1$ **51.** $\frac{x^2}{702.25} + \frac{y^2}{210.25} = 1$ **53.** $\frac{x^2}{256} + \frac{y^2}{324} = 1$ **55.** $\frac{x^2}{16} + \frac{y^2}{12} = 1$ **57.** $\frac{x^2}{36} + \frac{y^2}{27} = 1$ **59.** $\frac{x^2}{20} + \frac{y^2}{18} = 1$ **67.** $(x - 1)^2 + (y + 5)^2 = 9$ **68.** $(x + 2)^2 + (y - 4)^2 = 81$ **69.** $\frac{1}{2x - 3x^4}$; $x \neq 0, x \neq \sqrt[3]{\frac{2}{3}}$ **70.** $\frac{x - 6}{x - 1}$; $x \neq 1, x \neq -6$ **71.** $\frac{x - 5}{x^2 - 2x + 4}$; $x \neq -2$ **72.** log 15 **73.** $\log_3 6$ **74.** log 2 **75.** $y = 2x + 4$ **76.** $y = \frac{1}{3}x$

Lesson 10-5 pp. 645-652

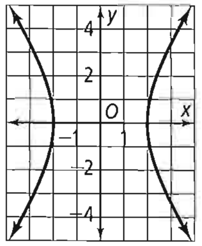
Got It? 1. a. $\frac{y^2}{16} - \frac{x^2}{9} = 1$

b.



c. when $a = b$

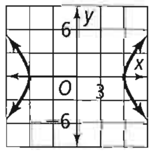
2. vertices: $(\pm 2, 0)$; foci: $(\pm\sqrt{13}, 0)$; asymptotes: $y = \pm\frac{3}{2}x$



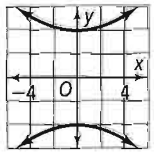
3. $\frac{x^2}{49} - \frac{y^2}{72} = 1$

Lesson Check

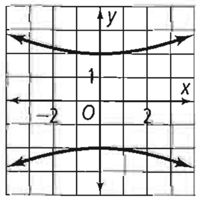
1. vertices: $(\pm 6, 0)$; foci: $(\pm\sqrt{61}, 0)$; slopes of asymptotes: $\pm\frac{5}{6}$



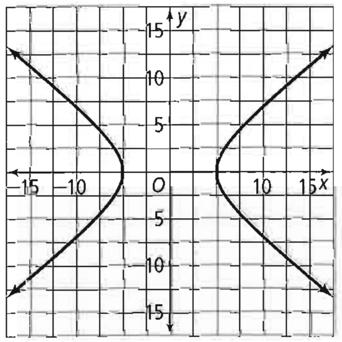
2. vertices: $(0, \pm 4)$; foci: $(0, \pm\sqrt{41})$; slopes of asymptotes: $\pm\frac{4}{5}$



3. vertices: $(0, \pm 2)$; foci: $(0, \pm 2\sqrt{5})$; slopes of asymptotes: $\pm\frac{1}{2}$



4. vertices: $(\pm 5, 0)$; foci: $(\pm\sqrt{41}, 0)$; slopes of asymptotes: $\pm\frac{4}{5}$

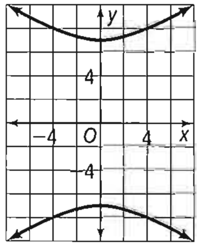


5. $\frac{x^2}{25} - \frac{y^2}{24} = 1$ **6.** Answers may vary. Sample: Similarities—Both have two axes of sym. that intersect at the center of the figure and two foci that lie on the same line as the two "principal" vertices. Differences—An ellipse consists of pts. whose distances from the foci have a constant sum, whereas a hyperbola consists of pts. whose distances from the foci have a constant diff. **7.** Answers may vary. Sample: A hyperbola is vert. or horizontal depending on whether it has a positive coefficient not because the larger denominator is under the y^2 term.

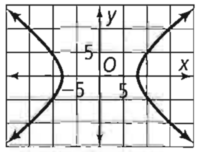
Exercises 9. $\frac{x^2}{144} - \frac{y^2}{25} = 1$ **11.** $\frac{x^2}{49} - \frac{y^2}{121} = 1$

13. $\frac{x^2}{4} - \frac{y^2}{5} = 1$

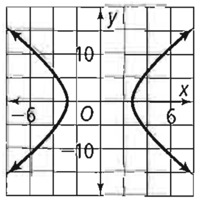
15. vertices: $(0, \pm 7)$; foci: $(0, \pm\sqrt{113})$; asymptotes: $y = \pm\frac{7}{8}x$



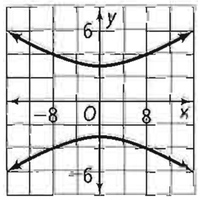
17. vertices: $(\pm 8, 0)$; foci: $(\pm 10, 0)$; asymptotes: $y = \pm\frac{3}{4}x$



19. vertices: $(0, \pm 3)$; foci: $(0, \pm 3\sqrt{10})$; asymptotes: $y = \pm\frac{1}{3}x$



21. vertices: $(\pm 2\sqrt{2}, 0)$; foci: $(\pm 2\sqrt{11}, 0)$; asymptotes: $y = \pm\frac{3\sqrt{2}}{2}x$

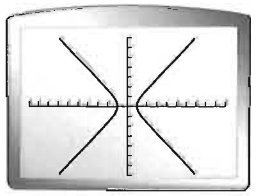


23. $x^2 - \frac{y^2}{6} = 1$ **25.** $\frac{x^2}{9} - \frac{y^2}{16} = 1$

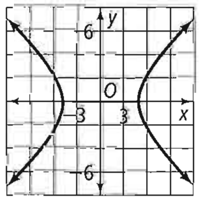
27. $y^2 - \frac{x^2}{3} = 1$ **29.** $\frac{y^2}{20.25} - \frac{x^2}{4} = 1$

31. $\frac{x^2}{32} - \frac{y^2}{64} = 1$

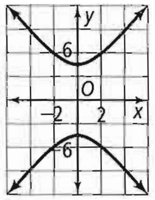
33. $y = \pm\sqrt{x^2 - 1}$; $(\pm 1, 0)$



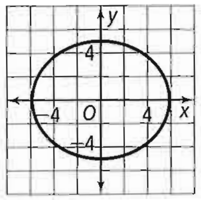
35.



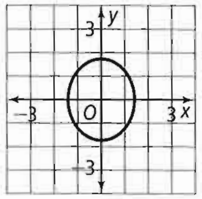
37.



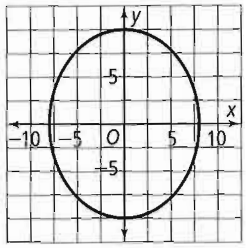
47. foci: (±3, 0)



48. foci: (0, ±1)



49. foci: (0, ±6)

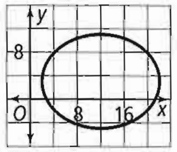


50. $\frac{1}{3}$ **51.** 125 **52.** 6 **53.** $y = (x - 3)^2 - 8$
54. $y = 2(x + 3)^2 - 18$ **55.** $y = 3(x + 4)^2 - 50$

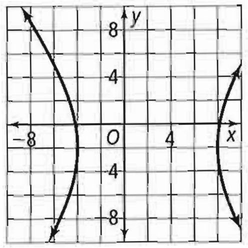
Lesson 10-6 pp. 653–660

Got It?

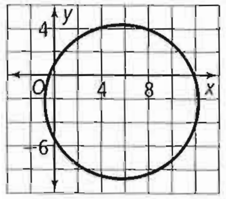
1. $\frac{(x - 12)^2}{100} + \frac{(y - 3)^2}{64} = 1$;



2. center (2, −2); vertices: (−4, −2), (8, −2); foci: (−8, −2), (12, −2); asymptotes: $y + 2 = \pm\frac{4}{3}(x - 2)$



3. a. circle with center (6, −2) and radius $4\sqrt{3}$



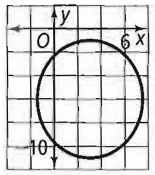
b. Replace $+y^2$ with $-y^2$ to get a new eq., $4x^2 - y^2 - 24x + 6y + 9 = 0$. The standard eq. of the hyperbola is $\frac{(x - 3)^2}{4.5} - \frac{(y - 3)^2}{18} = 1$.

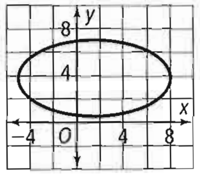
4. $\frac{(x + 1)^2}{9} + \frac{y^2}{8} = 1$

Lesson Check **1.** center (−7, −1); vertices: (−22, −1), (8, −1); foci: (−16, −1), (2, −1) **2.** center (1, 3); vertices: (4, 3), (−2, 3); foci: $(1 \pm \sqrt{13}, 3)$ **3.** $\frac{x^2}{48} + \frac{(y - 4)^2}{64} = 1$
4. $\frac{(y + 1)^2}{49} - \frac{(x + 3)^2}{51} = 1$ **5.** ellipse and hyperbola
6. Your friend used the center (−2, 1) instead of (2, −1). The vertices are (−10, −1) and (14, −1). **7.** The student didn't write the standard-form equation correctly. The standard-form equation is $(x + 4)^2 + (y - 5)^2 = 0$. This is an equation of a circle with center at (−4, 5) and radius 0. Since the radius is 0, the graph is a single point, (−4, 5).

Exercises

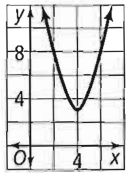
9. $\frac{(x - 3)^2}{21} + \frac{(y + 6)^2}{25} = 1$ **11.** $\frac{(x - 1.5)^2}{42.25} + \frac{(y - 4)^2}{12} = 1$



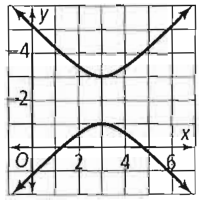


13. center: (3, 4); vertices: (3, 1), (3, 7); foci: $(3, 4 \pm \sqrt{13})$

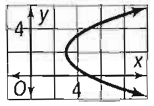
15. $y = (x - 4)^2 + 3$; parabola; vertex: (4, 3)



17. $(y - 2)^2 - (x - 3)^2 = 1$; hyperbola; center: (3, 2), foci: $(3, 2 \pm \sqrt{2})$;

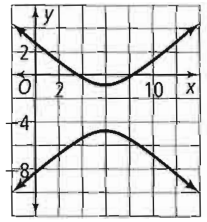


19. $x = \frac{1}{2}(y - 2)^2 + 3$; parabola; vertex: (3, 2)



21. a. hyperbola **b.** one focus; the other focus **c.** with the center at the origin, $x^2 - \frac{y^2}{3} = 1$
23. a. hyperbola **b.** horizontal line

25.

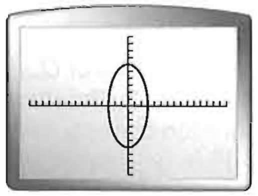


$\frac{(y + 3)^2}{4} - \frac{(x - 6)^2}{5} = 1$

27. $\frac{(x-1)^2}{9} + \frac{(y+1)^2}{16} = 1$ **29.** $x^2 + y^2 = 16$

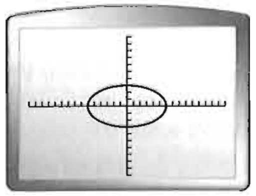
31. $y = 2(x+2)^2 + 4$

33.



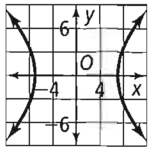
ellipse; $\frac{x^2}{9} + \frac{y^2}{36} = 1$

35.

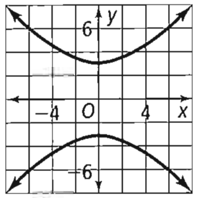


ellipse; $\frac{x^2}{36} + \frac{y^2}{9} = 1$

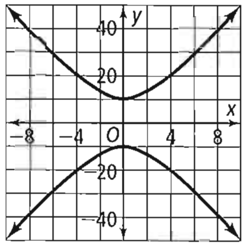
42. foci: $(\pm\sqrt{85}, 0)$



43. foci: $(0, \pm\sqrt{21})$



44. foci: $(0, \pm 2\sqrt{26})$

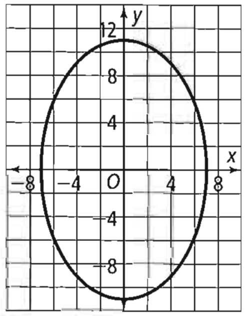


45. $\frac{3 \pm \sqrt{17}}{2}$ **46.** 5 **47.** $\frac{9 \pm \sqrt{201}}{12}$ **48.** 1 **49.** 2 **50.** 3 **51.** 8 **52.** −19 **53.** 6

Chapter Review pp. 663–666

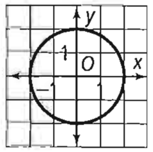
1. directrix **2.** major axis **3.** standard form of an eq. of a circle **4.** radius **5.** transverse axis

6.



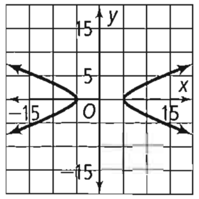
ellipse; lines of sym.: x-axis and y-axis; domain: $-7 \le x \le 7$, range: $-11 \le y \le 11$

7.



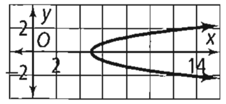
circle; lines of sym.: every line through the center; domain: $-2 \le x \le 2$, range: $-2 \le y \le 2$

8.



hyperbola; lines of sym.: x-axis and y-axis; domain: $x \le -5$ or $x \ge 5$, range: all real numbers

9.



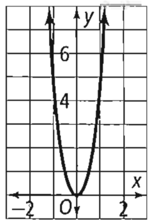
parabola; line of sym.: x-axis; domain: $x \ge 5$, range: all real numbers

10. center (0, 0); domain: $x \le -4$ or $x \ge 4$, range: all real numbers **11.** center (0, 0); domain: $-3 \le x \le 3$, range: $-2 \le y \le 2$ **12.** $x = \frac{1}{20}y^2$ **13.** $y = -\frac{1}{20}x^2$

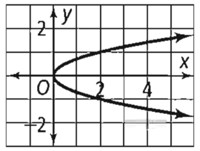
14. $y = \frac{1}{24}x^2$ **15.** $y = \frac{1}{10}x^2$ **16.** $y = 3x^2$

17. $y = \frac{1}{8}x^2 + 1$ **18.** $x = -\frac{1}{12}y^2 + 1$

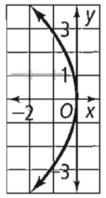
19. focus: $\left(0, \frac{1}{20}\right)$, directrix: $y = -\frac{1}{20}$



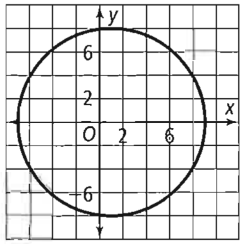
20. focus: $\left(\frac{1}{8}, 0\right)$, directrix: $x = -\frac{1}{8}$



21. focus: (−2, 0), directrix: $x = 2$

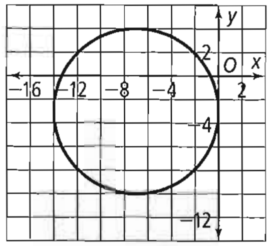


22. $x^2 + y^2 = 16$ **23.** $(x-8)^2 + (y-1)^2 = 25$
24. $(x+3)^2 + (y-2)^2 = 100$
25. $(x-5)^2 + (y+3)^2 = 64$
26. center (1, 0), radius 8



circle with radius 8 translated 1 unit to the rt.

27. center (−7, −3), radius 7

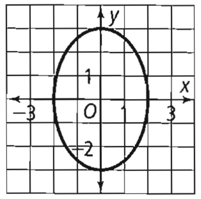


circle with radius 7 translated 7 units to the left and 3 units down

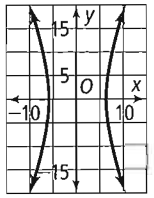
28. $\frac{x^2}{17} + \frac{y^2}{16} = 1$ **29.** $\frac{x^2}{25} + \frac{y^2}{29} = 1$ **30.** $\frac{x^2}{9} + \frac{y^2}{10} = 1$

31. $\frac{x^2}{40} + \frac{y^2}{36} = 1$ **32.** $\frac{x^2}{64} + \frac{y^2}{16} = 1$

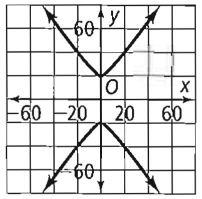
33. foci: $(0, \pm\sqrt{5})$



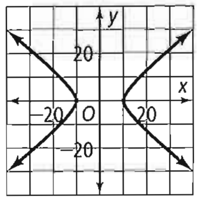
34. foci: $(\pm 3\sqrt{29}, 0)$



35. foci: $(0, \pm\sqrt{569})$



36. foci: $(\pm\sqrt{202}, 0)$



37. $\frac{x^2}{64} - \frac{y^2}{225} = 1$ **38.** $\frac{y^2}{49} - \frac{x^2}{576} = 1$

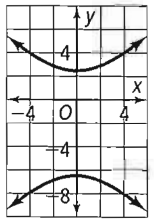
39. $\frac{x^2}{1.148 \times 10^{10}} - \frac{y^2}{3.395 \times 10^{10}} = 1$

40. $(x - 1)^2 + (y - 1)^2 = 25$

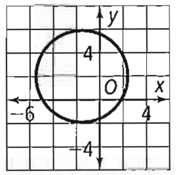
41. $\frac{(x - 3)^2}{4} + \frac{(y + 2)^2}{9} = 1$

42. $\frac{(x - 6)^2}{9} - \frac{(y - 3)^2}{16} = 1$

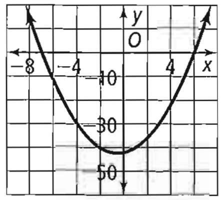
43. hyperbola; center (0, −2), foci: $(0, -2 \pm 2\sqrt{10})$



44. circle; center $\left(-\frac{3}{2}, 2\right)$, radius $\frac{\sqrt{61}}{2}$



45. parabola; vertex: $\left(-\frac{1}{2}, -\frac{169}{4}\right)$



Skills Handbook

p. 674 **1.** 46% **3.** 0.7% **5.** 1.035 **7.** 25% **9.** $66.\overline{6}\%$ **11.** 115% **13.** 12.5 **15.** 75 **17.** 20%

p. 675 **1.** $1\frac{2}{5}$ **3.** $6\frac{5}{6}$ **5.** $\frac{5}{21}$ **7.** $2\frac{19}{20}$ **9.** 8 **11.** $1\frac{1}{2}$ **13.** 2 **15.** 42

p. 676 **1.** 3 to 4 **3.** 19 g in 2 oz **5.** $\frac{14}{5}$ **7.** 8 **9.** 1.8 **11.** 1.95 **13.** 45.5 **15.** ±6

p. 677 **1.** 1 **3.** −38 **5.** −17 **7.** 4 **9.** 28 **11.** −12 **13.** −90 **15.** 12 **17.** 19 **19.** 9 **21.** −10

p. 678 **1.** 14 m^2 **3.** 30 cm^2 **5.** $91\frac{1}{8}$ ft^3 **7.** 100π $in.^3$ **9.** 110.5 $in.^2$ **11.** $121\frac{1}{2}$ ft^2

p. 679 **1.** I **3.** IV **5.** III **7.** $\frac{2}{3}$ **9.** 0 **11.** $\frac{1}{5}$ **13.** $\left(5, -\frac{3}{2}\right)$

p. 680 **1.** x^3 **3.** a^4b **5.** $\frac{1}{c^4}$ **7.** $\frac{x^5}{y^7z^3}$ **9.** d^8 **11.** c^6 **13.** $\frac{a^4}{b^5}$ **15.** $\frac{a^4}{b^4}$ **17.** c^{12} **19.** $u^{12}v^6$ **21.** a^3 **23.** $\frac{1}{mg^3}$ **25.** $\frac{a^5}{2}$

p. 681 **1.** $x^2 + 10x - 5$ **3.** $12x^4 - 20x^3 + 36x^2$ **5.** $x^2 - 2x - 15$ **7.** $(a - 6)(a - 2)$ **9.** $(x + 4)(x + 1)$ **11.** $(y + 8)(y - 3)$ **13.** $2x(x^2 + 2x - 4)$

p. 682 **1.** 1.34×10^6 **3.** 7.75×10^{-4} **5.** 111,300 **7.** 1.895×10^3 **9.** 1.234×10^5 **11.** 6.4×10^5 **13.** 8.52×10^2 **15.** 18 **17.** 8.95×10^{-12} **19.** 3.77×10^{10} **21.** 1.8×10^{-6}

p. 683 **1.** 10 **3.** 15 **5.** 54.7 **7.** 5 **9.** 13 **11.** 22.7 **13.** 2.8 **15.** 9 **17.** 7

p. 684

p. 685 **1.** $3.\overline{7}$; 5; 5 **3.** $3.9\overline{6}$; 2.4; 2.4 **5.** 1.5; 1.5; no mode

p. 686 **1.** $\frac{a}{3b^2}$ **3.** $\frac{1}{2}$ **5.** $4x$ **7.** $\frac{2}{h}$ **9.** $\frac{x}{10}$ **11.** $\frac{74x}{35}$ **13.** $\frac{4x^2}{5}$ **15.** $\frac{16}{x}$ **17.** $\frac{16}{5}$ **19.** 2x

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